

FOLIATED COHOMOLOGY AND INFINITESIMAL DEFORMATIONS OF DEVELOPABLE FOLIATIONS

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Abstract. A foliation (M, \mathcal{F}) is said to be *developable* if its pull-back $(\widehat{M}, \widehat{\mathcal{F}})$ to some covering $\pi : \widehat{M} \rightarrow M$ associated to a subgroup $\Gamma \subset \pi_1(M)$ is a fibration $D : \widehat{M} \rightarrow W$ on which Γ acts by automorphisms of $\widehat{\mathcal{F}}$ (and then on the basis W). Let $E \rightarrow M$ be a foliated vector bundle and $\widehat{E} \rightarrow \widehat{M}$ its pull-back to \widehat{M} by π . We suppose that $\widehat{E} = D^*(\overline{E})$ where \overline{E} is a vector bundle on W and on which Γ acts.

We prove: i) if the fibre \widehat{L} of $\widehat{\mathcal{F}}$ has trivial cohomology – it is the case for instance if \widehat{L} is contractible – the foliated cohomology $H_{\mathcal{F}}^*(M, E)$ of (M, \mathcal{F}) with values in E is isomorphic to the cohomology $H^*(\Gamma, C^\infty(\overline{E}))$ of the discrete group Γ with values in the space $C^\infty(\overline{E})$ of C^∞ -sections of \overline{E} ; ii) if $H^1(\widehat{L}) = 0$, the space $H_{\mathcal{F}}^*(M, \mathcal{V})$ (where $\mathcal{V} = TM/T\mathcal{F}$ is the normal bundle of \mathcal{F}) of infinitesimal deformations of (M, \mathcal{F}) is naturally identified to the space $H^1(\Gamma, \mathfrak{X}(W))$ where $\mathfrak{X}(W)$ is the Γ -module of smooth vector fields on W . Some examples and explicit computations are given.

0. Introduction

The study of deformations of foliations uses various mathematical tools, for instance that of differential geometry, global analysis, algebraic topology... They fall within the more general framework of deformation theory of geometric structures which was initiated by the works of Kodaira and Spencer [KS] in the 50's on the variation of complex structures. Even if there is still to be done, Kuranishi [Kur] has finalized the subject by showing the existence of a *versal space* through which transits any deformation of a given complex structure on a compact manifold. This result was a great advance in the development of complex geometry. Adapting these methods, Girbau, Haefliger and Sundararaman [GHS] established, in a similar way, the existence of a versal space for the deformations of transversely holomorphic foliations.

Unfortunately, the tools used in the complex case were not completely available for studying the deformations of real foliations. Moreover, at that time, the theory really lacked non-trivial examples. The first one which is differentiably stable and with an interesting dynamic (for instance all its leaves are dense) was given in [GS] by Ghys and Sergiescu in 1980. Their constructions and proofs are based on qualitative properties of codimension one foliations on 3-manifolds.

More or less around the same time, Hamilton gave a strong criterion for deciding whether a foliation on a compact manifold is differentiably stable. Unfortunately the

paper [Ham] containing this result (always highly requested by specialists in these matters) has never been published, probably by voluntary decision of the author who had as examples to which he could apply its criterion only Riemannian foliations with compact leaves. Later on, using this criterion, in [EN] the authors constructed a class of C^∞ -stable foliations (of arbitrary dimension and codimension) which includes the example of Ghys and Sergiescu.

A powerful tool in the study of the deformations of a foliation (M, \mathcal{F}) and which appears to be an essential ingredient in the Hamilton criterion is the *foliated cohomology* $H_{\mathcal{F}}^*(M, \mathcal{V})$ with values in the normal bundle \mathcal{V} of \mathcal{F} . This is the reason why it has since been the subject of works by many authors. In this paper, we explicitly describe the foliated cohomology with values in a foliated vector bundle for *developable foliations* and relate it, by spectral sequences, to the cohomology of a discrete group with coefficients in the Fréchet space of differentiable sections of a certain vector bundle over a manifold naturally associated to the foliation. This makes possible to transpose its calculation, usually carried out by differential forms, to that using different methods and which sometimes prove to be more effective. In particular, this gives a way to compute the space $H_{\mathcal{F}}^1(M, \mathcal{V})$ which contains the infinitesimal deformations of \mathcal{F} . Some explicit examples are given showing concretely the interest of this point of view.

