FOLIATED COHOMOLOGY AND INFINITESIMAL DEFORMATIONS OF DEVELOPABLE FOLIATIONS

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Abstract. Let $\mathcal{F}$ be a foliation on a connected manifold M and denote by $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$ and $\mathcal{V} = TM/T\mathcal{F}$ its normal bundle. We say that $\mathcal{F}$ is *developable* if there exists a normal Γ-covering $\pi : \widehat{M} \longrightarrow M$ such that the pull-back $\widehat{\mathcal{F}}$ of $\mathcal{F}$ by π is a locally trivial fibration $D : \widehat{M} \longrightarrow W$ on which Γ acts by automorphisms (and then on W). The fibre of D is denoted $\widehat{L}$.

Suppose M compact. We prove: i) if $H^1(\widehat{L}) = 0$, the space $H^1_{\mathcal{F}}(M, \mathcal{V})$ of infinitesimal deformations of $(M, \mathcal{F})$ is naturally identified to the first cohomology space $H^1(\Gamma, \mathfrak{X}(W))$ of the discrete group Γ with values in the Γ-module $\mathfrak{X}(W)$ of smooth vector fields on W; ii) if $\widehat{L}$ has trivial cohomology, the foliated cohomology $H^*_{\mathcal{F}}(M, \mathcal{V})$ of $(M, \mathcal{F})$ with values in $\mathcal{V}$ is isomorphic to the cohomology $H^*(\Gamma, \mathfrak{X}(W))$ of Γ with values in $\mathfrak{X}(W)$. Some examples and explicit computations are given.

0. Introduction

The study of deformations of foliations uses various mathematical tools, for instance that of differential geometry, global analysis, algebraic topology... They fall within the more general framework of deformation theory of geometric structures which was initiated by the works of Kodaira and Spencer [KS] in the 50's on the variation of complex structures. Even if there is still work to be done, Kuranishi [Kur] has finalized the subject by showing the existence of a *versal space* through which transits any deformation of a given complex structure on a compact manifold. This result was a great advance in the development of complex geometry. Adapting these methods, Girbau, Haefliger and Sundararaman [GHS] established, in a similar way, the existence of a versal space for the deformations of transversely holomorphic foliations.

Unfortunately, the tools used in the complex case were not completely available for studying the deformations of real foliations. Moreover, at that time, the theory really lacked non-trivial examples. The first one which is differentiably stable and with an interesting dynamic (for instance all its leaves are dense) was given in [GS] by Ghys and Sergiescu in 1980. Their constructions and proofs are based on qualitative properties of codimension one foliations on 3-manifolds. More or less around the same time, Hamilton gave a strong criterion for deciding whether a foliation on a compact manifold is differentiably stable. But the paper [Ham] containing this result (always

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highly requested by specialists in these matters) has never been published, probably by voluntary decision of the author who had as examples to which he could apply its criterion only Riemannian foliations with compact leaves. Later on, using this criterion, in [EN] the authors constructed a class of C^∞-stable foliations (of arbitrary dimension and codimension) which includes the example in [GS].

A powerful tool in the study of the deformations of a foliation $(M, \mathcal{F})$ and which appears to be an essential ingredient in the Hamilton criterion is the *foliated cohomology* $H^*_{\mathcal{F}}(M, \mathcal{V})$ with values in the normal bundle $\mathcal{V}$ of $\mathcal{F}$. This is the reason why it has since been the subject of works by many authors.

In this paper, we explicitly describe the foliated cohomology with values in the normal bundle for *developable foliations* and relate it, by spectral sequences, to the cohomology of a discrete group with coefficients in the Fréchet space of C^∞-vector fields over a manifold naturally associated to the foliation. This makes possible to transpose its calculation, usually carried out by differential forms, to that using different methods and which sometimes prove to be more effective. In particular, this gives a way to compute the space $H^1_{\mathcal{F}}(M, \mathcal{V})$ which contains the infinitesimal deformations of $\mathcal{F}$. Explicit examples are given showing concretely the interest of this point of view.

Of course, the category of developable foliations is particular but it includes enough examples, for instance foliations obtained by suspension of groups of diffeomorphisms and transversely homogeneous foliations. For these latter foliations, a fairly specific study of their deformations accompanied by those of their transverse homogeneous structures was carried out in [EGN].

All the manifolds and the different geometric structures (functions, vector fields, differential forms...) are assumed to be of class C^∞.

The following notion appears frequently in this article. We will fix once and for all our terminology that designates it (even though it may not be the usual one).

Let M be a manifold and Γ a countable discrete group. A *normal* Γ*-covering* of M is given by a manifold $\widehat{M}$ with a free and proper action of Γ by diffeomorphisms such that M is the quotient $\widehat{M}/\Gamma$.

1. Foliations

Let M be a connected manifold of dimension $m + n$. A *foliation* of *codimension* n on M is a geometric structure such that around each point one can cut a small open neighborhood which looks like the product $\mathbb{R}^m \times \mathbb{R}^n$ where the second factor is equipped with the discrete topology. More precisely we have the following:

1.1. Definition. *Let M be a manifold of dimension $m+n$. A codimension n* **foliation** *$\mathcal{F}$ on M is given by an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ and for each i, a diffeomorphism $\varphi_i : \mathbb{R}^{m+n} \longrightarrow U_i$ such that, on each non empty intersection $U_i \cap U_j$, the coordinate change:*

$$(1) \qquad \varphi_j^{-1} \circ \varphi_i : (x, y) \in \varphi_i^{-1}(U_i \cap U_j) \longrightarrow (x', y') \in \varphi_j^{-1}(U_i \cap U_j)$$

has the form $x' = \varphi_{ij}(x, y)$ and $y' = \gamma_{ij}(y)$.

The manifold M is decomposed into connected submanifolds of dimension m. Each of them is called a *leaf* of $\mathcal{F}$. A subset U of M is *saturated* for $\mathcal{F}$ if it is a union of leaves: if $x \in U$ then the leaf passing through x is contained in U.

Coordinate patches (U_i, φ_i) satisfying conditions of definition 1.1 are said to be *distinguished* for the foliation $\mathcal{F}$.

The following (second) definition is more appropriate for introducing the notion of transverse structure, which is crucial in the theory of foliations.

Let $\mathcal{F}$ be a codimension n foliation on M defined by a maximal atlas $\{(U_i, \varphi_i)\}_{i\in I}$ like in definition 1.1. Let $\pi : \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the second projection. Then the map $f_i : U_i \longrightarrow \mathbb{R}^n$ (where $f_i = \pi \circ \varphi_i^{-1}$) is a submersion. On $U_i \cap U_j \neq \emptyset$ we have $f_j = \gamma_{ij} \circ f_i$. The fibres of the submersion f_i are the $\mathcal{F}$-*plaques* of U_i. The submersions f_i and the local diffeomorphisms γ_{ij} of $\mathbb{R}^n$ give a complete characterization of $\mathcal{F}$.

1.2. Definition. *A codimension n* ***foliation*** *on M is given by an open cover $(U_i)_{i\in I}$, submersions $f_i : U_i \longrightarrow T$ over a n dimensional transverse manifold T and, for any non empty intersection $U_i \cap U_j$, a diffeomorphism:*

$$\gamma_{ij} : f_i(U_i \cap U_j) \subset T \longrightarrow f_j(U_i \cap U_j) \subset T$$

satisfying $f_j(z) = \gamma_{ij} \circ f_i(z)$ for $z \in U_i \cap U_j$. We say that $\{U_i, f_i, T, \gamma_{ij}\}$ is a ***foliated cocycle*** *defining $\mathcal{F}$.*

The foliation $\mathcal{F}$ is said to be *transversely orientable* if the transverse manifold T can be given an orientation preserved by all the local diffeomorphisms γ_{ij}.

1.3. Morphisms of foliations

Let M and M' be two manifolds endowed respectively with foliations $\mathcal{F}$ and $\mathcal{F}'$. A map $f : M \longrightarrow M'$ will be called *foliated* or a *morphism* between $\mathcal{F}$ and $\mathcal{F}'$ if, for every leaf L of $\mathcal{F}$, $f(L)$ is contained in a leaf of $\mathcal{F}'$; we say that f is an *isomorphism* if, in addition, f is a diffeomorphism whose restriction to any leaf $L \in \mathcal{F}$ is a diffeomorphism on the leaf $L' = f(L) \in \mathcal{F}'$.

Suppose now that f is a diffeomorphism of M. Then for every leaf $L \in \mathcal{F}$, $f(L)$ is a leaf of a codimension n foliation $\mathcal{F}'$ on M; we say that $\mathcal{F}'$ is the *image* of $\mathcal{F}$ by the diffeomorphism f and we write $\mathcal{F} = f^*(\mathcal{F}')$. Two foliations $\mathcal{F}$ and $\mathcal{F}'$ on M are said to be C^r-*conjugated* (*topologically* if $r = 0$, *differentiably* if $r = \infty$ and *analytically* in the case $r = \omega$) if there exists a C^r-homeomorphism $f : M \longrightarrow M$ such that $f^*(\mathcal{F}') = \mathcal{F}$.

The set of C^∞-diffeomorphisms of M which preserve the foliation $\mathcal{F}$ is a group denoted $\mathrm{Diff}(M, \mathcal{F})$.

The following definition introduces a very important property of a foliation. It describes a large part of its geometric structure.

1.4. Definition. *Let M be a manifold and $\mathcal{F}$ a codimension n foliation on M defined by a foliated cocycle $\{U_i, f_i, T, \gamma_{ij}\}$. A* ***transverse structure*** *to $\mathcal{F}$ is a geometric structure on T invariant by the local diffeomorphisms γ_{ij}.*

Examples of such structures will be given later.

