Publ. Mat. **53** (2009), 329–354

SKEW-PRODUCT FOR GROUP-VALUED EDGE LABELLINGS OF BRATTELI DIAGRAMS

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Abstract

We associate a Cantor dynamical system to a non-properly ordered Bratteli diagram. Group valued edge labellings λ of a Bratteli diagram B give rise to a skew-product Bratteli diagram $B(\lambda)$ on which the group acts. The quotient by the group action of the associated dynamics can be a nontrivial extension of the dynamics of B. We exhibit a Bratteli diagram for this quotient and construct a morphism to B with unique path lifting property. This is shown to be an isomorphism for the dynamics if a property "loops lifting to loops" is satisfied by $B(\lambda) \to B$.

Introduction

In this article we associate (Subsection 1.10) a Cantor dynamical system to a non-properly ordered Bratteli diagram. In other words, we are able to define dynamics without the assumption of unique maximal path and unique minimal path. If one had a Bratteli diagram with, say 2 max paths and 2 min paths, one can, of course, define the usual Vershik transformation on the path space, except the two max paths. The issue of whether this map extends continuously seems a very subtle issue. Generically, one should expect that it does not. But it is easy to get examples where it does: just begin with a Cantor minimal system and choose K-R partitions where the tops shrink to two points. Naturally our model is not the infinite path space, but it reduces to it in the properly ordered case.

Since Bratteli diagrams with group actions rarely have unique maximal and minimal paths, until now there was no dynamics associated to them. Since this is no more a hurdle, in Subsection 2.1 we construct skew-product dynamical systems associated to finite group valued edge labellings. Our main result Theorem 2.11 has two parts. The first part

2000 *Mathematics Subject Classification.* 37B10, 54H20.

Key words. Cantor minimal system, Bratteli diagram, substitutional system.

gives a sufficient condition: when a certain condition "loops lifting to loops" (see 2.9) is satisfied for the labelling the quotient by G of our skew-product dynamical system is the original system. In the second part of Theorem 2.11 we describe what happens when the "loops lifting to loops" property does not hold; we show that these skew-products also arise from an edge labelling enjoying the property "loops lifting to loops" of *another Bratteli diagram* which admits a morphism into the original Bratteli diagram with a certain "unique path lifting" property. This is achieved by employing a certain "tripling" construction (see Remark 2.14). Our attention has been drawn by the referee to the notion of "collaring" introduced by J. E. Anderson and I. F. Putnam [**AP**] in the context of tilings, which, as observed by the referee, is seemingly a procedure akin to what we refer to here as "tripling". We heartily thank the referee for this observation. However, our results here are different from [**AP**] as well as [**M**] below.

Matui [**M**] studied "skew-product dynamical systems", even though he did not consider edge labellings. We were inspired to consider edge labellings in the context of Bratteli diagrams by studying the abovementioned article of Matui and also the article [**KP**] of Kumjian and Pask who introduced skew-product C^*-algebras associated to group valued edge labellings of graphs. The obvious diagrams which arise in pursuing our curiosity are seldom properly ordered and this was our first stumbling block which we had to overcome. We remark at the outset that the skew-products which Matui associates to cocycles arise as our skew-products for particular edge labellings. For Matui, going modulo the group action on the skew-product system always produces the original dynamical system. The quotient by G of our skew-product systems would in general be nontrivial extensions of the original dynamical system. Even for an odometer system our skew-product associated to edge labellings arises from a cocycle for Matui only when the group is severely restricted. (See Example 2.10.2). We are further thankful to the referee for many valuable remarks. He/She also informed us about a paper by K. Medynets [**Me**].

We briefly recall some basic concepts and definitions which are fundamental in the theory of Cantor dynamical systems.

1. Preliminaries

A *topological dynamical system* is a pair (X, φ) where X is a compact metric space and φ is a homeomorphism in X. We say that φ is *minimal* if for any $x \in X$, the φ-orbit of $x := \{\varphi^n(x) \mid n \in \mathbb{Z}\}$ is dense in X.

We say that (X, φ) is a *Cantor dynamical system* if X is a Cantor set, i.e. X is totally disconnected without isolated points. (X, φ) is a *Cantor minimal dynamical system* if, in addition, φ is minimal. Some of the basic concepts of the theory are recalled below, mostly from the more detailed sources [**DHS**] and [**HPS**].

1.1. Bratteli diagram. A Bratteli diagram is an infinite directed graph (V, E), where V is the vertex set and E is the edge set. Both V and E are partitioned into non-empty disjoint finite sets

$$V = V_0 \cup V_1 \cup V_2 \cdots \text{ and } E = E_1 \cup E_2 \cup \cdots .$$

There are two maps $r, s\colon E \to V$ the *range* and *source* maps. The following properties hold:

(i) $V_0 = \{v_0\}$ consists of a single point, referred to as the "top vertex" of the Bratteli diagram.

(ii) $r(E_n) \subseteq V_n$, $s(E_n) \subseteq V_{n-1}$, $n = 1, 2, \ldots$. Also $s^{-1}(v) \neq \phi$, $\forall\, v \in V$ and $r^{-1}(v) \neq \phi$ for all $v \in V_1, V_2, \ldots$.

Maps between Bratteli diagrams are assumed to preserve gradings and intertwine the range and source maps. If $v \in V_n$ and $w \in V_m$, where $m > n$, then a path from v to w is a sequence of edges $(e_{n+1}, \ldots, e_m)$ such that $s(e_{n+1}) = v$, $r(e_m) = w$ and $s(e_{j+1}) = r(e_j)$. Infinite paths from $v_0 \in V_0$ are defined similarly. The Bratteli diagram is called *simple* if for any $n = 0, 1, 2, \ldots$ there exists $m > n$ such that every vertex of V_n can be joined to every vertex of V_m by a path.

1.2. Order. An *ordered* Bratteli diagram $(V, E, \geq)$ is a Bratteli diagram (V, E) together with a linear order on $r^{-1}(v)$, $\forall\, v \in V - \{v_0\} = V_1 \cup V_2 \cup V_3 \cdots$. We say that an edge $e \in E_n$ is a *maximal* edge (resp. *minimal* edge) if e is maximal (resp. minimal) with respect to the linear order in $r^{-1}(r(e))$.

Given $v \in V_n$, it is easy to see that there exists a unique path $(e_1, e_2, \ldots, e_n)$ from v_0 to v such that each e_i is maximal (resp. minimal).

Note that if $m > n$, then for any $w \in V_m$, the set of paths starting from V_n and ending at w obtains an induced (lexicographic) linear order:

$$(e_{n+1}, e_{n+2}, \ldots, e_m) > (f_{n+1}, f_{n+2}, \ldots, f_m)$$

if for some i with $n + 1 \leq i \leq m$, $e_j = f_j$ for $1 < j \leq m$ and $e_i > f_i$.

1.3. Proper order. A *properly ordered* Bratteli diagram is a simple ordered Bratteli diagram $(V, E \geq)$ which possesses a unique infinite path $x_{\max} = (e_1, e_2, \dots)$ such that each e_i is a maximal edge and a unique infinite path $x_{\min} = (f_1, f_2, \dots)$ such that each f_i is a minimal edge.

Given a properly ordered Bratteli diagram $B = (V, E, \geq)$ we denote by X_B its infinite path space. So

$$X_B = \{(e_1, e_2, \dots) \mid e_i \in E_i,\, r(e_i) = s(e_{i+1}),\, i = 1, 2, \dots\}.$$

For an initial segment $(e_1, e_2, \dots, e_n)$ we define the cylinder sets

$$U(e_1, e_2, \dots, e_n) = \{(f_1, f_2, \dots) \in X_B \mid f_i = e_i,\, 1 \leq i \leq n\}.$$

By taking cylinder sets to be a basis for open sets X_B becomes a topological space. We exclude trivial cases (where X_B is finite, or has isolated points). Thus, X_B is a Cantor set. X_B is a metric space, where for two paths x, y whose initial segments to level m agree but not to level $m+1$, $d(x, y) = 1/m + 1$.

1.4. Vershik map. If $x = (e_1, e_2, \dots, e_n, \dots) \in X_B$ and if at least one e_i is not maximal define

$$V_B(x) = y = (f_1, f_2, \dots, f_j, e_{j+1}, e_{j+2}, \dots) \in X_B$$

where $e_1, e_2, \dots, e_{j-1}$ are maximal, e_j is not maximal and has f_j as successor in the linearly ordered set $r^{-1}(r(e_j))$ and $(f_1, f_2, \dots, f_{j-1})$ is the minimal path from v_0 to $s(f_j)$. Extend the above V_B to all of X_B by setting $V_B(x_{\max}) = x_{\min}$. Then (X_B, V_B) is a Cantor minimal dynamical system. V_B is called a Vershik map.

Next, we describe the construction of a dynamical system associated to a non-properly ordered Bratteli diagram. The Bratteli diagram need not be simple. To motivate this construction, it is perhaps worthwhile to begin by indicating how it works in the case of an ordered Bratteli diagram associated to a nested sequence of Kakutani-Rohlin partitions of a Cantor dynamical system (X, T).

1.5. K-R partition. A *Kakutani-Rohlin partition* of the Cantor minimal system (X, T) is a clopen partition $\mathcal{P}$ of the kind

$$\mathcal{P} = \{T^j Z_k \mid k \in A \text{ and } 0 \leq j < h_k\}$$

where A is a finite set and h_k is a positive integer. The k^{th} *tower* $\mathcal{S}_k$ of $\mathcal{P}$ is $\{T^j Z_k \mid 0 \leq j < h_k\}$; its *floors* are $T^j Z_k$, $(0 \leq j < h_k)$. The *base* of $\mathcal{P}$ is the set $Z = \bigcup_{k \in A} Z_k$.

Let $\{\mathcal{P}_n\}$, $(n \in \mathbb{N})$ be a sequence of Kakutani-Rohlin partitions

$$\mathcal{P}_n = \{T^j Z_{n,k} \mid k \in A_n, \text{ and } 0 \leq j < h_{n,k}\},$$

with $\mathcal{P}_0 = \{X\}$ and with base $Z_n = \bigcup_{k \in A_n} Z_{n,k}$. We say that this sequence is *nested* if, for each n,

(i) $Z_{n+1} \subseteq Z_n$.
(ii) $\mathcal{P}_{n+1}$ refines the partition $\mathcal{P}_n$.

