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Results in Mathematics

On Some Holomorphic Cohomological Equations

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Abstract. Let Σ be a non compact Riemann surface and $\gamma : \Sigma \longrightarrow \Sigma$ an automorphism acting freely and properly such that the quotient $M = \Sigma/\gamma$ is a non compact Riemann surface. Using the fact that Σ and M are Stein manifolds, we prove that, for any holomorphic function $g : \Sigma \longrightarrow \mathbb{C}$ and any $\lambda \in \mathbb{C}$, there exists a holomorphic function $f : \Sigma \longrightarrow \mathbb{C}$ which is a solution of the holomorphic cohomological equation $f \circ \gamma - \lambda f = g$.

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1. Preliminaries

Let E be a complex Fréchet space whose topology is defined by a countable family of semi-norms $(p_n)_{n\in\mathbb{N}}$. Let $T : E \longrightarrow E$ be a bounded linear operator (that is a continuous linear map). We say that T is *inversible* if there exists a bounded operator $S : E \longrightarrow E$ such that $ST = TS = I$ (I is the identity of E).

1.1.

A *spectral value* of T is a complex number λ such that the operator $T - \lambda I$ is not inversible. The set of spectral values of T is the *spectrum* of T; it is denoted by $\sigma(T)$. A *regular value* of T is a complex number λ such that the operator $T - \lambda I$ is inversible. If E is a Banach space, $\sigma(T)$ is a compact set contained in the closed disc $D(0, |||T|||)$ where $|||T|||$ is the norm of T. It is not the case for a general Fréchet space.

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1.2.

An *eigenvalue* of T is a complex number λ such that the operator $T - \lambda I$ is not injective. Any non zero vector x of E satisfying the condition $Tx = \lambda x$ is called an *eigenvector* associated to the eigenvalue λ. Of course, any eigenvalue is a spectral value.

1.3. The Problem

Let Σ be a non compact Riemann surface and denote by $\mathcal{H}(\Sigma)$ the space of holomorphic functions $\Sigma \longrightarrow \mathbb{C}$. Let $(C_n)_{n\geq 0}$ be an increasing sequence of compact sets whose union is Σ. For any $f \in \mathcal{H}(\Sigma)$, we set:

$$p_n(f) = \sup_{z\in C_n} |f(z)|.$$

This defines a semi-norm on $\mathcal{H}(\Sigma)$. The family $(p_n)_{n\geq 0}$ define a distance:

$$\delta(f,g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \inf\left(1, p_n(f-g)\right)$$

on $\mathcal{H}(\Sigma)$; this distance is invariant by translation and makes $\mathcal{H}(\Sigma)$ a Fréchet space. Let γ be an automorphism of Σ. Then γ defines a *composition operator*:

$$T : f \in \mathcal{H}(\Sigma) \longmapsto f \circ \gamma \in \mathcal{H}(\Sigma)$$

on $\mathcal{H}(\Sigma)$. This operator is a topological automorphism of $\mathcal{H}(\Sigma)$ and then $\lambda = 0$ is a regular value of T. The following questions are natural:

What is the spectrum $\sigma(T)$ of T? For which complex numbers λ the operators $T - \lambda I$ are surjective?

A priori the answer to the latter question is not so easy to give analytically (is at least I concluded from some discussions with analysts). But a judicious remark on the fact that the cohomology of a Stein manifold with coefficients in an analytic coherent sheaf is trivial enables one to answer this question. The goal of this note is to prove the:

1.4. Theorem

Let Σ be a non compact Riemann surface and $\gamma : \Sigma \longrightarrow \Sigma$ an automorphism acting freely and properly such that the quotient $M = \Sigma/\gamma$ is a non compact Riemann surface. Then:

(i) *For any $\lambda \in \mathbb{C}$, the operator $T - \lambda I$ is surjective, that is, the cohomological equation $Tf - \lambda f = g$ admits a solution $f \in \mathcal{H}(\Sigma)$ for any $g \in \mathcal{H}(\Sigma)$.*