2. Basic elements

Let us fix some notations. Let $\mathcal{F}$ be a codimension n foliation on M. We denote by $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$ and $\mathcal{V}$ its *normal bundle* that is the quotient $TM/T\mathcal{F}$.

2.1. Basic forms and basic vector fields

We denote by $\mathfrak{X}(\mathcal{F})$ the space of sections of $T\mathcal{F}$ (elements of $\mathfrak{X}(\mathcal{F})$ are vector fields $X \in \mathfrak{X}(M)$ tangent to $\mathcal{F}$).

A differential form $\alpha \in \Omega^r(M)$ is said to be *basic* if it satisfies $i_X\alpha = 0$ and $L_X\alpha = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$. (Here i_X and L_X denote respectively the inner product and the Lie derivative with respect to the vector field X.) For a function $f : M \longrightarrow \mathbb{R}$, these conditions are equivalent to $X \cdot f = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$ *i.e.* f is constant on the leaves of $\mathcal{F}$; we denote by $\Omega_b^r(M)$ the space of basic forms of degree r on the foliated manifold $(M, \mathcal{F})$; this is a module over the algebra $C_b^\infty(M)$ of basic functions.

A vector field $Y \in \mathfrak{X}(M)$ is said to be *foliated*, if for every $X \in \mathfrak{X}(\mathcal{F})$, the bracket $[X, Y] \in \mathfrak{X}(\mathcal{F})$. We see easily that the set $\mathfrak{X}(M, \mathcal{F})$ of foliated vector fields is a Lie algebra in which $\mathfrak{X}(\mathcal{F})$ is an ideal. The quotient $\mathfrak{X}_b(M) = \mathfrak{X}(M, \mathcal{F})/\mathfrak{X}(\mathcal{F})$ is called the Lie algebra of *basic* (or *transverse*) vector fields on the foliated manifold $(M, \mathcal{F})$. Also, it has a module structure over the algebra $C_b^\infty(M)$.

2.2. Foliated vector bundles and basic sections

1. Let $\tau : E \longrightarrow M$ be a vector bundle of rank N defined by a cocycle $\{U_i, g_{ij}\}$ where $\{U_i\}$ is an open cover of M and the g_{ij} are the (continuous) transition functions $g_{ij} : U_i \cap U_j \longrightarrow \mathrm{GL}(N, \mathbb{R})$ satisfying the cocycle condition:

$$g_{ij}(z) = g_{ik}(z) \cdot g_{kj}(z) \quad \text{for } z \in U_i \cap U_j \cap U_k. \tag{2}$$

We say that E is *foliated* if the functions g_{ij} are basic on $U_i \cap U_j$. For such vector bundle, the foliation $\mathcal{F}$ can be lifted to a same dimension foliation $\mathcal{F}_E$ on E such that the projection τ sends leaves of $\mathcal{F}_E$ into leaves of $\mathcal{F}$.

Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be two foliations and $\tau : E \longrightarrow M$ and $\tau' : E' \longrightarrow M'$ foliated vector bundles. A morphism of vector bundles $E \longrightarrow E'$ is a *foliated morphism* if it sends leaves of $\mathcal{F}_E$ into leaves of $\mathcal{F}'_{E'}$. Of course, it induces a foliated morphism $(M, \mathcal{F}) \longrightarrow (M', \mathcal{F}')$. Foliated vector bundles and their morphisms form a category.

2. The fundamental example of a foliated bundle over a foliated manifold $(M, \mathcal{F})$ is the normal bundle $\mathcal{V} = TM/T\mathcal{F}$ (by definition of the foliation $\mathcal{F}$). The bundles naturally associated to it, for instance its dual $\mathcal{V}^*$, all of its exterior and symmetric powers $\Lambda^*\mathcal{V}^*$ and $\mathcal{S}^*\mathcal{V}^*$... are foliated.

3. Recall that a section of E is a C^∞-map $\alpha : M \longrightarrow E$ such that $\tau \circ \alpha = id_M$. If $(U, (x_1, \cdots, x_m, y_1, \cdots, y_n))$ is a distinguished coordinates system on which E is trivial, α is represented by a C^∞-function $\alpha : U \longrightarrow \mathbb{R}^N$. We say that α is *basic*, if the function α_U is basic, that is, it depends only on the transverse coordinates $(y_1, \cdots, y_n)$. The space $C_b^\infty(E)$ of basic sections of E is a module over the algebra $A = C_b^\infty(M)$ of basic functions. For more details on these foliated objects see [Ek3].

For a foliated vector bundle E we denote by $\mathcal{E}_b$ the sheaf of germs of its basic sections. In general $\mathcal{E}_b$ is not fine (see subsection 4.2) and then it gives rise to a non trivial cohomology $H^*(M, \mathcal{E}_b)$. We shall see in section 4 how to give a fine resolution of this sheaf and compute $H^*(M, \mathcal{E}_b)$ by using special differential forms on M.

3. Developable foliations

3.1. Foliated covering

1. Let $\widehat{M}$ be a connected manifold of dimension $m+n$ endowed with a codimension n foliation $\widehat{\mathcal{F}}$. Suppose that a discrete group Γ acts on $\widehat{M}$ freely and properly by diffeomorphisms preserving $\widehat{\mathcal{F}}$. The quotient $M = \widehat{M}/\Gamma$ is a manifold of dimension $m+n$ and the canonical projection $\pi : \widehat{M} \longrightarrow M$ is a normal Γ-covering. Moreover, because elements of Γ preserve $\widehat{\mathcal{F}}$, this foliation induces on M a codimension n foliation $\mathcal{F}$ for which π is a morphism of foliations. We say that $\pi : (\widehat{M}, \widehat{\mathcal{F}}) \longrightarrow (M, \mathcal{F})$ is a *foliated normal* Γ*-covering.*

2. Conversely, let M be a connected manifold of dimension $m+n$ equipped with a codimension n foliation $\mathcal{F}$. For any normal Γ-covering $\widehat{M} \xrightarrow{\pi} M = \widehat{M}/\Gamma$, the pull-back $\widehat{\mathcal{F}} = \pi^*(\mathcal{F})$ of $\mathcal{F}$ is a codimension n foliation such that $\pi : (\widehat{M}, \widehat{\mathcal{F}}) \longrightarrow (M, \mathcal{F})$ is a foliated normal Γ-covering.

This enables us to give the definition of the category of foliations we will be interested on in all this paper.

3.2. Definition. *A codimension n foliation $\mathcal{F}$ on a connected manifold M is said* ***developable*** *if there exists a normal* Γ*-covering* $\widehat{M} \xrightarrow{\pi} M$ *such that the leaves of the pull-back $\widehat{\mathcal{F}}$ of $\mathcal{F}$ are the fibres of a locally trivial fibration $D : \widehat{M} \longrightarrow W$. This fibration is called the* ***developing map*** *of $\mathcal{F}$.*

The group Γ acts on $\widehat{M}$ by automorphisms of $\widehat{\mathcal{F}}$ and then by diffeomorphisms on W. For any such automorphism $\gamma \in \Gamma$, we denote by $\overline{\gamma}$ the induced diffeomorphism on W; we have a commutative diagram:

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\gamma} & \widehat{M} \\ D \downarrow & & \downarrow D \\ W & \xrightarrow{\overline{\gamma}} & W. \end{array} \tag{3}$$

From now on, the examples we will give in this section are all developable foliations.

3.3. Transversely homogeneous foliations

Let M be a manifold of dimension $m+n$ endowed with a codimension n foliation $\mathcal{F}$ defined by a foliated cocycle $\{U_i, f_i, T, \gamma_{ij}\}$. We say that $\mathcal{F}$ is *transversely homogeneous* if T is a homogeneous space G/H and the diffeomorphisms γ_{ij} are induced by restriction of left translations on the Lie group G. (Here G is a connected Lie group and H is a connected closed subgroup.) Also we say that $\mathcal{F}$ is a G/H*-foliation.*

The structure of such foliation on a compact manifold is given by the following theorem due to R. Blumenthal [Blu].

Let $\mathcal{F}$ be a G/H-foliation on a connected compact manifold M. Then, there exist a normal Γ-covering $\widehat{M} \longrightarrow M = \widehat{M}/\Gamma$ (where Γ is a countable discrete group), an injective homomorphism $h : \Gamma \longrightarrow G$ and a locally trivial fibration $D : \widehat{M} \longrightarrow G/H$ whose fibres are the leaves of the lifted foliation $\widehat{\mathcal{F}}$ of $\mathcal{F}$ to $\widehat{M}$ and such that, for every $\gamma \in \Gamma$, the following diagram is commutative:

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\gamma} & \widehat{M} \\ D\downarrow & & \downarrow D \\ G/H & \xrightarrow{h(\gamma)} & G/H. \end{array} \tag{4}$$

Here the first line denotes the deck transformation of $\gamma \in \Gamma$ on $\widehat{M}$ and $h(\gamma)$ is the diffeomorphism of G/H induced by the left translation by $h(\gamma)$ on G.

The subgroup $h(\Gamma)$ of the Lie group G which we will denote also Γ, is called the *holonomy group* of $\mathcal{F}$.

The case where the subgroup H is trivial corresponds to a Lie *G-foliation*. The structure of such foliation is given by a result of Fédida [Féd] for which the theorem of Blumenthal is a generalization to transversely homogeneous foliations.