For the Bratteli-Vershik system (X_B, V_B) of Subsections 1.3–1.4, one obtains a Kakutani-Rohlin partition $\mathcal{P}_n$ for each n by taking the sets in the partition to be the cylinder sets $U(e_1, e_2, \ldots, e_n)$ of Subsection 1.3 and taking as the base of the partition the union $\bigcup U(e_1, e_2, \ldots, e_n)$ over minimal paths (i.e., each e_i is a minimal edge). This is a nested sequence.

1.6. To any nested sequence $\{\mathcal{P}_n\}$, $(n \in \mathbb{N})$ of Kakutani-Rohlin partitions we associate an ordered Bratteli diagram $B = (V, E, \geq)$ as follows (see [**DHS**, Section 2.3]): the $|A_n|$ towers in $\mathcal{P}_n$ are in 1-1 correspondence with V_n, the set of vertices at level n. Let $v_{n,k} \in V_n$ correspond to the tower $\mathcal{S}_{n,k} = \{T^j Z_{n,k} \mid 0 \leq j < h_{n,k}\}$ in $\mathcal{P}_n$. We refer to $T^j Z_{n,k}$, $0 \leq j < h_{n,k}$ as *floors* of the tower $\mathcal{S}_{n,k}$ and to $h_{n,k}$ as the *height* of the tower. *We will exclude nested sequences of K-R partitions where the infimum (over k for fixed n) of the height $h_{n,k}$ does not go to infinity with n.* Let us view the tower $\mathcal{S}_{n,k}$ against the partition $\mathcal{P}_{n-1} = \{T^j Z_{n-1,k} \mid k \in A_{n-1}, \text{ and } 0 \leq j < h_{n-1,k}\}$. As the floors of $\mathcal{S}_{n,k}$ rise from level $j = 0$ to level $j = h_{n,k} - 1$, $\mathcal{S}_{n,k}$ will start traversing a tower $\mathcal{S}_{n-1,i_1}$ from the bottom to the top floor, then another tower $\mathcal{S}_{n-1,i_2}$ from the bottom to the top floor, then another tower $\mathcal{S}_{n-1,i_3}$ likewise and so on till a final segment of $\mathcal{S}_{n,k}$ traverses a tower $\mathcal{S}_{n-1,i_m}$ from the bottom to the top. Note that in this final step the top floor $T^j Z_{n,k}$ for $j = h_{n,k} - 1$ of $\mathcal{S}_{n,k}$ reaches the top floor $T^q Z_{n-1,i_m}$ for $q = h_{n-1,i_m} - 1$ of $\mathcal{S}_{n-1,i_m}$ as a consequence of the assumption $Z_n \subset Z_{n-1}$ and the fact that T^{-1} (union of bottom floors) = union of top floors. Bearing in mind this order in which $\mathcal{S}_{n,k}$ traverses $\mathcal{S}_{n-1,i_1}, \mathcal{S}_{n-1,i_2}, \ldots, \mathcal{S}_{n-1,i_m}$ we associate m edges, ordered as $e_{1,k} < e_{2,k} < \cdots < e_{m,k}$ and we set the range and source maps for edges by $r(e_{j,k}) = v_{n,k}$ and $s(e_{j,k}) = v_{n-1,i_j}$. Note that m depends on the index $k \in A_n$ (and that by convention the indexing sets A_n are disjoint). E_n is the disjoint union over $k \in A_n$ of the edges having range in V_n.

1.7. For $x \in X$, we define $x_n \in \mathcal{P}_n^{\mathbb{Z}}$, $n \in \mathbb{N}$ as follows: $x_n = (x_{n,i})_{i \in \mathbb{Z}}$, where $x_{n,i} \in \mathcal{P}_n$ is the unique floor in $\mathcal{P}_n$ to which $T^i(x)$ belongs. If $m >$

n, let $j_{m,n}\colon \mathcal{P}_m \to \mathcal{P}_n$ be the unique map defined by $j_{m,n}(F) = F'$ if $F \subseteq F'$. (By abuse of notation, we use the same symbol F to denote a point of the finite set $\mathcal{P}_m$ and also to denote the subset of X, in the partition $\mathcal{P}_n$, which F represents.) An important property of the map

$$X \longrightarrow \prod_n (\mathcal{P}_n^{\mathbb{Z}}),\ x \longmapsto (x_1, x_2, \dots),\ x_n = (x_{n,i})_{i\in\mathbb{Z}},$$

defined above is the following:

1.8. If F and TF are two successive floors of a $\mathcal{P}_n$-tower and if $x_{n,i} = F$ then $x_{n,i+1} = TF$. If $x_{n,i}$ is the top floor of a $\mathcal{P}_n$-tower, then $x_{n,i+1}$ is the bottom floor of a $\mathcal{P}_n$-tower. More importantly, given integers K and n, there exist $m > n$ and a single tower $\mathcal{S}_{m,k}$ of level m such that the finite sequence $(x_{n,i})_{-K\le i\le K}$ is an interval segment contained in

$$\{j_{m,n}(T^\ell(Z_{m,k})) \mid 0 \le \ell < h_{m,k}\}.$$

This is a consequence of the assumption that the infimum of the heights of level-n towers goes to infinity. It is true that $x_{n,i} = j_{m,n}(x_{m,i})$, but the sequence $(x_{m,i})_{-K\le i\le K}$ need not be an interval segment of $\{T^\ell(Z_{m,k}) \mid 0 \le \ell < h_{m,k}\}$.

The foregoing observations in the case of an ordered Bratteli diagram associated to a nested sequence of Kakutani-Rohlin partitions gives us the hint to define a dynamical system (X_B, T_B) of a non properly ordered Bratteli diagram $B = (V, E, \ge)$ as follows:

1.9. $\boldsymbol{\varpi_n}$-tower. For each n define $\varpi_n =$ the set of paths from V_0 to V_n. There is an obvious truncation map $j_{m,n}\colon \varpi_m \to \varpi_n$ which truncates paths from V_0 to V_m to the initial segment ending in V_n. For each $v \in V_n$, the set $\varpi(v)$ of paths from $\{*\} \in V_0$ ending at v will be called a "*ϖ_n-tower parametrised by v*"; the cardinality $|\varpi(v)|$ will be referred to as the height of this tower. Each tower is a linearly ordered set (whose elements may be referred to as floors of the tower) since paths from v_0 to v acquire a linear order (*cf.* 1.2). *We will exclude unusual examples of ordered Bratteli diagram where the infimum of the height of level-n towers does not go to infinity with n*, (for example like [**HPS**, Example 3.2]). Now, we define

1.10. Underlying space of the dynamics. $X_B = \{x = (x_1, x_2, \dots, x_n, \dots)\}$ where

(i) $x_n = (x_{n,i})_{i\in\mathbb{Z}} \in \varpi_n^{\mathbb{Z}}$,

(ii) $j_{m,n}(x_{m,i}) = x_{n,i}$ for $m > n$ and $i \in \mathbb{Z}$ and

(iii) given n and K there exists m such that $m > n$ and a vertex $v \in V_m$, such that the interval segment $x_n[-K, K] := (x_{n,-K}, x_{n,-K+1}, \dots, x_{n,K})$ is obtained by applying $j_{m,n}$ to an interval segment of the linearly ordered set of paths from v_0 to v.

The condition (iii) is the crucial part of the definition. Without it what one gets is an inverse system.

The condition (iii) implies that a property similar to 1.8 holds. Since each ϖ_n is a finite set $\varpi_n^{\mathbb{Z}}$ has a product topology which makes it a compact set —in fact a Cantor set. Likewise, $\prod_n(\varpi_n^{\mathbb{Z}})$ is again a Cantor set. Thus, $X_B \subseteq \prod_n(\varpi_n^{\mathbb{Z}})$ has an induced topology. The lemma below and the following proposition are analogous to corresponding facts for the Vershik model associated to properly ordered Bratteli diagrams.

Lemma 1.11. *The topological space X_B is compact.*

Proof: We show that X_B is closed in $\prod_n \varpi_n^{\mathbb{Z}}$. Let $z = (z_1, z_2, \dots, z_n, \dots)$, where $z_n \in \varpi_n^{\mathbb{Z}}$. Assume that $z = \lim_{m\to\infty} w_m$, where $w_m \in \prod_n \varpi_n^{\mathbb{Z}}$ and moreover $w_m \in X_B$. Let $K_1, K_2, \dots,$ be a strictly increasing sequence of positive integers. Define neighbourhoods $U_1, U_2, \dots, U_m, \dots$ of z in $\prod_n \varpi_n^{\mathbb{Z}}$ shrinking to z by $U_m = \{z' = (z'_1, z'_2, \dots, z'_m, \dots) \in \prod_n \varpi_n^{\mathbb{Z}}\}$ where z'_k has the same coordinates as z_k in the range $[-K_m, K_m]$, i.e. $z'_k[-K_m, K_m] = z_k[-K_m, K_m]$ for $1 \le k \le m$. Since $z = \lim w_j$, for any given m, $\exists\ J(m)$, such that "$j > J(m)$" $\Rightarrow w_j \in U_m$. But, since $w_j \in X_B$, we conclude that for $1 \le k \le m$, the interval segments $z_k[-K_m, K_m]$ are obtained by applying $j_{M,k}$ to an interval segment of the sequence of floors of a single ϖ_M-tower (for some M). This shows that $z \in X_B$. □

Denote by T_B the restriction of the shift operator to X_B. So, if $x = (x_1, x_2, \dots)$, where $x_n = (x_{n,i})_{i\in\mathbb{Z}} \in \varpi_n^{\mathbb{Z}}$, then $T_B(x) = (x'_1, x'_2, \dots, x'_n, \dots)$, where $x'_n = (x'_{n,i})_{i\in\mathbb{Z}} \in \varpi_n^{\mathbb{Z}}$ and $x'_{n,i} = x_{n,i+1}$.

(X_B, T_B) will be called the dynamical system associated to $B = (V, E, \geq)$.