(ii) *Suppose Σ simply connected. Then any non zero complex number λ is an eigenvalue of the composition operator T induced by γ on $\mathcal{H}(\Sigma)$. The eigenspace $\mathcal{H}_\lambda(\Sigma)$ associated to λ is isomorphic to $\mathcal{H}(M)$.*

The case where $\Sigma = \mathbb{C}, \gamma(z) = z + 1$ and $\lambda = 1$, is exactly the situation studied by Guichard in [2].

1.5. Remark

Suppose that γ is conjugated to σ by an automorphism ϕ, that is we have $\gamma = \phi \circ \sigma \circ \phi^{-1}$. Then $f \in \mathcal{H}(\Sigma)$ is a solution of $f \circ \gamma - \lambda f = g$ if, and only if, $f_1 = f \circ \phi$ is a solution of $f_1 \circ \sigma - \lambda f_1 = g_1$ where $g_1 = g \circ \phi$. Hence the space $\mathcal{S}_\gamma$ of solutions of the equation $f \circ \gamma - \lambda f = g$ is naturally isomorphic to the space $\mathcal{S}_\sigma$ of solutions of the equation $f \circ \sigma - \lambda f = g$, the isomorphism is given by $f \in \mathcal{S}_\gamma \longmapsto f \circ \phi \in \mathcal{S}_\sigma$.

The next paragraph will be devoted to the proof of this theorem. We need to introduce some material on the complex geometry of Riemann surfaces.

2. Proof of the Theorem

We recall that Σ is non compact and that the group Γ generated by γ is isomorphic to $\mathbb{Z}$. The quotient $M = \Sigma/\gamma$ is then a Riemann surface (non compact by hypothesis) and the canonical projection $\pi : \Sigma \longrightarrow M$ is a holomorphic covering with group $\mathbb{Z}$. Then the Riemann surfaces Σ and M are Stein manifolds.

For $\lambda = 0$ the situation is immediate: T is a topological automorphism of the Fréchet space $\mathcal{H}(\Sigma)$, that is, 0 is a regular value. So we assume $\lambda \neq 0$.

2.1. Surjectivity of $T - \lambda I$

• First we recall a method which is very useful for computing cohomology in some specific situations; in general, it concerns coverings between topological spaces but we will focus our attention just on holomorphic coverings between complex manifolds.

Let $\widetilde{M}$ be a complex manifold and Γ a finitely generated group acting on it by complex automorphisms. If the action is free and proper the quotient $M = \widetilde{M}/\Gamma$ is a complex manifold and the canonical projection $\pi : \widetilde{M} \longrightarrow M$ is Galois holomorphic covering. Let $\widetilde{E} \longrightarrow \widetilde{M}$ be a holomorphic vector bundle on which the group Γ acts holomorphically. We suppose that $\widetilde{E}$ induces a complex vector bundle $E \longrightarrow M$ such that $\pi^*(E) = \widetilde{E}$. The group Γ acts on the space $\mathcal{H}(\widetilde{E})$ of holomorphic sections of $\widetilde{E}$:

$$(\gamma, \sigma) \in \Gamma \times \mathcal{H}(\widetilde{E}) \longmapsto \gamma \cdot \sigma \in \mathcal{H}(\widetilde{E}).$$