Of course, the category of developable foliations is particular but it includes enough examples, for instance foliations obtained by suspension of groups of diffeomorphisms and transversely homogeneous foliations. For these latter foliations, a fairly specific study of their deformations accompanied by those of their transverse homogeneous structures was carried out in [EGN].

1. Foliations

Let M be a (connected) manifold of dimension $m + n$. A *foliation of dimension m or codimension n* on M is a geometric structure such that around each point one can cut a small piece (that is, an open neighborhood) which looks like the product $\mathbb{R}^m \times \mathbb{R}^n$ where the second factor is equipped with the discrete topology. More precisely we have the following:

1.1. Definition. *Let M be a manifold of dimension $m+n$. A codimension n **foliation** \mathcal{F} on M is given by an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ and for each i , a diffeomorphism $\varphi_i : \mathbb{R}^{m+n} \rightarrow U_i$ such that, on each non empty intersection $U_i \cap U_j$, the coordinate change:*

$$(1) \quad \varphi_j^{-1} \circ \varphi_i : (x, y) \in \varphi_i^{-1}(U_i \cap U_j) \longrightarrow (x', y') \in \varphi_j^{-1}(U_i \cap U_j)$$

has the form $x' = \varphi_{ij}(x, y)$ and $y' = \gamma_{ij}(y)$.

The manifold M is decomposed into connected submanifolds of dimension m . Each of them is called a *leaf* of \mathcal{F} . A subset U of M is *saturated* for \mathcal{F} if it is a union of leaves: if $x \in U$ then the leaf passing through x is contained in U .

Coordinate patches (U_i, φ_i) satisfying conditions of definition 1.1 are said to be *distinguished* for the foliation \mathcal{F} .

The following (second) definition is more appropriate for introducing the notion of transverse structure, which is crucial in the theory of foliations.

Let \mathcal{F} be a codimension n foliation on M defined by a maximal atlas $\{(U_i, \varphi_i)\}_{i \in I}$ like in definition 1.1. Let $\pi : \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the second projection. Then the map $f_i : U_i \longrightarrow \mathbb{R}^n$ (where $f_i = \pi \circ \varphi_i^{-1}$) is a submersion. On $U_i \cap U_j \neq \emptyset$ we have $f_j = \gamma_{ij} \circ f_i$. The fibres of the submersion f_i are the \mathcal{F} -plaques of U_i . The submersions f_i and the local diffeomorphisms γ_{ij} of \mathbb{R}^n give a complete characterization of \mathcal{F} .

1.2. Definition. A codimension n **foliation** on M is given by an open cover $(U_i)_{i \in I}$, submersions $f_i : U_i \longrightarrow T$ over a n dimensional transverse manifold T and, for any non empty intersection $U_i \cap U_j$, a diffeomorphism: $\gamma_{ij} : f_i(U_i \cap U_j) \subset T \longrightarrow f_j(U_i \cap U_j) \subset T$ satisfying $f_j(z) = \gamma_{ij} \circ f_i(z)$ for $z \in U_i \cap U_j$. We say that $\{U_i, f_i, T, \gamma_{ij}\}$ is a **foliated cocycle** defining \mathcal{F} .

The foliation \mathcal{F} is said to be *transversely orientable* if T can be given an orientation preserved by all the local diffeomorphisms γ_{ij} .

1.3. Morphisms of foliations

Let M and M' be two manifolds endowed respectively with foliations \mathcal{F} and \mathcal{F}' . A map $f : M \longrightarrow M'$ will be called *foliated* or a *morphism* between \mathcal{F} and \mathcal{F}' if, for every leaf L of \mathcal{F} , $f(L)$ is contained in a leaf of \mathcal{F}' ; we say that f is an *isomorphism* if, in addition, f is a diffeomorphism; in this case the restriction of f to any leaf $L \in \mathcal{F}$ is a diffeomorphism on the leaf $L' = f(L) \in \mathcal{F}'$.