Proposition 1.12. *If $B = (V, E, \geq)$ is a simple ordered Bratteli diagram, then (X_B, T_B) is a Cantor minimal dynamical system.*

Proof: Let $x, y \in X_B$, where $x = (x_1, \dots, x_m, \dots)$ and $y = (y_1, \dots, y_m, \dots)$ satisfy the conditions (i), (ii) and (iii) of 1.10. We will show that x belongs to the closure of the orbit $\{T_B^i(y) \mid i \in \mathbb{Z}\}$. For integers n and K

define the neighbourhood $U(n,K,x)$ of x to be the set

$$\{z=(z_1,z_2,\dots,z_m,\dots)\in X_B \mid z_m[-K,K]=x_m[-K,K] \quad \text{for } 1\le m\le n\}.$$

Given any neighbourhood U of x, choose n and K such that $U \supseteq U(n,K,x)$. Since $x\in X_B$, we can choose J and a $\mathcal{P}_J$-tower $\mathcal{S}_v$ of paths from v_0 ending at a fixed $v\in V_J$, such that $x_n[-K,K]$ is obtained by applying $j_{J,n}$ to an interval segment of $\mathcal{S}_v$. Since the Bratteli diagram $(V,E,\geq)$ is simple, we can choose $L>J$ such that every point of V_J is connected to every point of V_L by a path. This implies that $x_n[-K,K]$ occurs as an interval segment of the sequence obtained by applying $j_{L,n}$ to the sequence of floors of *any* $\mathcal{P}_L$-tower. Now, choose $a<b$ such that $y_L[a,b]$ is the sequence of all floors of a $\mathcal{P}_L$-tower. From the preceding observation, we can choose d such that $a\le d<d+2K\le b$ and $x_n[-K,K]=j_{L,n}(y_L[d,d+2K])$. But since $y\in X_B$, y satisfies condition (ii) in 1.10. So $x_n[-K,K]=y_n[d,d+2K]$ and we conclude $T_B^{d+K}(y)\in U(n,K,x)$. □

In 1.7, given a nested sequence of Kakutani-Rohlin partitions of (X,T), we defined a map from (X,T) to the dynamical system (X_B,T_B) of the associated ordered Bratteli diagram. It follows that if (X,T) is minimal, and if the Bratteli diagram of the nested sequence of K-R partitions is a simple Bratteli diagram, then $(X,T)\to(X_B,T_B)$ is onto. If the topology of (X,T) is spanned by the collection of the clopen sets belonging to the K-R partitions then clearly the map $(X,T)\to(X_B,T_B)$ is injective. In particular, if the Bratteli diagram is properly ordered then the Bratteli-Vershik system is naturally isomorphic to the system given by our construction in 1.10.

Note that the same term "*towers*" has been used to denote two separate but related objects (in 1.5 and 1.9). For $v\in V_n$, let y be a path from $\{*\}$ to v in $(V,E,\geq)$. So, y is a "floor" (consisting of the single element y) belonging to the ϖ_n- tower $\varpi(v)$ (a finite set) parametrized by $v\in V_n$ —all in the sense of 1.9. Here, $\varpi(v)$= all paths from $\{*\}$ to v. Put $\mathcal{F}_y=\{x=(x_1,x_2,\dots,x_n,\dots)\in X_B \mid x_{n,0}=y\}$. $\mathcal{F}_y$ is a clopen set of the Cantor set X_B. Put $\mathcal{P}_n=\{\mathcal{F}_y \mid y\in\varpi(v),\ v\in V_n\}$. Then, in the sense of 1.5 $\mathcal{P}_n$ is a K-R partition of X_B whose base is the union of $\bigcup\mathcal{F}_y$, (y minimal $\in\varpi(v)$, $v\in V_n$). Its towers S_v are parametrized by $v\in V_n : S_v=\{\mathcal{F}_y \mid y\in\varpi(v)\}$. $\mathcal{F}_y$, ($y\in\varpi(v)$) are the floors of the tower S_v. (We encountered this K-R partition earlier in the case of the Bratteli-Vershik system at the end of 1.5.) It is easy to see that the ordered Bratteli diagram obtained from $\{\mathcal{F}_y \mid y\in\varpi(v),\ v\in V_n\}$

is $(V, E, \geq)$ (compare the last two lines in the proof of [**HPS**, Theorem 4.5] and also the opening observation in the proof of [**DHS**, Proposition 16]).

2. Skew-product dynamical systems

2.1. Let $(V, E, \geq)$ be an ordered Bratteli diagram. Let G be a finite group and $\lambda\colon E \to G$ a map defined on the edges with values in G ($\lambda \stackrel{\text{def.}}{=}$ the "*labels*" on edges). We define a new Bratteli diagram $B(\lambda) \stackrel{\text{def.}}{=} (V_\lambda, E_\lambda)$ as follows: $V_{0,\lambda} = \{*\}$, $V_{n,\lambda} = V_n \times G$, $(n \geq 1)$ and $E_{n,\lambda} = E_n \times G$, $(n \geq 1)$. The source and range maps on E_λ are defined by $r(e, g) = (r(e), g)$, and $s(e, g) = (s(e), g\lambda(e))$. If $e \in E_1$, we define $s(e, g) = \{*\}$. We define a map $\pi\colon (V_\lambda, E_\lambda) \to (V, E)$ by $\pi(v_{\lambda,0}) = v_0$, $\pi(v, g) = v$ if $v \in V_n$, $n \geq 1$ and $\pi(e, g) = e$. One sees that π commutes with the range and source map. It is easy to see that $\pi|_{r^{-1}(v,g)}$ maps bijectively onto $r^{-1}(v)$, $\forall\, v \in V - \{v_0\}$ and $\forall\, g \in G$. Thus if $(V, E, \geq)$ is an ordered Bratteli diagram there is a unique order in E_λ such that $\pi\colon (V_\lambda, E_\lambda, \geq) \to (V, E, \geq)$ is order preserving. In the sequel we assume that (V_λ, E_λ) is equipped with this order. The ordered Bratteli diagram $(V_\lambda, E_\lambda, \geq)$ will be denoted by $B(\lambda)$. The dynamical system constructed in 1.10 for this ordered Bratteli diagram $B(\lambda)$ will be denoted by (X_λ, T_λ). The dynamical system (X_λ, T_λ) will be called the *skew-product system* for the edge labelling λ. We also remark that

$$\pi\colon (V_\lambda, E_\lambda) \longrightarrow (V, E) \text{ has the "\textit{unique path lifting}" property}$$

in the following sense. If $m > n \geq 1$, and $(e_n, e_{n+1}, \ldots, e_m)$ is a path in (V, E) from V_{n-1} to V_m with $r(e_m) = v$ then for any $g \in G$, there is a unique path $(\tilde{e}_n, \tilde{e}_{n+1}, \ldots, \tilde{e}_m)$ in (V_λ, E_λ) which maps onto $(e_n, e_{n+1}, \ldots, e_m)$ under π and such that $r(\tilde{e}_m) = (v, g)$.

[Caution: Even in the presence of unique path lifting property two different edges on the left with the same source may map into the same edge on the right. See 2.8 below. What the property asserts is that two different edges on the left with the same range cannot map to the same edge on the right.]

The group G acts on $(V_\lambda, E_\lambda, \geq)$ by $\gamma_g(v, h) = (v, gh)$ and $\gamma_g(e, h) = (e, gh)$ for $v \in V - \{v_0\}$ and of course $\gamma_g(v_{0,\lambda}) = v_{0,\lambda}$.

Given the labelling λ we can extend the labelling to paths from V_{n-1} to V_m, for $m > n$. With notation as above, we define:

$$\lambda(e_n, e_{n+1}, \ldots, e_m) = \lambda(e_m)\lambda(e_{m-1}) \ldots \lambda(e_n).$$

Let $\{n_k\}_{k=0}^{\infty}$ be a subsequence of $\{0,1,2,\dots\}$ where we assume $n_0 = 0$. A Bratteli diagram (V', E') is called a "*telescoping*" of (V, E) if $V'_k = V_{n_k}$ and E'_k consists of paths $(e_{n_{k-1}+1}, \dots, e_{n_k})$ from $V_{n_{k-1}}$ to V_{n_k} in (V, E), the range and source maps being the obvious ones. Thus, for every telescoping the labelling λ on the edges of (V, E) gives rise to a labelling on the edges of (V', E').

Remark 2.2. Suppose there is a telescoping (V', E') of (V, E) such that the induced labelling λ on (V', E') has the following property: given k, there exists $v' \in V'_k$ and $w' \in V'_{k+1}$, such that $\lambda(s^{-1}(v') \cap r^{-1}(w')) = G$. If in addition (V, E) is simple, then (V_λ, E_λ) is simple.

2.3. Stationary Bratteli diagrams. A Bratteli diagram is stationary if the diagram repeats itself after level 1. (One may relax by allowing a period from some level onwards; but, a telescoping will be stationary in the above restricted sense.) If $(V, E, \geq)$ is an ordered Bratteli diagram and the diagram together with the order repeats itself after level 1, then $(V, E, \geq)$ will be called a stationary ordered Bratteli diagram. We refer the reader to [**DHS**, Section 3.3] for the usual definition of a substitutional system and how they give rise to stationary Bratteli diagrams. Some details are recalled below. Let $(V, E, \geq)$ be as above and suppose moreover that it is a simple Bratteli diagram and that λ is a labelling of the edges with values in a finite group G. Assume that the labelling is stationary: so, we have

(1) an enumeration $\{v_{n,1}, v_{n,2}, \dots, v_{n,L}\}$ of V_n, $\forall\, n \geq 1$,
(2) for $n > 1$ and $1 \leq j \leq L$ an enumeration $\{e_{n,j,1}, e_{n,j,2}, \dots, e_{n,j,a_j}\}$ of $r^{-1}(e_{n,j})$ which is assumed to be listed in the linear order in $r^{-1}(v_{n,j})$,
(3) in the enumerations above, L does not depend on n and a_j depends only on j and not on n. Moreover, if $n, m > 1$, if $1 \leq j \leq L$, $1 \leq k \leq L$, $1 \leq i \leq a_j$, then "$s(e_{n,j,i}) = v_{n-1,k}$" $\Rightarrow$ "$s(e_{m,j,i}) = v_{m-1,k}$",
(4) with notation as in (3) above, $\lambda(e_{n,j,i}) = \lambda(e_{m,j,i})$.