So $\mathcal{H}(\widetilde{E})$ is a Γ-module. The subspace $\mathcal{H}_\Gamma(\widetilde{E})$ of the Γ-invariant sections of $\widetilde{E}$ is canonically identified to the space $\mathcal{H}(E)$ of holomorphic sections of E. Let $\widetilde{\mathcal{E}}$ and $\mathcal{E}$ be the sheaves of germs on $\widetilde{M}$ and M associated respectively to $\widetilde{E}$ and E. The direct image $\pi_*(\widetilde{\mathcal{E}})$ by π of the sheaf $\widetilde{\mathcal{E}}$ is exactly $\mathcal{E}$. Then there exists a spectral sequence whose term E_2 is given by:

$$E_2^{k\ell} = H^k(\Gamma, H^\ell(\widetilde{M}, \widetilde{\mathcal{E}}))$$

and which converges to $H^*(M,\mathcal{E})$. (A good reference for this fact is [1].) If $\widetilde{M}$ is acyclic, that is:

$$H^\ell(\widetilde{M},\widetilde{\mathcal{E}}) = \begin{cases} \mathcal{H}(\widetilde{E}) & \text{if } \ell = 0 \\ 0 & \text{if } \ell \geq 1 \end{cases}$$

the spectral sequence E_r converges at the term E_2 and $H^k(M,\mathcal{E}) = H^k(\Gamma,\mathcal{H}(\widetilde{E}))$. For $\Gamma = \mathbb{Z}$ generated by γ, all the cohomology groups $H^k(\Gamma,\mathcal{H}(\widetilde{E}))$ are trivial for $k \geq 2, H^0(\Gamma,\mathcal{H}(\widetilde{E})) = \mathcal{H}_\Gamma(\widetilde{E})$ (the space of holomorphic Γ-invariant sections of $\widetilde{E}$) and $H^1(\Gamma,\mathcal{H}(\widetilde{E})) = \mathcal{H}(\widetilde{E})/\mathcal{C}$ where $\mathcal{C}$ is the subspace of $\mathcal{H}(\widetilde{E})$ generated by elements of the form $\sigma - \gamma\cdot\sigma$ with $\sigma \in \mathcal{H}(\widetilde{E})$. This introduces naturally the discrete cohomological equation $\sigma - \gamma\cdot\sigma = \alpha$ in the space $\mathcal{H}(\widetilde{E})$.

• Let $\widetilde{E} = \Sigma \times \mathbb{C}$ be the trivial holomorphic line bundle over Σ and ρ be the morphism of $\mathbb{Z}$ in $\mathrm{Aut}(\widetilde{E})$ (automorphism group of $\widetilde{E}$) defined by $\rho(1) = \rho$ where $\rho(z,\sigma) = (\gamma(z), u(\sigma))$ and $u(\sigma) = \lambda\sigma$ with $\lambda \in \mathbb{C}^*$. The quotient $E = \widetilde{E}/\rho$ is a holomorphic line bundle over M. It is flat with monodromy $\sigma \in \mathbb{C} \longmapsto \lambda\sigma \in \mathbb{C}$.

We will be interested by the cohomology of the discrete group $\mathbb{Z}$ with values in the space $\mathcal{H}(\widetilde{E})$ of holomorphic sections of $\widetilde{E}$; the action of $\mathbb{Z}$ on $\mathcal{H}(\widetilde{E})$ induced by ρ is:

$$(\rho\cdot\sigma)(z) = u(\sigma(\gamma^{-1}z)) = \lambda\sigma(\gamma^{-1}z).$$

Since $\widetilde{E}$ is trivial, $\mathcal{H}(\widetilde{E})$ is a free $\mathcal{H}(\Sigma)$-module with basis the constant section σ_0 which associates to any $z \in \Sigma$ the vector 1. Thus, any section $\sigma \in \mathcal{H}(\widetilde{E})$ must be written $\sigma(z) = f(z)\sigma_0$ where $f \in \mathcal{H}(\Sigma)$. The action of ρ on a section $\sigma = f\sigma_0$ of $\mathcal{H}(\widetilde{E})$ is:

$$(\rho\cdot\sigma)(z) = \lambda f(\gamma^{-1}z)\sigma_0.$$

• We have $H^0(\mathbb{Z},\mathcal{H}(\widetilde{E})) = \{\rho\text{-invariant sections}\}$; $f\sigma_0 \in \mathcal{H}(\widetilde{E})$ is ρ-invariant if and only if $\lambda f(\gamma^{-1}z) = f(z)$. This means that λ is an eigenvalue of the composition operator $T : f \in \mathcal{H}(\Sigma) \longmapsto f\circ\gamma \in \mathcal{H}(\Sigma)$ for which f is an eigenvector, that is, $f\circ\gamma = \lambda f$. The space $H^0(\mathbb{Z},\mathcal{H}(\widetilde{E}))$ is then "formed" by the eigenvectors associated to the eigenvalue λ of the operator T.