A particular case is given by a *homogeneous foliation* which can be defined as follows. Let G be a connected Lie group of dimension $m + n$ and H a connected closed subgroup of dimension m. Then the right action of H on G defines a foliation $\widehat{\mathcal{F}}$ of dimension m: its leaves are the orbits and also the fibres of the principal bundle $H \hookrightarrow G \xrightarrow{D} W$ where W is the homogeneous space G/H. Now, let Γ be a cocompact lattice of G. Since the left action of Γ and the right action of H commute, the action of H induces a locally free action on the quotient $M = \Gamma\backslash G$ which defines a foliation $\mathcal{F}$. (Locally free means that the isotropy subgroups are discrete.)

The foliation $\mathcal{F}$ on M is developable. Here $\widehat{M} = G$ and the developing map is just the locally trivial fibration $D : \widehat{M} = G \longrightarrow W$. The pull-back of $\mathcal{F}$ by the normal Γ-covering projection $\pi : G \longrightarrow M$ is exactly the foliation $\widehat{\mathcal{F}}$.

3.4. Suspension of a group of diffeomorphisms

Let L and W be two manifolds, respectively of dimensions m and n. Suppose that the fundamental group $\pi_1(L)$ of L is finitely generated. Let $\rho : \pi_1(L) \longrightarrow \mathrm{Diff}(W)$ be a representation, where $\mathrm{Diff}(W)$ is the diffeomorphism group of W. Denote by $\widehat{L}$ the universal covering of L and $\widehat{\mathcal{F}}$ the horizontal foliation on $\widehat{M} = \widehat{L} \times W$, i.e., the foliation whose leaves are the subsets $\widehat{L} \times \{y\}$, $y \in W$. This foliation is invariant by all the transformations $T_\gamma : \widehat{M} \longrightarrow \widehat{M}$ defined by $T_\gamma(\widehat{x}, y) = (\gamma \cdot \widehat{x}, \rho(\gamma)(y))$ where $\gamma \cdot \widehat{x}$ is the natural action of $\gamma \in \pi_1(L)$ on $\widehat{L}$; then $\widehat{\mathcal{F}}$ induces a codimension n foliation $\mathcal{F}_\rho$ on the quotient manifold:

$$M = \widehat{M}/(\widehat{x}, y) \sim (\gamma \cdot \widehat{x}, \rho(\gamma)(y)).$$

We say that $\mathcal{F}_\rho$ is the *suspension* of the representation $\rho : \pi_1(L) \longrightarrow \mathrm{Diff}(W)$. The leaves of $\mathcal{F}_\rho$ are transverse to the fibres of the natural fibration induced by the projection on the first factor $\widehat{L} \times W \longrightarrow \widehat{L}$.

The geometric transverse structures of the foliation $\mathcal{F}$ are exactly the geometric structures on the manifold W invariant under the action of Γ.

4. Cohomologies

4.1. Cohomology of a discrete group

Let Γ be a (countable) discrete group and V a vector space on which Γ acts by automorphisms; this makes V a Γ-*module.* The action of $\gamma \in \Gamma$ on $v \in V$ is denoted $\gamma \cdot v$.

1. For each integer $p \geq 1$, let $C^p(\Gamma, V)$ be the vector space of maps from Γ^p to V; an element of $C^p(\Gamma, V)$ is called a *p-cochain* on Γ with values in V. By convention $C^0(\Gamma, V) = V$. Define the linear map $d : C^p(\Gamma, V) \longrightarrow C^{p+1}(\Gamma, V)$ by:

$$\begin{aligned} dc(\gamma_1, \ldots, \gamma_{p+1}) &= \gamma_1 \cdot c(\gamma_2, \ldots, \gamma_{p+1}) \\ &\quad + \sum_{i=1}^{p} (-1)^i c(\gamma_1, \ldots, \gamma_{i-1}, \gamma_i\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_{p+1}) \\ &\quad + (-1)^{p+1} c(\gamma_1, \ldots, \gamma_p). \end{aligned} \tag{5}$$

An element of the kernel $Z^p(\Gamma, V)$ of $d : C^p(\Gamma, V) \longrightarrow C^{p+1}(\Gamma, V)$ is called a *cocycle* and an element of the image $B^p(\Gamma, V)$ of $d : C^{p-1}(\Gamma, V) \longrightarrow C^p(\Gamma, V)$ is called a *coboundary.* The operator d satisfies $d^2 = 0$ and then $B^p(\Gamma, V)$ is a subspace of $Z^p(\Gamma, V)$. The quotients $H^p(\Gamma, V) = Z^p(\Gamma, V)/B^p(\Gamma, V)$ for $p \in \mathbb{N}$ are called the *cohomology spaces* of Γ with values in the Γ-module V.

2. An element c of $C^0(\Gamma, V)$ is just a vector in V and $dc(\gamma) = \gamma \cdot c - c$. So $H^0(\Gamma, V)$ is the subspace V^Γ of elements of V which are invariant under the action of Γ.

3. If $\Gamma = \mathbb{Z}$ and its action is generated by an automorphism γ of V, an easy computation shows that:

$$H^p(\Gamma, V) = \begin{cases} V^\Gamma & \text{if } p = 0 \\ V/\langle v - \gamma \cdot v \rangle & \text{if } p = 1 \\ 0 & \text{if } p \geq 2 \end{cases} \tag{6}$$

where $\langle v - \gamma \cdot v \rangle$ is the subspace of V generated by elements of the form $v - \gamma \cdot v$ with v varying in V. If, for instance, V is finite dimensional and γ does not fix any vector, the linear map $v \in V \longmapsto v - \gamma \cdot v \in V$ is an isomorphism and then $H^0(\Gamma, V) = H^1(\Gamma, V) = 0$.

4.2. Sheaf cohomology

Let $\mathcal{E}$ be a sheaf of vector spaces on M and $\mathcal{U} = \{U_i\}_{i \in I}$ a locally finite open cover of M. For any multi-index $(i_0, \cdots, i_q)$ in I, we denote $U_{i_0 \cdots i_q}$ the intersection

$U_{i_0} \cap \cdots \cap U_{i_q}$ and Σ_q the set of multi-indices for which $U_{i_0 \cdots i_q} \neq \emptyset$. Let $C^q(\mathcal{U}, \mathcal{E})$ be the set of all collections $(f_{i_0 \cdots i_q})_{(i_0, \cdots, i_q) \in \Sigma_q}$ where $f_{i_0 \cdots i_q}$ is an element of $\mathcal{E}(U_{i_0 \cdots i_q})$ (space of sections of $\mathcal{E}$ over the open set $U_{i_0 \cdots i_q}$); it is a $\mathbb{K}$-vector space. An element f of $C^q(\mathcal{U}, \mathcal{E})$ is called *q-cochain* on $\mathcal{U}$ with values in $\mathcal{E}$. Define the cobord operator $\delta : C^q(\mathcal{U}, \mathcal{E}) \longrightarrow C^{q+1}(\mathcal{U}, \mathcal{E})$ by:

$$(\delta f)_{i_0 \cdots i_{q+1}} = \sum_{j=0}^{q+1} (-1)^j f_{i_0 \cdots \hat{i}_j \cdots i_{q+1}} \tag{7}$$

where $f_{i_0 \cdots \hat{i}_j \cdots i_{q+1}}$ is the section $f_{i_0 \cdots \hat{i}_j \cdots i_{q+1}}$ restricted to the open set $U_{i_0 \cdots i_{q+1}}$. For instance, for a 0-cochain $f = (f_i)$, the 1-cochain $\delta f = (f_{ij})$ is given by $f_{ij} = f_j - f_i$; if $q = 1$ and $f = (f_{ij})$, then $\delta f = (f_{ijk})$ with $f_{ijk} = f_{jk} - f_{ik} + f_{ij}$. One can verify that:

$$\delta \circ \delta : C^{q-1}(\mathcal{U}, \mathcal{E}) \longrightarrow C^q(\mathcal{U}, \mathcal{E}) \longrightarrow C^{q+1}(\mathcal{U}, \mathcal{E})$$

is zero. Then the kernel $Z^q(\mathcal{U}, \mathcal{E})$ of $\delta : C^q(\mathcal{U}, \mathcal{E}) \longrightarrow C^{q+1}(\mathcal{U}, \mathcal{E})$ contains the image $B^q(\mathcal{U}, \mathcal{E})$ of $\delta : C^{q-1}(\mathcal{U}, \mathcal{E}) \longrightarrow C^q(\mathcal{U}, \mathcal{E})$; the quotient:

$$H^q(\mathcal{U}, \mathcal{E}) = Z^q(\mathcal{U}, \mathcal{E}) / B^q(\mathcal{U}, \mathcal{E})$$

is the q^{th} *cohomology space* of the cover $\mathcal{U}$ with coefficients in the sheaf $\mathcal{E}$.

If $\mathcal{U}' = \{U'_j\}_{j \in J}$ is an open cover *finer* than $\mathcal{U}$, that is, for $j \in J$, there exists $i \in I$ such that $U'_j \subset U_i$, we have a restriction morphism $\rho_q : C^q(\mathcal{U}, \mathcal{E}) \longrightarrow C^{q+1}(\mathcal{U}', \mathcal{E})$ which induces a morphism $\rho_q^* : H^q(\mathcal{U}, \mathcal{E}) \longrightarrow H^q(\mathcal{U}', \mathcal{E})$. The inductive limit of the system $\left\{\rho_q^* : H^q(\mathcal{U}, \mathcal{E}) \longrightarrow H^q(\mathcal{U}', \mathcal{E})\right\}_{\mathcal{U}' \prec \mathcal{U}}$ is a vector space denoted $H^q(M, \mathcal{E})$ and called the q^{th} *cohomology* space of M with coefficients in the sheaf $\mathcal{E}$ (since we work on paracompact Hausdorff spaces).