If S is a set of generators for G and if in addition to the above, we also assume that $\lambda(\{r^{-1}(v_{n,j})\} \cap \{s^{-1}(v_{n-1,k})\}) \supseteq S \cup \{e\}$, $\forall\, j, k$ between 1 and L then (V_λ, E_λ) is simple.

2.4. Substitutional systems. Let A be an alphabet set. Write A^+ for the set of words of finite length in the alphabets of A. Let $\sigma\colon A \to A^+$ be a substitution, written, $\sigma(a) = \alpha\beta\gamma\dots$. The stationary ordered Bratteli diagram $B = (V, E, \geq)$ associated to (A, σ) (*cf.* [**DHS**, Section 3.3] can be described as

$$V_n = A, \forall\, n \geq 1,\ V_0 = \{*\}$$

$$E_n = \{(a,k,b) \mid a,b \in A,\ k \in \mathbb{N},\ a \text{ is the } k^{\text{th}} \text{ alphabet in the word } \sigma(b)\}.$$

(If one prefers one can introduce an extra factor "$\times\{n\}$" so that vertices and edges at different levels are seen to be disjoint.) The source and range maps s and r are defined by $s(a,k,b) = a$, $r(a,k,b) = b$. In the linear order in $r^{-1}(b)$, (a,k,b) is the k^{th} edge. If λ is a stationary labelling on edges with values in a finite group G, then define a new "skew-product" substitutional system $(A_\lambda, \sigma_\lambda)$ as follows:=

$$A_\lambda = A \times G.$$

$$E_{n,\lambda} = \{[(a,g),k,(b,h)] \mid \text{(i) } (a,k,b) \in E_n,\ \text{(ii) } g = h \cdot \lambda(a,k,b)\}.$$

Define $\sigma_\lambda \colon A_\lambda \to A_\lambda^+$ by the rule that

$$(a,g) \text{ is the } k^{\text{th}} \text{ alphabet in the word } \sigma_\lambda(b,h)$$

if

$$[(a,g),k,(b,h)] \in E_{n,\lambda}.$$

Set $V_\lambda = \{*\} \cup \{\bigcup_{n\in\mathbb{N}} V_{n,\lambda}\}$ where each $V_{n,\lambda}$ is a (disjoint) copy of A_λ and likewise, $E_\lambda = \cup_n E_{n,\lambda}$. The source and range maps are defined by $s([(a,g),k,(b,h)]) = (a,g)$ and $r([(a,g),k,(b,h)]) = (b,h)$. Then $(V_\lambda, E_\lambda, \geq)$ is the stationary ordered Bratteli diagram arising from the substitutional system $(A_\lambda, \sigma_\lambda)$. The group G acts on $(V_\lambda, E_\lambda, \geq)$ by

$$g(a,g_1) = (a, gg_1)$$

$$g[(a,g_1),k,(b,g_2)] = [(a,gg_1),k,(b,gg_2)].$$

The ordered Bratteli diagram $(V_\lambda, E_\lambda, \geq)$ thus obtained is the same as the skew-product of 2.1 if one starts with the $(V,E,\geq)$ in the beginning of 2.4.

To the stationary ordered Bratteli diagram B of (A,σ) (which may not be properly ordered unless σ is a primitive, aperiodic, proper substitution, see [**DHS**, Section 3]) we can associate a dynamical system X_B following the construction of 1.10; we will see that this is naturally isomorphic to the substitutional dynamical system X_σ associated to (A,σ) defined for example in [**DHS**, Section 3.3.1].

2.5. *For a stationary ordered Bratteli diagram B the dynamical system 1.10 is identical to the usual substitutional dynamical system.*

Let us use the nested sequence $(\mathcal{P}_n)$ of Kakutani-Rohlin partitions in X_σ defined in [**DHS**, Corollary 13]; the ordered Bratteli diagram associated by Subsection 1.6 to this nested sequence $(\mathcal{P}_n)$ is the same as the stationary ordered Bratteli diagram B in the beginning of this section. (This is the opening observation in the proof of [**DHS**, Proposition 16], but, this observation also remains valid for improper substitutions.) Next apply the map $X_\sigma \to X_B$ that was defined in 1.7. As remarked after the proof of Proposition 1.12, this map is onto. That this map is also injective can be seen as follows: let $x \in X_\sigma$. So, $x = (a_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}}$. Let the image of x in X_B by the map of 1.7 be denoted by $(x_1, x_2, \dots)$ where $x_n = (x_{n,i})_{i\in\mathbb{Z}} \in \mathcal{P}_n^{\mathbb{Z}}$. The sequence $x = (a_i)_i \in A^{\mathbb{Z}}$ can be read off from the first co-ordinate $x_1 = (x_{1,i})_i \in \mathcal{P}_1^{\mathbb{Z}}$. Choose any element $z \in x_{1,i}$. Writing, uniquely, as in [**DHS**, Corollary 12(ii)] $z = T_\sigma^k(\sigma(y))$ where $y \in X_\sigma$ and $0 \le k < |\sigma(y_0)|$, it is at once seen that a_i is the k^{th} alphabet in $\sigma(y_0)$ and is independent of $z \in x_{1,i}$. In fact it is a part of the definition [**DHS**, Corollary 13] of the Kakutani-Rohlin partition $\mathcal{P}_n$ that y_0 and k are independent of $z \in x_{1,i}$.

2.6. Example where *G* is infinite. Let G be a residually finite group. Let $G \supset N_0 \supset N_1 \supset \cdots \supset N_i \supset \cdots$ be a decreasing sequence of cofinite normal subgroups of G such that $\bigcap_i (N_i) = \{e\}$. Let $\varphi_i\colon G \to G/N_i$ be the canonical projection. We use freely the notation for a substitutional system (A, σ) set up in 2.4. Let λ be a stationary labelling on $E = \{(a, k, b) \mid a \text{ is the } k^{\text{th}} \text{ alphabet in } \sigma(b)\}$ with values in G. The skew-product system in this example is defined as follows:

$$\begin{aligned} V_{n,\lambda} &= A \times (G/N_n),\ \forall\, n \\ E_{n,\lambda} &= \{[(a, \varphi_{n-1}(g)), k, (b, \varphi_n(h))]\} \end{aligned}$$

where $a, b \in A, (a, k, b) \in E$ and $\varphi_{n-1}(g) = \varphi_{n-1}(h \cdot \lambda(a, k, b))$.

In the statements below, we do not assume that $(V_\lambda, E_\lambda, \ge)$ arises from a stationary labelling of a stationary ordered Bratteli diagram. Nor do we assume that $(V_\lambda, E_\lambda, \ge)$ is simple. We only assume that

(1) $(V, E, \ge)$ is a simple ordered Bratteli diagram and
(2) λ is a labelling with values in G.

2.7. Let (X, T) and (X_λ, T_λ) be the Cantor dynamical systems associated respectively to $(V, E, \ge)$ and $(V_\lambda, E_\lambda, \ge)$ as in Subsection 1.10. The map $\pi\colon (V_\lambda, E_\lambda, \ge) \to (V, E, \ge)$ sends paths from $\{*\}$ to (v, g) in (V_λ, E_λ) to paths from $\{*\}$ to v in (V, E) and respects truncation (*cf.* 1.9). The unique path lifting property implies that π maps a $\varpi_{n,\lambda}$-tower in $(V_\lambda, E_\lambda, \ge)$ parametrised by $(v, g) \in V_{n,\lambda}$ bijectively onto the

ϖ_n-tower in $(V, E, \geq)$ parametrised by $v \in V_n$ (and of course, respects the linear order of the floors). Thus π induces a map $\pi\colon (X_\lambda, T_\lambda) \to (X,T)$ between the two dynamical systems. The action of G on $(V_\lambda,E_\lambda,\geq)$ given by $\{\gamma_g\}_{g\in G}$ gives rise to a free action $\{\gamma_g\}_{g\in G}$ of G on (X_λ, T_λ) such that $\pi \circ \gamma_g = \pi$.

2.8. In the next few sections, we describe an ordered Bratteli diagram for the quotient of (X_λ, T_λ) by the G-action $\{\gamma_g\}_{g\in G}$. The Bratteli diagram thus constructed naturally maps to $(V, E, \geq)$ with unique path lifting property.

Recall that by definition $X_\lambda = \{x = (x_1, x_2, \ldots, x_n, \ldots)\}$ where

(i) $x_n = (x_{n,i})_{i\in\mathbb{Z}} \in (\varpi_{n,\lambda})^{\mathbb{Z}}$.

(ii) $j_{m,n}(x_{m,i}) = x_{n,i}$ for $m > n$ and $i \in \mathbb{Z}$.

(iii) Given n and K, $\exists\ m > n$ and a vertex $v \in V_{m,\lambda}$ such that the interval segment $x_n[-K, K] := (x_{n,-K}, x_{n,-K+1}, \ldots, x_{n,K})$ is obtained by applying $j_{m,n}$ to an interval segment of the linearly ordered set of paths from v_0 to v.