• Recall that $H^1(\mathbb{Z},\mathcal{H}(\widetilde{E}))$ is the quotient of $\mathcal{H}(\widetilde{E})$ by the subspace whose elements are $\rho\cdot\sigma - \sigma$. Calculate this space amounts to solving in $\mathcal{H}(\widetilde{E})$ the equation $\rho\cdot\sigma - \sigma = \xi$ that is the equation $\frac{f(\gamma z)}{\lambda} - f(z) = h(z)$ in $\mathcal{H}(\Sigma)$ or:

$$f(\gamma z) - \lambda f(z) = g(z)$$

with a given $g \in \mathcal{H}(\Sigma)$. We then easily see that $T - \lambda I$ will be surjective if $H^1(\mathbb{Z},\mathcal{H}(\widetilde{E}))$ is trivial. It will be the case!

Denote by $\widetilde{\mathcal{E}}$ and $\mathcal{E}$ the sheaves of germs of holomorphic sections associated respectively to the line bundles $\widetilde{E}$ and E. They are coherent analytic sheaves (cf. [3]) respectively over the Stein manifolds Σ and M. Then we have (cf. [3]):

$$H^1(\Sigma, \widetilde{\mathcal{E}}) = 0 \quad \text{and} \quad H^1(M, \mathcal{E}) = 0.$$

Now, we have a holomorphic covering $\pi : \Sigma \longrightarrow M$ with group $\mathbb{Z}$ and the direct image $\pi_*(\widetilde{\mathcal{E}})$ by π of the sheaf $\widetilde{\mathcal{E}}$ is exactly the sheaf $\mathcal{E}$. Since $(\Sigma, \widetilde{\mathcal{E}})$ is acyclic, the spectral sequence of the covering $\pi : \Sigma \longrightarrow M$ gives:

$$H^1(\mathbb{Z}, \mathcal{H}(\widetilde{E})) = H^1(M, \mathcal{E}) = 0.$$

2.2. Eigenvalues and Eigenspaces

We shall prove assertion ii) of the theorem. Since Σ is non compact and simply connected, it is holomorphically equivalent to the complex plane $\mathbb{C}$ or to the half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$. In the two cases the Riemann surface $M = \Sigma/\gamma$ is non compact (it is diffeomorphic to $\mathbb{C}^*$).

- *The case* $\Sigma = \mathbb{C}$

The automorphism group $\mathrm{Aut}(\mathbb{C})$ of the Riemann surface $\mathbb{C}$ is the semi-direct product of $\mathbb{C}$ by $\mathbb{C}^*$ where $\mathbb{C}^*$ acts on $\mathbb{C}$ by $(a, z) \in \mathbb{C}^* \times \mathbb{C} \longmapsto az \in \mathbb{C}$. More precisely $\gamma \in \mathrm{Aut}(\mathbb{C})$ if and only if $\gamma(z) = az + b$ where $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$ are constants. If γ has no fixed point it is necessarily of the form $\gamma(z) = z + b$. In this case the associated composition operator T maps the function $f \in \mathcal{H}(\mathbb{C})$ to $Tf \in \mathcal{H}(\mathbb{C})$ given by $Tf(z) = f(z + b)$. Any non zero complex number $\lambda = e^{\xi b}$ (with $\xi \in \mathbb{C}$) is an eigenvalue of T associated to the eigenvector $e_\xi(z) = e^{\xi z}$. Then $\sigma(T) = \mathbb{C}^*$ and formed only by eigenvalues. Any other eigenvector associated to λ is a holomorphic function ϕ of the form $\phi(z) = h(z)e^{\xi z}$ where h is a γ-invariant holomorphic function, that is, satisfying the condition $h(z + b) = h(z)$ for any $z \in \mathbb{C}$. (The γ-invariant holomorphic functions on Σ are exactly the holomorphic functions on the quotient Riemann surface $\mathbb{C}/\gamma$ which is isomorphic to $\mathbb{C}^*$.)