We have the following properties.

1. $H^0(M, \mathcal{E})$ is the space $\mathcal{E}(M)$ of global sections of $\mathcal{E}$.

2. Let M and M' be two manifolds, $f : M \longrightarrow M'$ a continuous map, $\mathcal{E}$ a sheaf on M and $\mathcal{E}'$ its direct image by f. Then, for any integer $q \geq 0$, f induces a morphism $f^* : H^q(M', \mathcal{E}') \longrightarrow H^q(M, \mathcal{E})$. If M'' is an other manifold, $g : M' \longrightarrow M''$ a continuous map and $\mathcal{E}''$ the direct image of $\mathcal{E}'$ by g, then $(g \circ f)^* = f^* \circ g^*$. Furthermore if $M = M'$ and f is the identity map, then f^* is the identity of $H^q(M, \mathcal{E})$.

3. We say that a sheaf $\mathcal{E}$ is *fine* if, for any open locally finite cover $\mathcal{U} = \{U_i\}$ there are endomorphisms $h_i : \mathcal{E} \longrightarrow \mathcal{E}$ such that the support of h_i is a subset of U_i and $\sum_{i \in I} h_i$ is the identity of $\mathcal{E}$. (Recall that the *support* of a morphism $h : \mathcal{E} \longrightarrow \mathcal{E}$ is the set $\mathrm{supp}(h) = \overline{\{x \in M : h(\mathcal{E}_x) \neq 0\}}$ where $\mathcal{E}_x$ is the fibre of $\mathcal{E}$ at x.) For instance, the sheaf $\mathcal{C}^k$ of germs of functions of class C^k ($k \in \mathbb{N} \cup \{\infty\}$) is fine.

We have the following assertion (see [God] for the proof): *Suppose $\mathcal{E}$ is a fine sheaf. Then $H^q(M, \mathcal{E}) = 0$ for $q \geq 1$.*

4.3. Spectral sequence of a covering

Let Γ be a countable discrete group and $\pi : \widehat{M} \longrightarrow M$ a normal Γ-covering. Let $\widehat{\mathcal{E}}$ be a sheaf of vector spaces over $\widehat{M}$ on which Γ acts and $\mathcal{E}$ its direct image by π on M.

Let $\mathcal{U} = \{U_i\}$ be a locally finite open cover of M; the pull-back $\widehat{\mathcal{U}} = \{\widehat{U}_i\}$ of $\mathcal{U}$ by π is an open cover of $\widehat{M}$. For each $q \in \mathbb{N}$, Γ acts on the space $C^q(\widehat{\mathcal{U}}, \widehat{\mathcal{E}})$ of q-cochains on $\widehat{\mathcal{U}}$ with values in $\widehat{\mathcal{E}}$; $C^q(\widehat{\mathcal{U}}, \widehat{\mathcal{E}})$ is therefore a Γ-module. One can then consider the cohomology $H^p(\Gamma, H^q(\widehat{M}, \widehat{\mathcal{E}}))$. According to [Gro] page 204, this is the term E_2^{pq} of a spectral sequence:

$$E_2^{pq} = H^p(\Gamma, H^q(\widehat{M}, \widehat{\mathcal{E}})) \tag{8}$$

converging to $H^*(M, \mathcal{E})$. A construction of this spectral sequence can also be found in [Bro], Chapter VII, Sections 5 and 7.

If $(\widehat{M}, \widehat{\mathcal{E}})$ is acyclic, that is $H^q(\widehat{M}, \widehat{\mathcal{E}}) = 0$ for $q \geq 1$, the spectral sequence E_r converges at the E_2 term and then:

$$H^p(M, \mathcal{E}) = H^p(\Gamma, H^0(\widehat{M}, \widehat{\mathcal{E}})) = H^p(\Gamma, \widehat{\mathcal{E}}(\widehat{M})) \tag{9}$$

where $\widehat{\mathcal{E}}(\widehat{M})$ is the space of global sections of the sheaf $\widehat{\mathcal{E}}$.

4.4. Foliated cohomologies

Let M be a connected differentiable manifold supporting a foliation $\mathcal{F}$ of dimension m (and codimension n).

1. For any $r \in \mathbb{N}$, we denote $\Lambda^r(T^*\mathcal{F})$ the bundle of exterior algebras of degree r over $T\mathcal{F}$ (tangent bundle to $\mathcal{F}$). Its sections are the *foliated forms* of degree r; they form a vector space $\Omega^r_{\mathcal{F}}(M)$. We have an operator (exterior derivative along the leaves) $d_{\mathcal{F}} : \Omega^r_{\mathcal{F}}(M) \longrightarrow \Omega^{r+1}_{\mathcal{F}}(M)$ defined (as in the classical case) by the formula:

$$\begin{aligned} d_{\mathcal{F}}\alpha(X_1, \cdots, X_{r+1}) = &\sum_{i=1}^{r+1} (-1)^{i+1} X_i \cdot \alpha(X_1, \cdots, \widehat{X}_i, \cdots, X_{r+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots, X_{r+1}) \end{aligned}$$

where $\widehat{X}_i$ means that the argument X_i is omitted. We easily verify that $d^2_{\mathcal{F}} = 0$. So we obtain a differential complex (called the *de Rham foliated complex* of $\mathcal{F}$):

$$0 \longrightarrow \Omega^0_{\mathcal{F}}(M) \xrightarrow{d_{\mathcal{F}}} \Omega^1_{\mathcal{F}}(M) \xrightarrow{d_{\mathcal{F}}} \cdots \xrightarrow{d_{\mathcal{F}}} \Omega^{m-1}_{\mathcal{F}}(M) \xrightarrow{d_{\mathcal{F}}} \Omega^m_{\mathcal{F}}(M) \longrightarrow 0.$$

Let $Z^r_{\mathcal{F}}(M)$ be the kernel of $d_{\mathcal{F}} : \Omega^r_{\mathcal{F}}(M) \longrightarrow \Omega^{r+1}_{\mathcal{F}}(M)$ and $B^r_{\mathcal{F}}(M)$ the image of $d_{\mathcal{F}} : \Omega^{r-1}_{\mathcal{F}}(M) \longrightarrow \Omega^r_{\mathcal{F}}(M)$. The quotient $H^r_{\mathcal{F}}(M) = Z^r_{\mathcal{F}}(M)/B^r_{\mathcal{F}}(M)$ is the r^{th} vector space of *foliated cohomology* of $(M, \mathcal{F})$.

Foliated cohomology was substantially used in the study of the parametric rigidity of some Lie group actions. (See [MM] and [Asa] for an account on the subject).

2. Let $\tau : E \longrightarrow M$ be a $\mathcal{F}$-foliated vector bundle. Then $d_{\mathcal{F}}$ extends to the space $\Omega^*_{\mathcal{F}}(M,E)$ of foliated forms with values in E and gives rise to a differential complex:

$$(10)\quad 0 \longrightarrow \Omega^0_{\mathcal{F}}(M,E) \xrightarrow{d_{\mathcal{F}}} \Omega^1_{\mathcal{F}}(M,E) \xrightarrow{d_{\mathcal{F}}} \cdots \xrightarrow{d_{\mathcal{F}}} \Omega^{m-1}_{\mathcal{F}}(M,E) \xrightarrow{d_{\mathcal{F}}} \Omega^m_{\mathcal{F}}(M,E) \longrightarrow 0$$

whose cohomology $H^*_{\mathcal{F}}(M,E)$ is called the *foliated cohomology* of the foliated manifold $(M,\mathcal{F})$ with values in E. If $\mathcal{E}_b$ is the sheaf of germs of basic sections of E and $\widetilde{\Omega}^r_{\mathcal{F}}(E)$ the sheaf of germs of foliated r-forms with values in E, we have a fine resolution:

$$(11)\qquad 0 \longrightarrow \mathcal{E}_b \hookrightarrow \widetilde{\Omega}^0_{\mathcal{F}}(E) \xrightarrow{d_{\mathcal{F}}} \widetilde{\Omega}^1_{\mathcal{F}}(E) \xrightarrow{d_{\mathcal{F}}} \cdots \xrightarrow{d_{\mathcal{F}}} \widetilde{\Omega}^{m-1}_{\mathcal{F}}(E) \xrightarrow{d_{\mathcal{F}}} \widetilde{\Omega}^m_{\mathcal{F}}(E) \longrightarrow 0$$

first established by Vaisman in [Vai] Chapter 5 Section 2. Then:

$$H^*_{\mathcal{F}}(M,E) = H^*(M,\mathcal{E}_b).$$

3. If E is the trivial vector bundle of rank 1, $\mathcal{E}_b$ is the sheaf of germs of basic functions and so $H^r(M,\mathcal{E}_b) = H^r_{\mathcal{F}}(M)$ for $r \in \mathbb{N}$.

4. As our main interest in this work is the study of infinitesimal deformations of a developable foliation $(M,\mathcal{F})$, our $\mathcal{F}$-bundle E will now be the normal bundle $\mathcal{V}$ of $\mathcal{F}$.

The following proposition will be an essential ingredient in the proof of Theorem 5.2 which is the main result of this paper.

4.5. Proposition. *Suppose that the foliation $\mathcal{F}$ is a (locally trivial) fibration $M \xrightarrow{\pi} W$ with fibre L. If $H^q(L) = 0$ (for some $q \geq 1$) then $H^q_{\mathcal{F}}(M,\mathcal{V}) = 0$.*

Proof. Note that $\mathcal{V}$ is the pull-back by π of the tangent bundle TW of W. Then it is trivial over each leaf.