2.9. Definition: "Loops lift to loops". If $(V, E, \geq)$ and $(V', E', \geq)$ are two ordered Bratteli diagrams and $\pi\colon (V, E, \geq) \to (V', E', \geq)$ is a morphism with unique path-lifting property we say that π "lifts loops to loops" (or, that π has the "loops lifting to loops" property) if the following condition is satisfied:

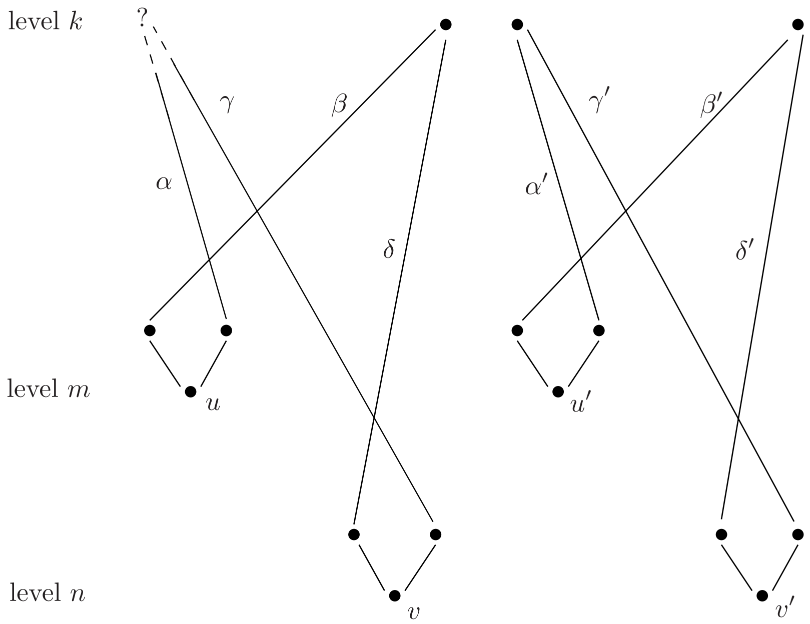
("Loops on the right"). Let $u' \in V'_m$, $v' \in V'_n$. Let $k \leq m$ and $k \leq n$. Suppose α' and β' are paths from V'_k ranging at u'. Suppose γ' and δ' are paths from V'_k ranging at v'. In the lexicographic order induced on paths from V'_k to u' assume that α' is the successor of β'. Similarly for paths from V'_k to v' assume that γ' is the successor of δ'. Assume that α' and γ' have the same source in V'_k and that, likewise, β' and δ' have the same source in V'_k. (Thus, one has a loop: going from the range of α' to the source of β' via β', then to the range of δ' via δ', then to the source of α' via γ' and then to the range of α' via α'.)

("Pull-back of the above loop to the left"). Now, let $u \in V_m$, $v \in V_n$ and assume $\pi(u) = u'$, $\pi(v) = v'$. Let α (resp. β) be the unique path from V_k to V_m lying above α', (resp. β') and ranging at u. Similarly, let γ (resp. δ) be the unique path from V_k to V_n lying above γ' (resp. δ') and ranging at v. If π has the property that under the above conditions,

$$\text{“}\{\text{source of } \beta = \text{source of } \delta\}\text{”} \implies \text{“}\{\text{source of } \alpha = \text{source of } \gamma\}\text{”}$$

then we say that π has the "loops lifting to loops" property.

Diagram 1



Finally, we say that the labelling $\lambda\colon E \to G$ has the "loops lifting to loops" property if $(V_\lambda, E_\lambda, \geq) \to (V, E, \geq)$ has the above property.

Example 2.10.1. Let $\beta\colon V \to G$ be a function. The labelling λ defined on E by $\lambda(e) = \beta(r(e))^{-1}\beta(s(e))$ has the "loops lifting to loops" property.

Example 2.10.2. Let $s, t \in G$. Let λ be a G-valued stationary labelling on the 2-adic odometer system (*cf.* [**GJ**, pp. 1691, 1695]) whose image consists of $\{s, t\}$.

The Bratteli diagram of the odometer has one vertex at each level and two edges, say, min and max from level k to $k+1$ for $k \geq 1$, with $\lambda(\min) = s$ and $\lambda(\max) = t$. In Theorem 2.11, we take

$$\begin{aligned}
\alpha' &= \max && \text{from level 1 to 2,}\\
\beta' &= \min && \text{from level 1 to 2,}\\
\gamma' &= (\min, \max) && \text{from level 1 to 3,}\\
\delta' &= (\max, \min) && \text{from level 1 to 3.}
\end{aligned}$$

Denote by w', u', v' the vertices at levels 1,2,3. Following the description of vertices and edges for (V_λ, E_λ) as in 2.5,

first (resp. second) edge ranging at (u', g) has source (w', gs) (resp. (w', gt)),
first (resp. second) edge ranging at (v', h) has source (u', hs) (resp. (u', ht)),

source (w', gt) to range (u', g) is a lift α of α',
source (w', gs) to range (u', g) is a lift β of β',
(w', hts) to (u', ht) to (v', h) is a lift γ of γ',
(w', hst) to (u', hs) to (v', h) is a lift δ of δ'.

If the "loops lifting to loops" property holds then for the above choice of α', β', γ', δ' the condition that

$$\text{“}\{\text{source of } \beta = \text{source of } \delta\ \}\text{”} \Longrightarrow \text{“}\{\text{source of } \alpha = \text{source of } \gamma\ \}\text{”}$$

yields the property in the group

$$\text{“}\{(w', gs) = (w', hst)\}\text{”} \Longrightarrow \text{“}\{(w', gt) = (w', hts)\}\text{”}.$$

This reduces to the condition that $st^{-1}s^{-1} = ts^{-1}t^{-1}$. One can easily see that analogous conditions with other choices of α', β', γ', δ' are all consequences of the property $st^{-1}s^{-1} = ts^{-1}t^{-1}$.

Assume that G is generated by s, t and that $st^{-1}s^{-1} = ts^{-1}t^{-1}$. Then either $s = t$, (so G is cyclic), or, G has the following simple description:

Put $x = ts^{-1}$. Then, $x^2 = ts^{-1}ts^{-1} = ts^{-1}ts^{-1}t^{-1}t = ts^{-1}st^{-1}s^{-1}t = s^{-1}t$. So, $t^{-1}xt = s^{-1}t = x^2$. Let H be the subgroup of G generated by x. Let J be the subgroup of G generated by t. Note that elements of J normalize H. Also, $J \cap H = \{e\}$ the unit element. In fact since x and x^2 are conjugate, the order of x has to be $2k+1$ an odd number. If $x^\ell = t^m \in J \cap H$, where, $0 \le \ell < 2k+1$ then $x^\ell = t^m = t^{-1}x^\ell t = x^{2\ell}$, which implies that $x^\ell = \{e\}$ which is not possible (unless $\ell = 0$) since $\ell <$ order of x.

Hence, G is the semi-direct product of J and H for the action given by $t^{-1}xt = x^2$. The skew-product Bratteli diagram $(V_\lambda, E_\lambda, \geq)$ has the following connectivity structure. Telescoping to levels $\{0, n_1 = 1, n_2, n_3, \ldots, n_i, \ldots\}$ where $n_i = 1 + (i-1) \times \{\text{order of } G\}$, a vertex (v, g) in level n_i is connected to a vertex (w, g') in level n_{i+1} by an edge if and only if g and g' belong to the same H-coset.

Our main result about the skew-product dynamical systems (X_λ, T_λ) which were constructed in 2.1 are contained in the following.

Theorem 2.11. (I) *Suppose that the map* $\pi\colon (V_\lambda, E_\lambda, \geq) \to (V, E, \geq)$ *has the property "loops lift to loops". Then the quotient of* (X_λ, T_λ) *by the action of* G *is canonically isomorphic to* (X, T).

(II) *In general, there is a commutative diagram*

$$\begin{array}{ccc} (V_\lambda, E_\lambda, \geq) & \xrightarrow{\pi} & (V, E, \geq) \\ {\scriptstyle\rho'}\big\uparrow & & \big\uparrow{\scriptstyle\rho} \\ (\widetilde{V}_\mu, \widetilde{E}_\mu, \geq) & \xrightarrow{\widetilde{\pi}} & (\widetilde{V}, \widetilde{E}, \geq) \end{array}$$

and the induced diagram

$$\begin{array}{ccc} (X_\lambda, T_\lambda) & \xrightarrow{\pi_B} & (X, T) \\ {\scriptstyle\rho'_B}\big\uparrow & & \big\uparrow{\scriptstyle\rho_B} \\ (\widetilde{X}_\mu, \widetilde{T}_\mu) & \xrightarrow{\widetilde{\pi}_B} & (\widetilde{X}, \widetilde{T}) \end{array}$$

between the corresponding dynamical systems, where

(i) *All the horizontal and vertical arrows in the first diagram have the "unique path lifting property",*

(ii) $(\widetilde{V}_\mu, \widetilde{E}_\mu, \geq)$ *is the skew-product Bratteli diagram associated to the* G*-valued labelling* μ *on* $(\widetilde{V}, \widetilde{E}, \geq)$ *defined by* $\mu = \lambda \circ \rho$,

(iii) ρ' *commutes with the* G*-action and* ρ'_B *is an isomorphism, and*

(iv) *the map* $\widetilde{\pi}\colon (\widetilde{V}_\mu, \widetilde{E}_\mu, \geq) \to (\widetilde{V}, \widetilde{E}, \geq)$ *has the property "loops lift to loops". (Consequently, by Part (I), the quotient of* (X_λ, T_λ) *by the action of* G *is isomorphic to* $(\widetilde{X}, \widetilde{T})$.*)*

Remark. Note that Part (II) is almost a converse to Part (I). If we are given that the quotient of (X_λ, T_λ) by the action of G is isomorphic to (X, T), we are not deducing from this that $\pi\colon (V_\lambda, E_\lambda, \geq) \to (V, E, \geq)$ has the property "loops lift to loops". Instead, Part (II) gives us a closely related morphism with the property "loops lift to loops" which can be nicely fit into a square with the given morphism π as one of its sides and in which all the arrows have the unique path lifting property; the vertical arrows are compatible with the given labelling and its pull-back and induce isomorphisms in the corresponding dynamical systems.

Proof of Part (I): By hypothesis $\pi\colon (V_\lambda, E_\lambda, \geq) \to (V, E, \geq)$ has the property "loops lifts to loops". Suppose that $\pi_B\colon (X_\lambda, E_\lambda, \geq) \to (X, E, \geq)$ is not an isomorphism, modulo the G-action on X_λ. Let y and z be two different G-orbits in X_λ, which have the same image x in X. We employ the notation of 2.7 and 2.8. Write

$$
\begin{aligned}
y &= (y_n)_n, \quad y_n = (y_{n,i})_{i\in\mathbb{Z}}, \quad y_{n,i} \in \varpi_{n,\lambda}\\
z &= (z_n)_n, \quad z_n = (z_{n,i})_{i\in\mathbb{Z}}, \quad z_{n,i} \in \varpi_{n,\lambda}\\
x &= (x_n)_n, \quad x_n = (x_{n,i})_{i\in\mathbb{Z}}, \quad x_{n,i} \in \varpi_{n}.
\end{aligned}
$$

By the hypothesis, y and z have the image x in X and also the G-orbits of y and z are different. By shifting y and z by a suitable power of the shift operator T_λ and replacing z if necessary by another element within the orbit $\{\gamma_g \cdot z \mid g \in G\}$, we can assume that, for some $k \in \mathbb{N}$, $y_{k,0} = z_{k,0}$ and $y_{k,1} \neq z_{k,1}$. In view of 1.8 it follows that

2.11.1. $y_{k,0}$ is the top floor of a $\varpi_{k,\lambda}$-tower $\varpi(u, g_1)$

and

2.11.2. $y_{k,1}$ is the lowest floor of a $\varpi_{k,\lambda}$-tower $\varpi(w, g_2)$.