Suppose now that f is a diffeomorphism of M . Then for every leaf $L \in \mathcal{F}$, $f(L)$ is a leaf of a codimension n foliation \mathcal{F}' on M ; we say that \mathcal{F}' is the *image* of \mathcal{F} by the diffeomorphism f and we write $\mathcal{F}' = f^*(\mathcal{F})$. Two foliations \mathcal{F} and \mathcal{F}' on M are said to be C^r -conjugated (*topologically* if $r = 0$, *differentially* if $r = \infty$ and *analytically* in the case $r = \omega$) if there exists a C^r -homeomorphism $f : M \longrightarrow M$ such that $f^*(\mathcal{F}') = \mathcal{F}$.

The set of C^r -diffeomorphisms of M which preserve the foliation \mathcal{F} is a group denoted $\text{Diff}^r(M, \mathcal{F})$.

The following definition introduces a notion which is very important for a foliation. It describes a large part of its geometric structure.

1.4. Definition. Let M be a manifold and \mathcal{F} a codimension n foliation on M defined by a foliated cocycle $\{U_i, f_i, T, \gamma_{ij}\}$. A **transverse structure** to \mathcal{F} is a geometric structure on T invariant by the local diffeomorphisms γ_{ij} .

Examples of such structures will be given later.

2. Basic elements

Let us fix some notations. Let \mathcal{F} be a codimension n foliation on M . We denote by $T\mathcal{F}$ the tangent bundle to \mathcal{F} and \mathcal{V} the quotient $TM/T\mathcal{F}$ which is its *normal bundle*.

2.1. Basic forms and basic vector fields

We denote by $\mathfrak{X}(\mathcal{F})$ the space of sections of $T\mathcal{F}$ (elements of $\mathfrak{X}(\mathcal{F})$ are vector fields $X \in \mathfrak{X}(M)$ tangent to \mathcal{F}).

A differential form $\alpha \in \Omega^r(M)$ is said to be *basic* if it satisfies $i_X\alpha = 0$ and $L_X\alpha = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$. (Here i_X and L_X denote respectively the inner product and the Lie derivative with respect to the vector field X .) For a function $f : M \rightarrow \mathbb{R}$, these conditions are equivalent to $X \cdot f = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$ *i.e.* f is constant on the leaves of \mathcal{F} ; we denote by $\Omega_b^r(M)$ the space of basic forms of degree r on the foliated manifold (M, \mathcal{F}) ; this is a module over the algebra $C_b^\infty(M)$ of basic functions.

A vector field $Y \in \mathfrak{X}(M)$ is said to be *foliated*, if for every $X \in \mathfrak{X}(\mathcal{F})$, the bracket $[X, Y] \in \mathfrak{X}(\mathcal{F})$. We see easily that the set $\mathfrak{X}(M, \mathcal{F})$ of foliated vector fields is a Lie algebra in which $\mathfrak{X}(\mathcal{F})$ is an ideal. The quotient $\mathfrak{X}_b(M) = \mathfrak{X}(M, \mathcal{F})/\mathfrak{X}(\mathcal{F})$ is called the Lie algebra of *basic* (or *transverse*) vector fields on the foliated manifold (M, \mathcal{F}) . Also, it has a module structure over the algebra $C_b^\infty(M)$.

2.2. Foliated vector bundles and basic sections

Let $\tau : E \rightarrow M$ be a vector bundle of rank N defined by a cocycle $\{U_i, g_{ij}\}$ where $\{U_i\}$ is an open cover of M and the g_{ij} are the (continuous) transition functions $g_{ij} : U_i \cap U_j \rightarrow \text{GL}(N, \mathbb{R})$ satisfying the cocycle condition:

$$(2) \quad g_{ij}(z) = g_{ik}(z) \cdot g_{kj}(z) \quad \text{for } z \in U_i \cap U_j \cap U_k.$$

We say that E is *foliated* if the functions g_{ij} are basic on $U_i \cap U_j$. For such vector bundle, the foliation \mathcal{F} can be lifted to a same dimension foliation \mathcal{F}_E on E such that the projection τ sends leaves of \mathcal{F}_E on leaves of \mathcal{F} .