Let $\{U_i\}_{i\in I}$ be a locally finite open cover of W trivializing both the fibration π and the vector bundle TW. For each $i \in I$, we set $V_i = \pi^{-1}(U_i) \simeq U_i \times L$; then $\{V_i\}$ is an open cover of M and the vector bundle $\mathcal{V}$ is trivial over V_i. Let $\overline{\rho}_i$ be a partition of unity associated with the cover $\{U_i\}_{i\in I}$ and let $\rho_i = \overline{\rho}_i \circ \pi$. Then ρ_i is a partition of unity associated with the cover $\{V_i\}_{i\in I}$; in addition each of the functions ρ_i is basic *i.e.* $d_{\mathcal{F}}\rho_i = 0$.

Let $\alpha \in \Omega^q_{\mathcal{F}}(M,\mathcal{V})$ be a foliated q-form with values in $\mathcal{V}$ such that $d_{\mathcal{F}}\alpha = 0$. Denote by α_i its restriction to the open set V_i; α_i is a foliated q-form on the foliated manifold $(V_i,\mathcal{F})$ with values in $\mathcal{V}$ and it satisfies $d_{\mathcal{F}}\alpha_i = 0$. As the fibration $\pi : V_i \longrightarrow U_i$ and the vector bundle $\mathcal{V} \longrightarrow V_i$ are trivial and $H^q(L) = 0$ (by hypothesis), there exists $\beta_i \in \Omega^{q-1}_{\mathcal{F}}(V_i,\mathcal{V})$ such that $d_{\mathcal{F}}\beta_i = \alpha_i$. Define $\beta \in \Omega^{q-1}_{\mathcal{F}}(M,\mathcal{V})$ by $\beta = \sum_{i\in I}\rho_i\beta_i$. Then, taking into account the fact that $d_{\mathcal{F}}\rho_i = 0$, we have:

$$d_{\mathcal{F}}\beta = d_{\mathcal{F}}\left(\sum_{i\in I}\rho_i\beta_i\right) = \sum_{i\in I} d_{\mathcal{F}}(\rho_i\beta_i) = \sum_{i\in I}\rho_i(d_{\mathcal{F}}\beta_i) = \sum_{i\in I}\rho_i\alpha_i = \alpha.$$

Thus we have shown that the vector space $H^q_{\mathcal{F}}(M, \mathcal{V})$ is trivial. □

4.6. Integrable homotopy

This property is for foliated cohomology of a foliation what the ordinary homotopy is for de Rham cohomology of a manifold.

Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be two foliated manifolds. We equip $M \times \mathbb{R}$ with the foliation $\mathfrak{F}$ whose leaves are $L \times \mathbb{R}$ where L is a leaf of $\mathcal{F}$. Let $f, g : M \longrightarrow M'$ be two differentiable maps. An *integrable homotopy* from f to g is a differentiable foliated map $H : (M \times \mathbb{R}, \mathfrak{F}) \longrightarrow (M', \mathcal{F}')$ such that:

$$H(z, t) = \begin{cases} f(z) \text{ for } t \leq 0 \\ g(z) \text{ for } t \geq 1. \end{cases}$$

We have the following assertions (see [Ek1]) established in the case of foliated cohomology but easily generalized in the case where it is valued in the normal bundle.

1. *If there exists an integrable homotopy from f to g then the induced morphisms $f^*, g^* : H^*_{\mathcal{F}'}(M') \longrightarrow H^*_{\mathcal{F}}(M)$ are the same.*

2. *Suppose that the two foliated manifolds $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ have the same integrable homotopy type. Then their foliated cohomologies $H^*_{\mathcal{F}}(M)$ and $H^*_{\mathcal{F}'}(M')$ are isomorphic.*

5. Infinitesimal deformations of developable foliations

In all this section, M will be a connected compact manifold of dimension $d = m + n$. As usual, all the objects we will consider are supposed to be of class C^∞.

5.1. Preliminaries

All that we are going to expose in this subsection is taken from the paper [Ham].

For any $x \in M$, we denote $G_x(M, m)$ the Grassmannian of m-planes of T_xM. We obtain a locally trivial bundle $\mathcal{G}(M, m) \longrightarrow M$ whose typical fibre is the Grassmannian $G(d, m)$ of the vector space $\mathbb{R}^d$. A *C^∞-field* (or just a *field*) of m-planes on M is nothing but a section of $\mathcal{G}(M, m) \longrightarrow M$. Let τ be a field of m-planes and $(\tau_1, \cdots, \tau_m)$ a basis of local sections of τ. If $X = \sum_{i=1}^m a_i\tau_i$ and $Y = \sum_{j=1}^m b_j\tau_j$ are two local sections of τ, we have:

$$[X, Y] = \sum_{i,j=1}^m a_ib_j[\tau_i, \tau_j] + \sum_{i,j=1}^m \{a_i(\tau_i \cdot b_j)\tau_j - b_j(\tau_j \cdot a_i)\tau_i\}. \tag{12}$$

In the quotient $\mathcal{V} = TM/\tau$, the value of $[X, Y]$ at a point $x \in M$ depends only on the values of X and Y at this point and not on the values of their derivatives. This gives rise to a 2-form $Q(\tau) : \tau \times \tau \longrightarrow \mathcal{V}$ whose value at X_x and Y_x is the class in the quotient $\mathcal{V}_x = T_xM/\tau_x$ of the vector $[X, Y]_x$.

The 2-form $Q(\tau)$ is an element of the space $C^\infty(\Lambda^2(\mathcal{G}(TM, m))$ of the C^∞-sections of the bundle $\Lambda^2(\mathcal{G}(TM, m)) \longrightarrow \mathcal{G}(TM, m)$ whose fibre over a m-plane τ of TM is the space of 2-skew-symmetric forms $\Lambda^2(\tau, TM/\tau)$. By Frobenius theorem, τ is integrable if and only if the $Q(\tau)$ is identically zero. In this case, τ defines a dimension m foliation $\mathcal{F}$ on M. The C^∞-map $Q : \tau \in C^\infty(\mathcal{G}(M,m)) \longmapsto Q(\tau) \in C^\infty(\Lambda^2(\mathcal{G}(TM,m))$ plays a fundamental role in the theory of foliations.

The space $C^\infty(\mathcal{G}(M,m))$ of C^∞-sections of the bundle $\mathcal{G}(M,m)$ equipped with its natural C^∞-topology is a Fréchet manifold [Ham]. The subset $\mathfrak{F}(M,m)$ of foliations of dimension m on M ("zero set of Q") is closed; we equip it with the induced topology.

Let $\mathcal{F}$ be a foliation of dimension m, that is, $\mathcal{F}$ is an integrable field of m-planes on M. For any diffeomorphism φ of M, the pull-back $\varphi^*(\mathcal{F})$ of $\mathcal{F}$ by φ is a dimension m foliation on M *i.e.* an element of $\mathcal{G}(M,m)$ such that its associated 2-form $Q(\mathcal{F})$ is identically 0. Then we have a C^∞-map $\Theta_{\mathcal{F}} : \varphi \in \mathrm{Diff}(M) \longmapsto \varphi^*(\mathcal{F}) \in C^\infty(\mathcal{G}(M,m))$ (with values in fact in $\mathfrak{F}(M,m)$) and a sequence:

$$\mathrm{Diff}(M) \xrightarrow{\Theta_{\mathcal{F}}} C^\infty(\mathcal{G}(M,m)) \xrightarrow{Q} C^\infty(\Lambda^2(\mathcal{G}(TM,m)) \tag{13}$$

satisfying $Q \circ \Theta_{\mathcal{F}} = 0$. This sequence is called the *non-linear deformation complex* of the foliation $\mathcal{F}$. We say that $\mathcal{F}$ is C^∞-*stable* if there exist an open neighborhood $\mathfrak{U}$ of $\mathcal{F}$ in $C^\infty(\mathcal{G}(M,m))$ and an open neighborhood $\mathfrak{W}$ of the identity in $\mathrm{Diff}(M)$ with $\Theta_{\mathcal{F}}(\mathfrak{W}) \subset \mathfrak{U}$ and such that the complex:

$$\mathfrak{W} \xrightarrow{\Theta_{\mathcal{F}}} \mathfrak{U} \xrightarrow{Q} C^\infty(\Lambda^2(\mathcal{G}(TM,m)) \tag{14}$$

is exact. This means that any foliation $\mathcal{F}'$ of dimension m on M sufficiently close (in the C^∞-topology) to $\mathcal{F}$ is conjugate to $\mathcal{F}$ by a diffeomorphism φ of M close to the identity *i.e.* $\varphi^*(\mathcal{F}') = \mathcal{F}$.

The study of the stability of $\mathcal{F}$ requires the linearization of the nonlinear complex of its deformations. But this one is exactly the foliated de Rham complex (up to the degree 2) with values in the normal bundle (cf. [Ham] for calculations):

$$0 \longrightarrow \Omega^0_{\mathcal{F}}(M,\mathcal{V}) \xrightarrow{d_{\mathcal{F}}} \Omega^1_{\mathcal{F}}(M,\mathcal{V}) \xrightarrow{d_{\mathcal{F}}} \Omega^2_{\mathcal{F}}(M,\mathcal{V})... \tag{15}$$

An element of $H^1_{\mathcal{F}}(M,\mathcal{V})$ (cohomology at degree 1 of the complex (15)) is called an *infinitesimal deformation* of $\mathcal{F}$. So in all the following sections we will focus our attention mostly on the vector spaces $H^*_{\mathcal{F}}(M,\mathcal{V})$.