In the same way,

2.11.3. $z_{k,0}$ is the top floor of the $\varpi_{k,\lambda}$-tower $\varpi(u, g_1)$

and

2.11.4. $z_{k,1}$ the lowest floor of a $\varpi_{k,\lambda}$-tower $\varpi(w, g_3)$.

By property 1.10 (iii) there exists $m \geq k$, a vertex $(a, h) \in V_{m,\lambda}$ and two successive floors F^0, F^1 of the $\varpi_{m,\lambda}$ tower $\varpi(a, h)$ such that $j_{m,k}(F^0) = y_{k,0}$ and $j_{m,k}(F^1) = y_{k,1}$. Similarly, there exists $n \geq k$, a vertex $(b, \tilde{h}) \in V_{n,\lambda}$ and two successive floors $\widetilde{F}^0$, $\widetilde{F}^1$ of the $\varpi_{n,\lambda}$-tower $\varpi(b, \tilde{h})$ such that $j_{m,k}(\widetilde{F}^0) = y_{k,0}$ and $j_{m,k}(\widetilde{F}^1) = y_{k,1}$. The floors F^0 and F^1 correspond to two successive paths from $\{*\} \in V_{0,\lambda}$ to $(a, h) \in V_{m,\lambda}$ in the linearly ordered set of paths from $\{*\}$ to (a, h). The path F^0 is traced by first tracing a path $F^0[0, k]$ from level 0 to level k and following it by a path from level k to level m. The part $F^0[0, k]$ being the truncation $j_{m,k}(F^0)$ represents the floor $y_{k,0}$. By 2.11.1, $y_{k,0}$ is the unique maximal path from $\{*\} \in V_{0,\lambda}$ to $(u, g_1) \in V_{k,\lambda}$. Similarly, the path F^1 from level 0 to level m initially traces the truncation $F^1[0, k] = j_{m,k}(F^1)$ which represents the floor $y_{k,1}$ followed by $F^1[k, m]$ from level k to level m. By 2.11.2, $y_{k,1}$ is the unique minimal path (lowest floor) from $\{*\} \in V_{0,\lambda}$ to $(w, g_2) \in V_{k,\lambda}$. Since F^1 is the successor of F^0 in the lexicographic order of paths from $\{*\}$ to $(a,h) \in V_{m,\lambda}$, we conclude from the above observation that

2.11.5. $F^1[k,m]$ is the successor of $F^0[k,m]$ in the lexicographic order of paths from $V_{k,\lambda}$ to $(a,h) \in V_{m,\lambda}$.

Similarly, if $\widetilde{F}^0[k,n]$ and $\widetilde{F}^1[k,n]$ represent the truncation of $\widetilde{F}^0$ and $\widetilde{F}^1$ respectively from level k to level n then

2.11.6. $\widetilde{F}^1[k,n]$ is the successor of $\widetilde{F}^0[k,n]$ in the lexicographic order of paths from $V_{k,\lambda}$ to $(a,\tilde{h}) \in V_{n,\lambda}$.

We show how this leads to a contradiction of the property "loops lift to loops". Denote the paths $F^0[k,m]$, $F^1[k,m]$, $\widetilde{F}^0[k,n]$ and $\widetilde{F}^1[k,n]$ by β, α, δ and γ respectively. Let their images under π be denoted by β', α', δ' and γ' respectively.

We will observe that with the above data we have something on the left which is almost a loop, but not actually a loop eventhough its image on the right is a loop (*cf.* 2.9).

(Object on the right which is a loop). α' and β' are paths from V_k ranging at a. Also γ' and δ' are paths from V_k ranging at b. α' is the successor of β'. For paths from V_k to V_n, γ' is the successor of δ'. The paths α' and γ' have the same source in V_k, β' and δ' have the same source in V_k. (Thus, one has a loop: going from the range of α' to the source of β' via β', then to the range of δ' via δ', then to the source of α' via γ' and then to the range of α' via α'.)

(Object on the left lying over the above loop). $(a,h) \in V_{m,\lambda}$, $(b,\tilde{h}) \in V_{n,\lambda}$, $\pi(a,h) = a$, $\pi(b,\tilde{h}) = b$. α (resp. β) is the unique path from $V_{k,\lambda}$ to $V_{m,\lambda}$ lying above α', (resp. β') and ranging at (a,h). Similarly, γ (resp. δ) is the unique path from $V_{k,\lambda}$ to $V_{n,\lambda}$, lying above γ' (resp. δ') and ranging at $(b,\tilde{h})$. α is the successor of β. γ is the successor of δ.

(Object on the left is almost a loop). Moreover source of $\beta =$ source of δ.

But source of $\alpha \neq$ source of γ, which contradicts the "loops lifting to loops" property. $\square$

Now we begin the proof of the Part (II) in the statement of the theorem.

2.12. We will now define two nested sequences of K-R partitions of X_λ, both acted on by the group G. Recall $V_{n,\lambda} = V_n \times G$ (for $n \geq 1$). For $(v,g) \in V_n \times G$, let y be a path from $\{*\}$ to (v,g) in $(V_\lambda, E_\lambda, \geq)$. So, y is a "floor" belonging to the $\varpi_{n,\lambda}$- tower $\varpi(v,g)$ parametrized by (v,g). Put $\mathcal{F}_y = \{x = (x_1, x_2, \dots, x_n, \dots) \in X_\lambda \mid x_{n,0} = y\}$

$$\mathcal{P}_{n,\lambda} = \{\mathcal{F}_y \mid y \in \varpi(v,g),\, (v,g) \in V_{n,\lambda}\}.$$

Then $\{\mathcal{P}_{n,\lambda}\}_n$ is a nested sequence of K-R partitions of X_λ acted on by G. But, the topology of X_λ need not be spanned by the collection of clopen sets $\{\mathcal{F}_y\}$, $(y \in \varpi(v,g),\, (v,g) \in V_{n,\lambda},\, n \in \mathbb{N})$. In contrast, the topology of X_λ is indeed spanned by the collection of clopen sets in another nested sequence $\{\mathcal{Q}_{n,\lambda}\}_n$ of K-R partitions, defined below. Let $\varpi = \varpi(u,g_1)$, $\varpi' = \varpi(v,g_2)$, $\varpi'' = \varpi(w,g_3)$ be three $\varpi_{n,\lambda}$-towers and y a floor of ϖ'. For any $x \in X_\lambda$ and for any n if $x_{n,i}$ is a floor of a $\varpi_{n,\lambda}$-tower $\overline{\varpi}$, then for some $a, b \in \mathbb{Z}$ such that $a \leq i \leq b$, the segment $x_n[a,b]$ is just the sequence of floors in $\overline{\varpi}$. We define $\mathcal{F}(\varpi, \varpi', \varpi''; y)$ = the clopen subset of $\mathcal{F}_y$ consisting of the elements $x = (x_1, x_2, \dots, x_n, \dots)$ with the property that for some $a_1 < a_2 \leq 0 < a_3 < a_4 \in \mathbb{Z}$, the segment $x_n[a_1, a_2-1]$ is the sequence of floors of ϖ, the segment $x_n[a_2, a_3-1]$ is the sequence of floors of ϖ' and the segment $x_n[a_3, a_4]$ is the sequence of floors of ϖ''. Some of the sets $\mathcal{F}(\varpi, \varpi', \varpi''; y)$ may be empty, but the non-empty sets $\mathcal{F}(\varpi, \varpi', \varpi''; y)$ form a K-R partition which we denote by $\mathcal{Q}_{n,\lambda}$. For fixed ϖ, ϖ', ϖ'' the subcollection $\{\mathcal{F}(\varpi, \varpi', \varpi''; y)\}$ as y varies through the floors of ϖ', is a $\mathcal{Q}_{n,\lambda}$-tower parametrized by $[(u,g_1),(v,g_2),(w,g_3)]$. We denote this $\mathcal{Q}_{n,\lambda}$-tower by $\mathcal{S}_{(\varpi,\varpi',\varpi'')}$. The floors of the tower $\mathcal{S}_{(\varpi,\varpi',\varpi'')}$ are $\{\mathcal{F}(\varpi, \varpi', \varpi''; y)\}$ as y runs through the sequence of floors of ϖ'.

Lemma 2.13. *$\{\mathcal{Q}_{n,\lambda}\}_n$ is a nested sequence of K-R partitions of X_λ acted on by G. The topology of X_λ is spanned by the clopen sets in this sequence of partitions.*

Proof: Each $\mathcal{Q}_{n,\lambda} = \{\mathcal{F}(\varpi, \varpi', \varpi''; y)\}$ (for a fixed n) is a K-R partition. This is evident on going through the definitions. Since the infimum of the height of $\varpi_{n,\lambda}$-towers (ϖ, ϖ' etc.) goes to infinity as $n \to \infty$, it is evident from the definition of the topology of X_λ that the clopen sets $\mathcal{F}(\varpi, \varpi', \varpi''; y)$, (for various n, $\varpi_{n,\lambda}$- towers ϖ, ϖ', ϖ", floors y of ϖ') span the topology of X_λ. ☐

Remark 2.14. The above construction involving "triples" can be introduced starting with any simple ordered Bratteli diagram, (not necessarily the diagram $(V_\lambda, E_\lambda, \geq)$ which had a G-action). For this, let $(V, E, \geq)$ be an arbitrary simple, ordered Bratteli diagram. Define $(V^{\mathcal{Q}}, E^{\mathcal{Q}}, \geq)$ as follows: $V_0^{\mathcal{Q}} = \{*\}$, a single point.