Let (M, \mathcal{F}) and (M', \mathcal{F}') be two foliations and $\tau : E \rightarrow M$ and $\tau' : E' \rightarrow M'$ foliated vector bundles. A morphism of vector bundles $E \rightarrow E'$ is a *foliated morphism* if it sends leaves of \mathcal{F}_E into leaves of $\mathcal{F}'_{E'}$. Foliated vector bundles and their morphisms form a category.

Recall that a section of E is a C^∞ -map $\alpha : M \rightarrow E$ such that $\tau \circ \alpha = id_M$. If $(U, (x_1, \dots, x_m, y_1, \dots, y_n))$ is a distinguished coordinates system on which E is trivial, α is represented by a C^∞ -function $\alpha : U \rightarrow \mathbb{R}^N$. We say that α is *basic*, if the function α_U is basic, that is, it depends only on the transverse coordinates (y_1, \dots, y_n) . The space $C_b^\infty(E)$ of basic sections of E is a module over the algebra $A = C_b^\infty(M)$ of basic functions. The pull-back of a basic section of E by a foliated morphism $E \rightarrow E'$ is a basic section of E' .

For a foliated vector bundle E we denote by \mathcal{E}_b the sheaf of germs of its basic sections. In general, \mathcal{E}_b is not fine (see subsection 4.2) and then it gives rise to a non trivial cohomology $H^*(M, \mathcal{E}_b)$. We shall see in section 4 how to give a fine resolution of this sheaf and compute $H^*(M, \mathcal{E}_b)$ by using special differential forms on M .

3. Developable foliations

3.1. Foliated covering

1. Let \widehat{M} be a connected manifold of dimension $m+n$ endowed with a codimension n foliation $\widehat{\mathcal{F}}$. Suppose that a discrete group Γ acts on \widehat{M} by diffeomorphisms preserving $\widehat{\mathcal{F}}$, that is, there exists an injective representation $\Gamma \rightarrow \text{Diff}(\widehat{M}, \widehat{\mathcal{F}})$.

If this action is free and proper, the quotient $M = \widehat{M}/\Gamma$ is a manifold of dimension $m + n$ and the canonical projection $\pi : \widehat{M} \rightarrow M$ is a covering. Moreover, because elements of Γ preserve $\widehat{\mathcal{F}}$, this foliation induces on M a codimension n foliation \mathcal{F} such that the projection π is a morphism of foliations. We say that $\pi : (\widehat{M}, \widehat{\mathcal{F}}) \rightarrow (M, \mathcal{F})$ is a *foliated covering*.

2. Conversely, let M be a connected manifold of dimension $m + n$ equipped with a codimension n foliation \mathcal{F} . For any Γ -covering $\widehat{M} \xrightarrow{\pi} M$, there exists on \widehat{M} a codimension n foliation $\widehat{\mathcal{F}}$ such that the covering projection $\pi : (\widehat{M}, \widehat{\mathcal{F}}) \rightarrow (M, \mathcal{F})$ is a morphism of foliation. We say that the foliation $\widehat{\mathcal{F}}$ is the pull-back by π of \mathcal{F} .

This enables us to give the definition of the category of foliations we will be interested on in all this paper.

3.2. Definition. A codimension n foliation \mathcal{F} on a connected manifold M is said **developable** if there exist a normal subgroup Γ of $\pi_1(M)$ and a Γ -covering $\widehat{M} \xrightarrow{\pi} M$ such that the leaves of the pull-back $\widehat{\mathcal{F}}$ of \mathcal{F} are the fibres of a locally trivial fibration $D : \widehat{M} \rightarrow W$. This fibration is called the **developing map** of \mathcal{F} .