Now the foliation $\mathcal{F}$ we will consider on our manifold M will be developable. Let $\pi : \widehat{M} \longrightarrow M = \widehat{M}/\Gamma$ be its normal Γ-covering and $D : \widehat{M} \longrightarrow W$ its developing map whose fibres are the leaves $\widehat{L}$ of the pull-back $\widehat{\mathcal{F}} = \pi^*(\mathcal{F})$.

Let $\widehat{\mathcal{V}} = T\widehat{M}/T\widehat{\mathcal{F}}$ denote the normal bundle of $\widehat{\mathcal{F}}$. Since the group Γ acts on $\widehat{M}$ by diffeomorphisms preserving the foliation $\widehat{\mathcal{F}}$, it preserves the tangent bundle $T\widehat{\mathcal{F}}$ and thus acts on the normal bundle $\widehat{\mathcal{V}}$: $\widehat{\mathcal{V}}$ is both a $\widehat{\mathcal{F}}$-foliated bundle and a Γ-bundle. Now because the leaf space of $\widehat{\mathcal{F}}$ is the manifold W, the action of Γ on $\widehat{\mathcal{V}}$ induces one on the tangent bundle of W. Therefore the space $C_b^\infty(\widehat{\mathcal{V}})$ of basic sections of $\widehat{\mathcal{V}}$ is the space $\mathfrak{X}(W)$ of vector fields on W and on which Γ acts.

Finally, our main result on foliated cohomology and infinitesimal deformations of developable foliations is given by the following theorem.

5.2. Theorem. *Let $\mathcal{F}$ be a developable foliation on a connected compact manifold M. The notations are those just given before. We have the following assertions.*

1. *If $H^1(\widehat{L}) = 0$, the space $H^1_{\mathcal{F}}(M, \mathcal{V})$ is isomorphic to $H^1(\Gamma, \mathfrak{X}(W))$.*

2. *If $H^q(\widehat{L}) = 0$ for all $q \geq 1$, each $H^q_{\mathcal{F}}(M, \mathcal{V})$ is isomorphic to $H^q(\Gamma, \mathfrak{X}(W))$.*

Proof. It is almost immediate since we have prepared in the preceding sections all the material for this purpose.

Let $\mathcal{E}_b$ and $\widehat{\mathcal{E}}_b$ denote the sheaves of germs of basic sections respectively of the bundles $\mathcal{V}$ and $\widehat{\mathcal{V}}$. By 4.3. we have a spectral sequence (E_r, d_r) converging to $H^*(M, \mathcal{E}_b)$ and whose E_2 term is:

$$E_2^{pq} = H^p(\Gamma, H^q(\widehat{M}, \widehat{\mathcal{E}}_b)). \tag{16}$$

1. Since the foliation $(\widehat{M}, \widehat{\mathcal{F}})$ is the fibration $D : \widehat{M} \longrightarrow W$ and $H^1(\widehat{L}) = 0$ by hypothesis, we get by applying Proposition 4.5. $H^1(\widehat{M}, \widehat{\mathcal{E}}_b) = 0$ and then $E_2^{01} = 0$.

On the other hand $H^1(M, \mathcal{E}_b) = E_\infty^{10} \oplus E_\infty^{01} = E_\infty^{10}$, where E_∞^{pq} is the limit of the spectral sequence. Now, for $r \geq 2$, the differentials:

$$d_r : E_r^{10} \longrightarrow E_r^{1+r,-r+1} \quad \text{and} \quad d_r : E_r^{1-r,r-1} \longrightarrow E_r^{10}$$

are zero since the vector spaces $E_r^{1+r,-r+1}$ and $E_r^{1-r,r-1}$ are trivial; therefore:

$$E_\infty^{10} = E_2^{10} = H^1(\Gamma, H^0(\widehat{M}, \widehat{\mathcal{E}}_b)).$$

But $H^0(\widehat{M}, \widehat{\mathcal{E}}_b)$ (space of global $\widehat{\mathcal{F}}$-basic sections of $\widehat{\mathcal{E}}_b$) is exactly the space $\mathfrak{X}(W)$ of global vector fields on W. Finaly we have:

$$H^1_{\mathcal{F}}(M, \mathcal{V}) = H^1(M, \mathcal{E}_b) = H^1(\Gamma, \mathfrak{X}(W)). \tag{17}$$

2. Since $H^q(\widehat{L}) = 0$ for $q \geq 1$, we get from Proposition 4.5. $H^q(\widehat{M}, \widehat{\mathcal{E}}_b) = 0$ for $q \geq 1$. Then (like in Subsection 4.3. Formula (9)) the spectral sequence converges at the E_2 term and then:

$$H^p(M, \mathcal{E}_b) = E_2^{p0} = H^p(\Gamma, \mathfrak{X}(W)).$$

This ends the proof of the theorem □

Equality (17) reduces the computation of the space of infinitesimal deformations $H^1_{\mathcal{F}}(M, \mathcal{V})$ of the foliation $(M, \mathcal{F})$ to that of the first cohomology space of the discrete group Γ with values in the Γ-module $\mathfrak{X}(W)$. We will see explicit examples in section 6.

Using this result, Stephane Geudens has constructed recently in [Geu] a class of infinitesimally rigid non-Hausdorff Riemannian foliations. The only ones known so far are Riemannian with compact leaves and whose first cohomology space is trivial.

6. Examples and explicit computations

Some of them are well known and have already been studied under a continuous aspect (using vector fields and differential forms). We take them back by applying the different methods of cohomology of discrete groups we have just exposed.

6.1. Example 1

Let M be the torus $\mathbb{T}^{m+n} = \mathbb{R}^{m+n}/\mathbb{Z}^{m+n}$ where m and n are positive integers. We can view it as the product $\mathbb{T}^m \times \mathbb{T}^n$ and then as the quotient of $\widehat{M} = \mathbb{R}^m \times \mathbb{T}^n$ by the free and proper action of the group $\Gamma = \mathbb{Z}^m$:

$$(\mathbf{k}, (x, y)) \in \mathbb{Z}^m \times (\mathbb{R}^m \times \mathbb{T}^n) \longmapsto (x + \mathbf{k}, y) \in \mathbb{R}^m \times \mathbb{T}^n. \tag{18}$$

The canonical projection $\pi : \widehat{M} \longrightarrow M$ is a Γ-covering. Denote by $(e_1, \cdots, e_m)$ the canonical basis of $\mathbb{R}^m$, that is e_i (for $i = 1, \cdots, m$) has its i^{th} component equal to 1 and all the others equal to 0. This basis is also a system of generators of Γ.

Let $\rho : \Gamma \longrightarrow \mathrm{Diff}(\mathbb{T}^n)$ be the representation which associates to each generator e_i the translation on $\mathbb{T}^n$ by a vector $\alpha_i = (\alpha_i^1, \cdots, \alpha_i^n)$. The suspension of ρ gives rise to a foliation $\mathcal{F}$ on M whose associated lifted foliation $\widehat{\mathcal{F}}$ to $\widehat{M}$ is just the one defined by the second projection $\widehat{M} = \mathbb{R}^m \times \mathbb{T}^n \longrightarrow \mathbb{T}^n = W$. The action of Γ on W is given by the representation ρ.

The normal bundle $\mathcal{V}$ is identified to the tangent bundle of $W = \mathbb{T}^n$ which is trivial as a foliated bundle: any linear vector field Y on W is basic. So $\mathcal{V}$ is trivial as foliated vector bundle and also as a Γ-bundle. So:

$$H^1_{\mathcal{F}}(M, \mathcal{V}) = H^1(\Gamma, C^\infty(\mathbb{T}^n)) \otimes \mathbb{R}^n. \tag{19}$$

Then to determine $H^1_{\mathcal{F}}(M, \mathcal{V})$ it is sufficient to determine the space $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ which is the ordinary foliated cohomology $H^1_{\mathcal{F}}(M)$ of the manifold $(M, \mathcal{F})$.

Let us compute the vector space $H^1(\Gamma, C^\infty(\mathbb{T}^n))$. We shall do it by induction on the integer m. Suppose that $m = 1$, that is, Γ is reduced to its first factor $\Gamma_1 = \mathbb{Z}$. Then by formulas (6), $H^r(\Gamma_1, C^\infty(\mathbb{T}^n)) = 0$ for $r \geq 2$, $H^0(\Gamma_1, C^\infty(\mathbb{T}^n))$ is the subspace of $C^\infty(\mathbb{T}^n))$ consisting of functions invariant by τ_1 and $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ is the cokernel of the operator:

$$\delta : f \in C^\infty(\mathbb{T}^n) \longmapsto (f - f \circ \tau_1) \in C^\infty(\mathbb{T}^n)$$

where τ_1 is the translation by the vector $\alpha_1 = (\alpha_1^1, \cdots, \alpha_1^n)$. Then we have to solve the cohomological equation:

$$f(y) - f(y + \alpha_1) = g(y) \quad \text{for all } y \in \mathbb{T}^n \tag{20}$$

where g is given in $C^\infty(\mathbb{T}^n)$.