$V_n^{\mathcal{Q}}$ consists of triples $(u, v, w) \in V_n \times V_n \times V_n$ such that for some $y \in V_m$ where $m > n$, the level-m tower ϖ_y passes successively through the level-n tower ϖ_u, then ϖ_v and then ϖ_w. An edge $\tilde{e} \in E_n^{\mathcal{Q}}$ is a triple (u, e, w) such that e is an edge of (V, E) and $(u, r(e), w) \in V_n^{\mathcal{Q}}$. Let

$$
\begin{aligned}
&\{e_1, e_2, \ldots, e_k\} \text{ be all the edges in } r^{-1}(r(e)),\\
&\{f_1, f_2, \ldots, f_\ell\} \text{ be all the edges in } r^{-1}(u) \qquad \text{and}\\
&\{g_1, g_2, \ldots, g_m\} \text{ be all the edges in } r^{-1}(w).
\end{aligned}
$$

The sources of $(u, e_1, w), (u, e_2, w), \ldots, (u, e_k, w)$ are defined to be

$$(s(f_\ell), s(e_1), s(e_2)), (s(e_1), s(e_2), s(e_3)), \ldots, (s(e_{k-1}), s(e_k), s(g_1))$$

respectively. The range of (u, e, w) is of course $(u, r(e), w)$.

The map $(u, v, w) \mapsto v, (u, e, w) \mapsto e$ from $(V^{\mathcal{Q}}, E^{\mathcal{Q}})$ to (V, E) has unique path lifting property; in particular it gives rise to the ordered Bratteli diagram $(V^{\mathcal{Q}}, E^{\mathcal{Q}}, \geq)$.

2.15. The ordered Bratteli diagrams $B^{\mathcal{Q}}(\lambda)$ and $\overline{B}^{\mathcal{Q}}(\lambda)$. The construction 1.6 of an ordered Bratteli diagram applied to the nested sequence of K-R partitions $\{\mathcal{Q}_{n,\lambda}\}_n$ gives rise to an ordered Bratteli diagram. The vertices $V_{n,\lambda}^{\mathcal{Q}}$ are in 1-1 correspondence with the towers $\mathcal{S}_{[\varpi(u,g_1),\varpi(v,g_2),\varpi(w,g_3)]}$ of $\mathcal{Q}_{n,\lambda}$. Note that not all choices $[(u, g_1), (v, g_2), (w, g_3)]$ may give rise to a non-empty set $\mathcal{S}_{[\varpi(u,g_1),\varpi(v,g_2),\varpi(w,g_3)]}$. We may denote the vertex corresponding to (non-empty) $\mathcal{S}_{[\varpi(u,g_1),\varpi(v,g_2),\varpi(w,g_3)]}$ by $[(u, g_1), (v, g_2), (w, g_3)]$. If $[(u, g_1), (v, g_2), (w, g_3)] \in V_{n,\lambda}^{\mathcal{Q}}$, then $[(u, gg_1), (v, gg_2), (w, gg_3)] \in V_{n,\lambda}^{\mathcal{Q}}$, $\forall\, g \in G$. So G acts on $V_{n,\lambda}^{Q}$ freely if $n \geq 1$. As in 1.6 the edges connecting $V_{n,\lambda}^{\mathcal{Q}}$ and $V_{n+1,\lambda}^{\mathcal{Q}}$ and the linear order between edges with the same range simply reflects the sequence of $\mathcal{Q}_{n,\lambda}$-towers traversed by a given $\mathcal{Q}_{n+1,\lambda}$-tower. It is then clear that G-acts on the ordered Bratteli diagram $(V_\lambda^{\mathcal{Q}}, E_\lambda^{\mathcal{Q}}, \geq)$. We denote by $(\overline{V}_\lambda^{\mathcal{Q}}, \overline{E}_\lambda^{\mathcal{Q}}, \geq)$ the ordered Bratteli diagram which is the quotient by G-action on $(V_\lambda^{\mathcal{Q}}, E_\lambda^{\mathcal{Q}}, \geq)$. We write $B^{\mathcal{Q}}(\lambda)$ and $\overline{B}^{\mathcal{Q}}(\lambda)$ for the Bratteli diagrams $(V_\lambda^{\mathcal{Q}}, E_\lambda^{\mathcal{Q}}, \geq)$ and $(\overline{V}_\lambda^{\mathcal{Q}}, \overline{E}_\lambda^{\mathcal{Q}}, \geq)$ respectively.

2.16. Tripling for a substitutional system (A, σ). If (A, σ) is a substitutional system, define $A^{\mathcal{Q}}$ to be the unique smallest subset of $A \times A \times A$ with the property that $(a, b, c) \in A^{\mathcal{Q}}$ if and only if the word abc occurs as a subword of $\sigma^n(d)$ for some $d \in A$ and some n. Define

$$\sigma^{\mathcal{Q}} \colon A^{\mathcal{Q}} \longrightarrow (A^{\mathcal{Q}})^+$$

by $\sigma^{\mathcal{Q}}[(a,b,c)] = (a_m, b_1, b_2) \cdot (b_1, b_2, b_3) \cdots (b_{n-2}, b_{n-1}, b_n) \cdot (b_{n-1}, b_n, c_1)$, where $\sigma(b) = b_1 \cdot b_2 \cdots b_n$, and a_m is the last alphabet in $\sigma(a)$, while c_1 is the first alphabet in $\sigma(c)$. Suppose moreover that the ordered Bratteli diagram associated to (A, σ) is equipped with a stationary labelling λ. Then following the construction of 2.4 we get a skew-product substitutional system $(A_\lambda, \sigma_\lambda)$. The tripling described above applied to this $(A_\lambda, \sigma_\lambda)$ gives a substitutional system $(A^{\mathcal{Q}}_\lambda, \sigma^{\mathcal{Q}}_\lambda)$.

2.17. It now remains to finish the proof of Part (II) of Theorem 2.11. Our construction of $\overline{B}^{\mathcal{Q}}(\lambda) = \left(\overline{V}^{\mathcal{Q}}_\lambda, \overline{E}^{\mathcal{Q}}_\lambda, \geq\right)$ in 2.15 was motivated precisely to serve as a candidate for the $\widetilde{B}$ in the statement of Part (II) of the theorem. Thus, we set $\widetilde{V} = \overline{V}^{\mathcal{Q}}_\lambda$, $\widetilde{E} = \overline{E}^{\mathcal{Q}}_\lambda$, $\widetilde{B} = (\widetilde{V}, \widetilde{E}, \geq) = \left(\overline{V}^{\mathcal{Q}}_\lambda, \overline{E}^{\mathcal{Q}}_\lambda, \geq\right) = \overline{B}^{\mathcal{Q}}(\lambda)$. A vertex of $\overline{V}^{\mathcal{Q}}_{n,\lambda}$ is represented by the G-orbit of a triple $[(u, g_1), (v, g_2), (w, g_3)]$. Define $\rho \colon \overline{V}^{\mathcal{Q}}_\lambda \to V$ by sending the above vertex to $v \in V_n$. The set $\overline{E}^{\mathcal{Q}}_{n,\lambda}$ of edges of $\overline{B}^{\mathcal{Q}}(\lambda)$ from $\overline{V}^{\mathcal{Q}}_{n-1,\lambda}$ to $\overline{V}^{\mathcal{Q}}_{n,\lambda}$ ranging at $[(u, g_1), (v, g_2), (w, g_3)]$ is represented by the triple $[(u, g_1), (e, g_2), (w, g_3)]$ where e is an edge from V_{n-1} to V_n ranging at v. Define $\rho \colon \overline{E}^{\mathcal{Q}}_\lambda \to E$ by sending the above edge (namely the G-orbit of $[(u, g_1), (e, g_2), (w, g_3)]$) to e.

Define a labelling μ on $\overline{E}^{\mathcal{Q}}_\lambda$ by $\mu = \lambda \circ \rho$. We show that the corresponding skew-product of $\left(\overline{V}^{\mathcal{Q}}_\lambda, \overline{E}^{\mathcal{Q}}_\lambda, \geq\right)$ by μ is precisely $(V^{\mathcal{Q}}_\lambda, E^{\mathcal{Q}}_\lambda, \geq)$. For this we define a map

$$\Phi \colon (V^{\mathcal{Q}}_\lambda, E^{\mathcal{Q}}_\lambda, \geq) \longrightarrow (\overline{V}^{\mathcal{Q}}_\lambda \times G, \overline{E}^{\mathcal{Q}}_\lambda \times G, \geq)$$

as follows: let $x = [(u, g_1), (v, g_2), (w, g_3)] \in V^{\mathcal{Q}}_\lambda$. Let the G-orbit of x be denoted by $\overline{x} \in \overline{V}^{\mathcal{Q}}_\lambda$. Then, define $\Phi(x) = (\overline{x}, g_2)$. Similarly, let $\eta = [(u, g_1), (e, g_2), (w, g_3)]$ be an edge of $E^{\mathcal{Q}}_\lambda$. Let the G-orbit of η be denoted by $\overline{\eta} \in \overline{E}^{\mathcal{Q}}_\lambda$. Then, define $\Phi(\eta) = (\overline{\eta}, g_2)$. Then Φ is an isomorphism between the two ordered Bratteli diagrams.

Referring to the requirements in the statement of Part (II) of Theorem 2.11, the map $\rho'\colon (V_\lambda^{\mathcal{Q}}, E_\lambda^{\mathcal{Q}}, \geq) \to (V_\lambda, E_\lambda, \geq)$ is easy to define:

$$\rho'[(u,g_1),(v,g_2),(w,g_3)] = (v,g_2) \quad \text{and}$$
$$\rho'[(u,g_1),(e,g_2),(w,g_3)] = (e,g_2).$$

The map $\rho'\colon B^{\mathcal{Q}}(\lambda) = (V_\lambda^{\mathcal{Q}}, E_\lambda^{\mathcal{Q}}, \geq) \to (V_\lambda, E_\lambda, \geq) = B(\lambda)$ induces a map ρ'_B between the corresponding dynamical systems $X_{B^{\mathcal{Q}}(\lambda)}$ and $X_{B(\lambda)}$. The inverse β of this map is defined as follows:

Let $z \in X_{B(\lambda)}$. So $z = (z_1, \dots, z_n, \dots)$, where $z_n = (z_{n,i})_{i\in\mathbb{Z}} \in (\varpi_{n,\lambda})^{\mathbb{Z}}$. Fix n and i. So $z_{n,i}$ is a floor of a $\varpi_{n,\lambda}$-tower $\mathcal{S}_{(v,g_2)}$. Choose $a,b,c,d \in \mathbb{Z}$ such that $a < b \leq i < c < d$, the interval segment $z_n[b, c-1]$ is the sequence of all the floors of the tower $\mathcal{S}_{(v,g_2)}$, $z_n[a, b-1]$ is the sequence of all the floors of some $\varpi_{n,\lambda}$-tower $\mathcal{S}_{(u,g_1)}$ and $z_n[c, d-1]$ is the sequence of all the floors of some $\varpi_{n,\lambda}$-tower $\mathcal{S}_{(w,g_3)}$. Define

$$y = (y_1, \dots, y_n, \dots) \in X_{B^{\mathcal{Q}}(\lambda)} \quad \text{where} \quad y_n = (y_{n,i})_{i\in\mathbb{Z}} \in \left(\varpi_{n,\lambda}^{\mathcal{Q}}\right)^{\mathbb{Z}}$$

by $y_{n,i} = ((u,g_1),(v,g_2),(w,g_3); z_{n,i})$. This map defines an inverse of ρ'_B.