- *The case* $\Sigma = \mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$

Any automorphism γ of $\mathbb{H}$ can be written $\gamma(z) = \frac{az+b}{cz+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix in $\mathrm{SL}(2, \mathbb{R})$. If γ has no fixed point, it is conjugated to a translation $z \longmapsto z + b$ where $b \in \mathbb{R}$ (we say that γ is *parabolic*) or to a transformation of the type $z \longmapsto az$ where $a \in \mathbb{R}_+^* \backslash \{1\}$ (we say that γ is *hyperbolic*). So by Remark 1.5 one can reduce the problem to these two transformations.

If γ is parabolic, we show easily, like in the case $\Sigma = \mathbb{C}$, that any non zero complex number $\lambda = e^{\xi b}$ (with $\xi \in \mathbb{C}$) is an eigenvalue of the composition operator T; an eigenvector associated to this value is $e_\xi(z) = e^{\xi z}$. Any other eigenvector associated to λ is a holomorphic function $\phi : \mathbb{H} \longrightarrow \mathbb{C}$ of the form $\phi(z) = h(z)e^{\xi z}$ where $h : \mathbb{H} \longrightarrow \mathbb{C}$ is a γ-invariant holomorphic function (*i.e.* satisfying the condition $h(z + b) = h(z)$ for any $z \in \mathbb{H}$).

Suppose that γ is hyperbolic of the form $\gamma(z) = az$ with $a \in \mathbb{R}_+^* \backslash \{1\}$. (We say that γ is a *loxodromy*.) It is immediate to see that any $\lambda = a^n$ (with $n \in \mathbb{N}$) is an eigenvalue of T associated to the eigenvector $e_n(z) = z^n$.

On the open set $\Omega = \mathbb{C}\backslash\{w = u + iv : v = 0 \,\text{and}\; u \geq 0\}$ the logarithm function is well defined by:

$$\text{Log}(w) = \text{Log}(|w|) + i\text{arg}(w).$$

We can remark that, for any $w \in \Omega$, the quantity $w' = \text{Log}(w)$ belongs also to the open set Ω. We will use this fact to construct eigenvectors associated to the complex numbers $\lambda \in \mathbb{C}^*\backslash\{a^n : n \in \mathbb{N}\}$.

Suppose $\lambda \in \mathbb{C}^*\backslash\{a^n : n \in \mathbb{N}\}$. So the function $w \in \mathbb{H} \longrightarrow \lambda^w$ is well defined and then the function $\psi(z) = \lambda^{\theta(z)}$ where $\theta(z) = \frac{\text{Log}(z)}{\text{Log}(a)}$ is also well defined for any $z \in \mathbb{H}$ and it is holomorphic. A simple computation shows that $T\psi = \lambda\psi$ *i.e.* ψ is an eigenvector associated to λ. Any other eigenvector (associated to λ) Ψ is of the form $\Psi(z) = h(z)\psi(z)$ where $h : \mathbb{H} \longrightarrow \mathbb{C}$ is a holomorphic function on Σ which is γ-invariant *i.e.* satisfying $h(az) = h(z)$ for any $z \in \mathbb{H}$; it is a holomorphic function on the quotient Riemann surface $M = \mathbb{H}/\gamma$. □

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