One can find the resolution of this equation in several works. We will quickly resume the one given in [EH] and focus on the case where the components of the vector $\alpha_1 = (\alpha_1^1, \cdots, \alpha_1^n)$ are linearly independent over $\mathbb{Q}$. We denote $\langle\ ,\ \rangle$ the usual scalar product in $\mathbb{R}^n$.

i) *We say that* α_1 *is* ***Diophantine*** *if there exist real numbers* $C > 0$ *and* $\eta > 0$ *such that* $|\langle \mathbf{m}, \alpha_1\rangle| \geq \frac{C}{|\mathbf{m}|^\eta}$ *for all* $\mathbf{m} \in \mathbb{Z}^n$ *different from* 0.

ii) *We say that* α_1 *is* ***Liouville*** *if there exists a real number* $C > 0$ *such that for any* $\eta > 0$, *there exists* $\mathbf{m}_\eta \in \mathbb{Z}^n$ *satisfying the inequality* $|\langle \mathbf{m}_\eta, \alpha_1\rangle| \leq \frac{C}{|\mathbf{m}_\eta|^\eta}$.

Any vector $\alpha_1 \in \mathbb{R}^n$ whose components $\alpha_1^1, \cdots, \alpha_1^n$ are algebraic numbers and linearly independent over $\mathbb{Q}$ is Diophantine (see [EH] for the proof).

Let $\mathcal{L} : C^\infty(\mathbb{T}^n) \longrightarrow \mathbb{C}$ be the functional defined by $\mathcal{L}(g) = \int_{\mathbb{T}^n} g(x)dx$. Its kernel $\mathcal{N}$ is a closed codimension one subspace of $C^\infty(\mathbb{T}^n)$. We have the following assertions:

iii) $H^0(\Gamma, C^\infty(\mathbb{T}^n))$ *consists of constant functions: it is isomorphic to* $\mathbb{R}$.

iv) *If the vector* α_1 *is Diophantine, then the image of the operator* δ *is equal to* $\mathcal{N}$. *In this case,* $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ *is a Hausdorff topological vector space of dimension* 1 *generated by the constant function equal to* 1.

v) *If the vector* α_1 *is Liouville then* $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ *is a non Hausdorff infinite dimensional topological vector space.*

Suppose now that $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_1 and Γ_2 are copies of $\mathbb{Z}$. By Künneth formula (see [Bro] page 120), the space $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ is:

$$H^0(\Gamma_1, C^\infty(\mathbb{T}^n)) \otimes H^1(\Gamma_2, C^\infty(\mathbb{T}^n)) \oplus H^1(\Gamma_1, C^\infty(\mathbb{T}^n)) \otimes H^0(\Gamma_2, C^\infty(\mathbb{T}^n)).$$

But $H^0(\Gamma_1, C^\infty(\mathbb{T}^n))$ and $H^0(\Gamma_2, C^\infty(\mathbb{T}^n))$ are isomorphic to $\mathbb{R}$; then:

$$H^1(\Gamma, C^\infty(\mathbb{T}^n)) = H^1(\Gamma_1, C^\infty(\mathbb{T}^n)) \oplus H^1(\Gamma_2, C^\infty(\mathbb{T}^n)).$$

So $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ is Hausdorff and isomorphic to $\mathbb{R}^2$ if both the vectors α_1 and α_2 are Diophantine and it is a non Hausdorff infinite dimensional topological vector space if one of these vector is Liouville.

A repeated application of Künneth's formula gives the following result for the case where the group Γ is $\mathbb{Z}^m$:

vi) *If all the vectors* $\alpha_1, \cdots, \alpha_m$ *are Diophantine,* $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ *is Hausdorff and isomorphic to* $\mathbb{R}^m$. Then:

$$H^1_{\mathcal{F}}(M, \mathcal{V}) = H^1(\Gamma, C^\infty(\mathbb{T}^n)) \otimes \mathbb{R}^n = \mathbb{R}^m \otimes \mathbb{R}^n. \tag{21}$$

In fact one can easily prove that:

$$H^q(\Gamma, C^\infty(\mathbb{T}^n)) = \mathbb{R}^{C^q_m} \tag{22}$$

for $q = 0, 1, \cdots, m$ where $C_m^q = \frac{m!}{q!(m-q)!}$.

vii) *If at least one of the vectors $\alpha_1, \cdots, \alpha_m$ is Liouville, $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ is a non Hausdorff infinite dimensional topological vector space. Therefore it is the same for the vector space $H^1_{\mathcal{F}}(M, \mathcal{V})$.*

6.2. Example 2

The construction of this example is given in [EN] where only the vector space $H^1_{\mathcal{F}}(M, \mathcal{V})$ was computed. We will take it back and calculate explicitly all spaces of its foliated cohomologies.

1. Let A be a square matrix with integer coefficients and determinant 1. We suppose that A is diagonalizable on the field of the real numbers and having all its eigenvalues $\mu_1, \ldots, \mu_m, \lambda_1, \ldots, \lambda_n$ such that: $0 < \mu_1 \ldots, \mu_m < 1 < \lambda_1, \ldots, \lambda_n$. Because the product of these positive numbers is 1, the integers m and n are all such that $m \geq 1$ and $n \geq 1$.

We can think of A as a diffeomorphism of $\mathbb{T}^{m+n}$. Let $u_1, \cdots, u_m, v_1, \cdots, v_n$ be the eigenvectors corresponding respectively to the eigenvalues $\mu_1, \ldots, \mu_m, \lambda_1, \ldots, \lambda_n$ and $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ be linear vector fields on $\mathbb{T}^{m+n}$ whose directions are given respectively by $u_1, \cdots, u_m, v_1, \cdots, v_n$. We have:

$$A_* X_i = \mu_i X_i, \quad A_* Y_j = \lambda_j Y_j \quad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n. \tag{23}$$

2. Denote by $\mathcal{F}_0$ the foliation on $\mathbb{T}^{m+n}$ defined by the vector fields $X_1, \ldots, X_m$. The product of $\mathcal{F}_0$ by $\mathbb{R}$ gives a codimension n foliation $\widehat{\mathcal{F}}$ on $\widehat{M} = \mathbb{T}^{m+n} \times \mathbb{R}$ which is invariant by the diffeomorphism ϕ of $\mathbb{T}^{m+n} \times \mathbb{R}$ sending (z, t) to $(A(z), t+1)$. So it induces a codimension n foliation $\mathcal{F}$ on the quotient manifold $\mathbb{T}_A^{m+n+1} = \mathbb{T}^{m+n} \times \mathbb{R}/\phi$. Notice that $\mathbb{T}_A^{m+n+1}$ is a flat bundle over the circle $\mathbb{S}^1$ with fibre $\mathbb{T}^{m+n}$. In fact the manifold $M = \mathbb{T}_A^{m+n+1}$ is the homogeneous space G/Γ where G is the semi-direct product $\mathbb{R}^{m+n} \rtimes \mathbb{R}$ given by the action:

$$(t, z) \in \mathbb{R} \times \mathbb{R}^{m+n} \longmapsto A^t z \in \mathbb{R}^{m+n}$$

and Γ is the discrete subgroup $\{(\mathbf{k}, s) \in G \mid \mathbf{k} \in \mathbb{Z}^{m+n}, s \in \mathbb{Z}\}$. The subgroup:

$$H = \left\{ \left(\sum_{i=1}^m a_i u_i, b \right) \in G \;\mid\; a_1, \ldots, a_m, b \in \mathbb{R} \right\}$$

is isomorphic to the semi-direct product $\mathbb{R}^m \rtimes \mathbb{R}_+^*$ where $\mathbb{R}_+^*$ acts on $\mathbb{R}^m$ by homothety in each eigendirection. The action of H on $\mathbb{T}_A^{m+n+1}$ induced by this identification is a locally free action whose orbits define the foliation $\mathcal{F}$.

3. From now on we shall assume that the matrix A fulfills the following condition.

(C) *The basis of $\mathbb{R}^{m+n}$ given by the eigenvectors $u_1, \ldots, u_m, v_1, \ldots, v_n$ is such that the coordinates $w^1, \ldots, w^{m+n}$ of any of its vectors w are linearly independent over $\mathbb{Q}$.*

Under this assumption any of the vectors $u_1, \dots, u_m$ fulfills the Diophantine condition (cf. [Sch]) we gave in the Example 5.1.

4. It is well known that if the eigenvalues of A are all different then condition (C) is fulfilled if and only if the characteristic polynomial $\chi_A(t)$ of A is irreducible over $\mathbb{Q}$. The following example shows the existence of matrices fulfilling the above properties.

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 0 & \dots & 0 \\ 1 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & d_{m+n} \end{pmatrix}$$

where the diagonal elements $d_1, \dots, d_{m+n}$ of A are inductively defined by $d_1 = 1$ and $d_{\ell+1} = 1 + d_1 \cdot d_2 \cdot \dots \cdot d_\ell$ for $\ell = 1, \dots, m+n-1$. In this case the matrix A has one eigenvalue ν_1 in the interval $]0,1[$ and exactly one eigenvalue $\nu_2, \dots, \nu_{m+n}$ in each of the intervals $]d_2, d_3[$, $]d_3, d_4[$, ..., $]d_{m+n}, \infty[$. In particular all the eigenvalues are different and there is only one eigenvalue smaller than 1. This implies that $\chi_A(t)$ is irreducible over $\mathbb{Z}$ and thus over $\mathbb{Q}$ because of the Gauss Lemma. Therefore the matrix A fulfills condition (C).