Finally, we prove the property "loops lift to loops" for the map

$$(V_\lambda^{\mathcal{Q}}, E_\lambda^{\mathcal{Q}}, \geq) \longrightarrow (\overline{V}_\lambda^{\mathcal{Q}}, \overline{E}_\lambda^{\mathcal{Q}}, \geq).$$

To verify the property "loops lift to loops" for the above map, we start with some data on the left-side which is "almost" a loop, we make a further assumption that the image is actually a loop and then we have to conclude that the data we started with on the left-side must in fact be a loop.

2.18. The data on the left which is *almost* a loop. Let $\hat{u} \in V_{m,\lambda}^{\mathcal{Q}}$ and $\hat{v} \in V_{n,\lambda}^{\mathcal{Q}}$. Let $k \leq m$ and $k \leq n$. Let α and β be paths from $V_{k,\lambda}^{\mathcal{Q}}$ ranging at $\hat{u}$. Let γ, δ be paths from $V_{k,\lambda}^{\mathcal{Q}}$ ranging at $\hat{v}$. Assume that β and δ have the same source in $V_{k,\lambda}^{\mathcal{Q}}$.

2.19. The data "image on the right is *actually* a loop". Denote by α', β', γ', δ' the paths in $(\overline{V}_\lambda^{\mathcal{Q}}, \overline{E}_\lambda^{\mathcal{Q}}, \geq)$ which are the images of α, β, γ, δ. Suppose that α' is the successor of β' and that γ' is the successor of δ'. Assume that α' and γ' have the same source in $\overline{V}_{k,\lambda}^{\mathcal{Q}}$ and that β' and δ' have the same source in $\overline{V}_{k,\lambda}^{\mathcal{Q}}$.

To conclude that the data on the left must in fact be a loop we have to show that α and γ have the same source in $V_{k,\lambda}^{\mathcal{Q}}$.

Let source of β = source of $\delta = [(a, g_1), (b, g_2), (c, g_3)] \in V^{\mathcal{Q}}_{k,\lambda}$. Since α is the successor of β, necessarily the source of α has the form $[(b, g_2), (c, g_3), (d, g_4)]$ for some $d \in V_k$ and $g_4 \in G$. Likewise, since γ is the successor of δ the source of γ must be of the form $[(b, g_2), (c, g_3), (d', g_5)]$, for some $d' \in V_k$ and $g_5 \in G$. But it is a consequence of the assumption "source of α' = source of γ'" that $[(b, g_2), (c, g_3), (d, g_4)]$ and $[(b, g_2), (c, g_3), (d', g_5)]$ must lie in the same G-orbit, i.e., for some $h \in G$, $[(b, g_2), (c, g_3), (d, g_4)] = [(b, hg_2), (c, hg_3), (d', hg_5)]$.

Therefore, we conclude that $d = d'$ and $g_4 = g_5$. Thus, the property "loops lift to loops" is verified.

This ends the proof of the theorem. □

2.20. Cohomologous labellings. Let λ and μ be two G-valued edge labellings of $(V, E, \geq)$. We say that λ and μ are cohomologous if there exists a function $\beta\colon V \to G$ such that

$$\beta(r(e))\mu(e) = \lambda(e)\beta(s(e)), \text{ for } e \in E_n, \quad (n \geq 1).$$

Such a map β gives rise to an isomorphism

$$\Phi_\beta\colon (V_\lambda, E_\lambda, \geq) \longrightarrow (V_\mu, E_\mu, \geq)$$

where $\Phi_\beta(v, g) = (v, g\beta(v))$ and $\Phi_\beta(e, g) = (e, g\beta(r(e)))$.

2.21. Initially we were hopeful that our work for stationary labellings will lead to an example of a substitutional system $(A_\lambda, \sigma_\lambda)$ (see 2.4) arising from a stationary labelling λ with values in any permutation group G, such that (A, σ) and $(A_\lambda, \sigma_\lambda)$ are both *Toeplitz flows* defined below as usual (*cf. eg.* [**GJ**, p. 1695]).

Definition 2.22. A Toeplitz sequence is a non-periodic sequence $\eta = (\eta_n)_{n\in\mathbb{Z}}$ in $A^{\mathbb{Z}}$, where A is any finite alphabet set, so that for each $m \in \mathbb{Z}$ there exists $n \in \mathbb{N}$ so that $\eta_m = \eta_{m+kn}$ for all $k \in \mathbb{Z}$.

The following gives such an example with G a cyclic group of order k. Let (A, σ) be a primitive aperiodic proper substituion of constant length n. By [**GJ**, Corollary 9, p. 1698] this gives rise to a Toeplitz flow. Assume that n is congruent to 1 mod k. Let z denote a generator for G. We take the stationary labelling λ which assigns z^i to the i^{th} edge ranging at any vertex of the stationary Bratteli diagram for (A, σ) that we described in Subsection 2.5. It is not hard to see that if $\eta = (a_i)_{i\in\mathbb{Z}}$ is a Toeplitz sequence for (A, σ) then $\widetilde{\eta} \overset{\text{def.}}{=} \{(a_i, z^i)\}_{i\in\mathbb{Z}}$ is a Toeplitz sequence for $(A_\lambda, \sigma_\lambda)$.

But outside the domain of cyclic groups it is not possible to have such examples as the following result shows.

Proposition 2.23. *For a non-cyclic group G, one cannot have Toeplitz sequences occurring in primitive substitutional systems $(A_\lambda, \sigma_\lambda)$ arising (*cf. *2.4) from G-valued stationary edge labellings λ of another substitutional system (A, σ).*

Proof: Suppose $\widetilde{\eta} \stackrel{\text{def.}}{=} (a_i, g_i)_{i\in\mathbb{Z}}$ is a Toeplitz sequence in $(A_\lambda, \sigma_\lambda)$ (which is assumed to be primitive). Then $\forall\ g \in G$, $\widetilde{\eta}_g \stackrel{\text{def.}}{=} (a_i, gg_i)_{i\in\mathbb{Z}}$ are also Toeplitz sequences. Further, they all admit the same period structure $p = (p_0, p_1, p_2, \dots)$ (*cf.* [**DKL**, p. 220], [**GJ**, p. 1695]). More importantly, all these Toeplitz sequences lie in the same (minimal) subshift dynamical system X_λ in the alphabets A_λ and the $\mathbb{Z}$- orbit of any one of these Toeplitz sequences $\widetilde{\eta}_g$ is dense in X_λ. Let $(G_p, 1)$ be the maximal uniformly continuous factor of (X_λ, T_λ) (*cf.* [**GJ**, p. 1696]) and let $\pi\colon (X_\lambda, T_\lambda) \to (G_p, 1)$ denote the corresponding factor map. For a Toeplitz sequence $\omega, \pi^{-1}(\pi(\omega)) = \{\omega\}$ (*cf.* [**W**], [**GJ**, p. 1696], [**DL**, Theorem 6, p. 167]). Define elements $h_g, (g \in G)$ of the monothetic group G_p by $h_g = \pi(\widetilde{\eta}_g)$. They are pairwise distinct. For $g \in G$, denote by $\tau_g\colon (X_\lambda, T_\lambda) \to (X_\lambda, T_\lambda)$ the unique isomorphism which replaces an alphabet (a, θ) by $(a, g\theta)$, for $\theta \in G$. Thus, for example, $\tau_g(\widetilde{\eta}_\theta) = (\widetilde{\eta}_{g\theta})$.

As observed in [**DKL**, Section 2, paragraph 1], we have the following commutative diagram

$$\begin{array}{ccc} (X_\lambda, T_\lambda) & \xrightarrow{\tau_g} & (X_\lambda, T_\lambda) \\ {\scriptstyle\pi}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi} \\ (G_p, 1) & \xrightarrow{h_g} & (G_p, 1) \end{array}$$

where the bottom map denotes group operation by h_g. Composition of two such diagrams for $g_1, g_2 \in G$ yields the diagram for $g_1 g_2$. Thus, the group G sits faithfully as a finite subgroup of G_p. Hence, it has to be isomorphic to a subgroup of the finite cyclic group $\mathbb{Z}_{p_i}$ for one of the essential periods in $p_0, p_1, p_2, \dots, p_i, \dots$. □

Going forward. The success of the Bratteli-Vershik model as originally envisaged (in the *properly ordered* case) was basically that, first, the space X could be seen as the path space of the diagram while the orbit relation appeared as cofinality (almost). Secondly, the K-theory invariant could be read off the diagram in the usual way. While neither of these is strictly true for our models here (except in the case it reduces to the old one), Ian Putnam and Christian Skau have pointed out that it does not seem to be far from being true and feel that sorting out the exact sense of this could be quite interesting. We thank them for this observation and for the encouraging letter to us wherein they noted that the construction of a dynamical system without the usual assumption of unique maximal and minimal path resolves many long-standing issues. In separate work the authors have resolved the K-theory aspect.

Note added in proof: For further comments and remarks apropos 1.10 see our forthcoming article *The K-group of substitutional systems* to appear in this journal.

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Primera versió rebuda el 22 de febrer de 2008,
darrera versió rebuda el 27 de gener de 2009.