5. To compute the two cohomologies $H^*_{\mathcal{F}}(M)$ and $H^*_{\mathcal{F}}(M, \mathcal{V})$ we shall use the spectral sequence (8) of the foliated $\mathbb{Z}$-covering $\pi : (\widehat{M}, \widehat{\mathcal{F}}) \longrightarrow (M, \mathcal{F})$. (Recall that $\mathbb{Z}$ acts on $\widehat{M} = \mathbb{T}^{m+n} \times \mathbb{R}$ by its generator $\phi(z,t) = (A(z), t+1)$.) We will do the calculation just for $H^*_{\mathcal{F}}(M, \mathcal{V})$, it is practically the same for $H^*_{\mathcal{F}}(M)$. The spectral sequence E_r in our situation has its term E_2:

$$E_2^{pq} = H^p(\mathbb{Z}, H^q_{\widehat{\mathcal{F}}}(\widehat{M}, \widehat{\mathcal{V}})) \tag{24}$$

and converges to the cohomology $H^*_{\mathcal{F}}(M, \mathcal{V})$. (Here $\widehat{\mathcal{V}}$ denotes the normal bundle of the foliation $\widehat{\mathcal{F}}$.) Since the retraction of $\widehat{M} = \mathbb{T}^{m+n} \times \mathbb{R}$ along the second factor is an integrable homotopy from the foliated manifold $(\widehat{M}, \widehat{\mathcal{F}})$ to the foliated manifold $(\mathbb{T}^{m+n}, \mathcal{F}_0)$ (see point 2 of subection 4.5), we have:

$$H^q_{\widehat{\mathcal{F}}}(\widehat{M}, \widehat{\mathcal{V}}) = H^q_{\mathcal{F}_0}(\mathbb{T}^{m+n}, \mathcal{V}_0)$$

where $\mathcal{V}_0$ is the normal bundle of the foliation $\mathcal{F}_0$. Since all the vectors $u_1, \cdots, u_m$ are diophantine:

$$H^q_{\mathcal{F}_0}(\mathbb{T}^{m+n}, \mathcal{V}_0) = \mathbb{R}^{C^q_m} \otimes \mathbb{R}^n. \tag{25}$$

6. Now we will describe the action of the diffeomorphism ϕ of $\widehat{M}$ on these two spaces. For this let us take again the vector fields $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ satisfying the relations (23). They constitute a basis of the module $\mathfrak{X}(\mathbb{T}^{m+n})$ of tangent vector fields to $\mathbb{T}^{m+n}$; the first ones $X_1, \cdots, X_m$ are those who are tangent to the foliation $\mathcal{F}_0$. Let $\theta_1, \cdots, \theta_m$ be the 1-forms (with constant coefficients) such that, for $i, k = 1, \cdots, m$:

$$\theta_i(X_k) = \delta_i^k \quad \text{and} \quad \theta_i(Y_j) = 0 \quad \text{for} \quad j = 1, \cdots, n$$

where δ_i^k is the Kronecker symbol. The family $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_q} : 1 \leq i_1 < \cdots < i_q \leq m\}$ is a basis of $H^q_{\mathcal{F}_0}(\mathbb{T}^{m+n})$ for $q \geq 1$; $H^0_{\mathcal{F}_0}(\mathbb{T}^{m+n}) = \mathbb{R}$ is generated by the constant function equal to 1. Then one can take:

$$\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_q} \otimes Y_j : 1 \leq i_1 < \cdots < i_q \leq m \ \text{ and } \ j = 1, \cdots, n\}$$

as a basis of $H^q_{\mathcal{F}_0}(\mathbb{T}^{m+n}, \mathcal{V}_0)$. The action of the matrix A on this space is given on the elements of this basis by:

$$\begin{aligned} A^*(\theta_{i_1} \wedge \cdots \wedge \theta_{i_q} \otimes Y_j) &= (A^{-1})^*(\theta_{i_1}) \wedge \cdots \wedge (A^{-1})^*(\theta_{i_q}) \otimes A_*(Y_j) \\ &= \left(\mu_{i_1}^{-1} \cdots \mu_{i_q}^{-1}\right) \left(\theta_{i_1} \wedge \cdots \wedge \theta_{i_q}\right) \otimes (\lambda_j Y_j) \\ &= \left(\mu_{i_1} \cdots \mu_{i_q}\right)^{-1} \lambda_j \left(\theta_{i_1} \wedge \cdots \wedge \theta_{i_q} \otimes Y_j\right). \end{aligned}$$

Since $0 < \mu_1, \cdots, \mu_m < 1$ and $1 < \lambda_1, \cdots, \lambda_n$, all the products $(\mu_{i_1} \cdots \mu_{i_q})^{-1}\lambda_j$ are greater than 1; then no one of the elements $\theta_{i_1} \wedge \cdots \wedge \theta_{i_q} \otimes v_j$ is invariant under the action of A. Therefore, for any $q \in \{0, 1, \cdots, m\}$:

$$E_2^{0q} = H^0(\mathbb{Z}, H^q_{\mathcal{F}_0}(\mathbb{T}^{m+n}, \mathcal{V}_0)) = 0.$$

7. Recall that if V is a vector space on which $\mathbb{Z}$ acts freely $H^p(\mathbb{Z}, V) = 0$ for $p \geq 2$. Applying this to our situation, we get, for $p \geq 2$:

$$E_2^{pq} = H^p(\mathbb{Z}, H^q_{\mathcal{F}_0}(\mathbb{T}^{m+n}, \mathcal{V}_0)) = 0.$$

Then the second differential $d_2 : E_2^{pq} \longrightarrow E_2^{p+2,q-1}$ is zero. This implies the convergence of the spectral sequence at the E_2 term, and then $H^r_{\mathcal{F}}(M, \mathcal{V}) = E_2^{0r} \oplus E_2^{1,r-1}$. But $E_2^{0r} = 0$, so:

$$H^r_{\mathcal{F}}(M, \mathcal{V}) = E_2^{1,r-1} = H^1(\mathbb{Z}, H^{r-1}_{\mathcal{F}_0}(\mathbb{T}^{m+n}, \mathcal{V}_0)) = H^1(\mathbb{Z}, \mathbb{R}^d \otimes \mathbb{R}^n) \tag{26}$$

where $d = C_m^{r-1} = \frac{n!}{(r-1)!(m-r+1)!}$. Because the action of $\mathbb{Z}$ is without fixed vector on the space $\mathbb{R}^d \otimes \mathbb{R}^n$ (which is finite dimensional), $H^1(\mathbb{Z}, \mathbb{R}^d \otimes \mathbb{R}^n)$ is trivial and then so is $H^r_{\mathcal{F}}(M, \mathcal{V})$ for any $r = 0, 1, \cdots, m$.

8. We just have showed that $H^r_{\mathcal{F}}(M, \mathcal{V}) = 0$ for any $r = 0, 1, \cdots, m$, in particular $H^1_{\mathcal{F}}(M, \mathcal{V}) = 0$. The foliation $\mathcal{F}$ is then infinitesimally stable. This result is not new: the C^∞-stability (which is stronger) was already proved in [EN].

9. In a similar way one can prove that the two spaces $H^0_{\mathcal{F}}(M)$ and $H^1_{\mathcal{F}}(M)$ are equal to $\mathbb{R}$ and that $H^r_{\mathcal{F}}(M) = 0$ for $r \geq 2$.

6.3. Example 3

Let M be a compact manifold equipped with a Lie G-foliation $\mathcal{F}$. Let $\pi : \widehat{M} \longrightarrow M$ be the corresponding Γ-covering and $D : \widehat{M} \longrightarrow G$ its developing map. As it was mentioned in the subsection 3.3, there exists an injective representation $\rho : \Gamma \hookrightarrow G$ such that, for any $\gamma \in \Gamma$, the following diagram is commutative:

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\gamma} & \widehat{M} \\ D\downarrow & & \downarrow D \\ G & \xrightarrow{\rho(\gamma)} & G. \end{array}$$

As a subgroup of G, Γ acts on it by left translations. If $H^1(\widehat{L}) = 0$, by Theorem 5.2, the space $H^1_{\mathcal{F}}(M, \mathcal{V})$ of infinitesimal deformations of $\mathcal{F}$ is identified to the space $H^1(\Gamma, \mathfrak{X}(G))$. But $\mathfrak{X}(G) = C^\infty(G) \otimes \mathcal{G}$ where $\mathcal{G}$ is the Lie algebra of the group G. Because the action of Γ on $\mathcal{G}$ is trivial, we immediately have:

$$H^1_{\mathcal{F}}(M, \mathcal{V}) = H^1(\Gamma, C^\infty(G)) \otimes \mathcal{G} \tag{27}$$

where the action of Γ on G is by left translations and by the induced usual one on the Fréchet space $C^\infty(G)$.

The following problem is of interest independently of that which it could have in deformation theory of Lie foliations.

Problem. *Let G be a connected Lie group and Γ a countable subgroup. Compute the cohomology $H^*(\Gamma, C^\infty(G))$.*

For Γ finite or discrete, we have (see [Ek2]) $H^r(\Gamma, C^\infty(G)) = 0$ for $r \geq 1$. The case $\Gamma \simeq \mathbb{Z}$ was studied in [EH]. The one where G is a torus and $\Gamma \simeq \mathbb{Z}^m$ is more or less our example 6.1.

When Γ is infinite and not closed, the quotient $Q = G/\Gamma$ is a Q-manifold in the sense of [Bar]. Then one can ask: *What relationship is there between $H^*(\Gamma, C^\infty(G))$ and the cohomology of Q-manifolds defined in* [Bar]?

In the general case of a transversely homogeneous G/H-foliation $\mathcal{F}$, with the hypothesis $H^1(\widehat{L}) = 0$, the vector space $H^1_{\mathcal{F}}(M, \mathcal{V})$ is $H^1(\Gamma, \mathfrak{X}(G/H))$ where the action of Γ on G/H is induced by its left action on G.

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