The group Γ acts on \widehat{M} by automorphisms of $\widehat{\mathcal{F}}$ and then by diffeomorphisms on W . For any such automorphism $\gamma \in \Gamma$, we denote by $\overline{\gamma}$ the induced diffeomorphism on W ; we have a commutative diagram:

$$(3) \quad \begin{array}{ccc} \widehat{M} & \xrightarrow{\gamma} & \widehat{M} \\ D \downarrow & & \downarrow D \\ W & \xrightarrow{\overline{\gamma}} & W. \end{array}$$

From now on, the examples we will give in this section are all developable foliations.

3.3. Transversely homogeneous foliations

Let M be a manifold of dimension $m + n$ endowed with a codimension n foliation \mathcal{F} defined by a foliated cocycle $\{U_i, f_i, T, \gamma_{ij}\}$. We say that \mathcal{F} is *transversely homogeneous* if T is a homogeneous space G/H and the diffeomorphisms γ_{ij} are induced by restriction of left translations on the Lie group G . (Here G is a connected Lie group and H is a closed subgroup.) Also we say that \mathcal{F} is a *G/H -foliation*.

The structure of such foliation on a compact manifold is given by the following theorem due to R. Blumenthal [Blu].

Let \mathcal{F} be a G/H -foliation on a compact manifold M . Then, there exist a Γ -covering $\widehat{M} \rightarrow M$, an injective homomorphism $h : \Gamma \rightarrow G$ and a locally trivial fibration $D : \widehat{M} \rightarrow G/H$ whose fibres are the leaves of the lifted foliation $\widehat{\mathcal{F}}$ of \mathcal{F} to \widehat{M} and such that, for every $\gamma \in \Gamma$, the following diagram is commutative:

$$(4) \quad \begin{array}{ccc} \widehat{M} & \xrightarrow{\gamma} & \widehat{M} \\ D \downarrow & & \downarrow D \\ G/H & \xrightarrow{h(\gamma)} & G/H. \end{array}$$

Here the first line denotes the deck transformation of $\gamma \in \Gamma$ on \widehat{M} and $h(\gamma)$ is the diffeomorphism of G/H induced by the left translation by γ on G .

The subgroup $h(\Gamma)$ of the Lie group G which we will denote also Γ , is called the *holonomy group* of \mathcal{F}

The case where the subgroup H is trivial corresponds to a Lie G -*foliation*. The structure of such foliation is given by a result of Fédida [Féd] for which the theorem of Blumenthal is just an immediate generalization to transversely homogeneous foliations.

A particular case is given by a *homogeneous foliation* which can be defined as follows. Let G be a connected Lie group of dimension $m + n$ and H a closed subgroup of dimension m . Then the right action of H on G defines a foliation $\widehat{\mathcal{F}}$ of dimension m : its leaves are the orbits and also the fibres of the principal bundle $H \hookrightarrow G \xrightarrow{D} W$ where W is the homogeneous space G/H . Now, let Γ a cocompact lattice of G . Since the left action of Γ and the right action of H commute, the action of H induces a locally free action on the quotient $M = \Gamma \backslash G$ which defines a foliation \mathcal{F} . (Locally free means that the isotropy subgroups are discrete.)

The foliation \mathcal{F} on M is developable. Here $\widehat{M} = G$ and the developing map is just the locally trivial fibration $D : \widehat{M} = G \rightarrow W$. The pull-back of \mathcal{F} by the Γ -covering projection $\pi : G \rightarrow M$ is exactly the foliation $\widehat{\mathcal{F}}$.

3.4. Suspension of a group of diffeomorphisms

Let L and W be two manifolds, respectively of dimensions m and n . Suppose that the fundamental group $\pi_1(L)$ of L is finitely generated. Let $\rho : \pi_1(L) \rightarrow \text{Diff}(W)$ be a representation, where $\text{Diff}(W)$ is the diffeomorphism group of W . Denote by \widehat{L} the universal covering of L and $\widehat{\mathcal{F}}$ the horizontal foliation on $\widehat{M} = \widehat{L} \times W$, i.e., the foliation whose leaves are the subsets $\widehat{L} \times \{y\}$, $y \in W$. This foliation is invariant by all the transformations $T_\gamma : \widehat{M} \rightarrow \widehat{M}$ defined by $T_\gamma(\widehat{x}, y) = (\gamma \cdot \widehat{x}, \rho(\gamma)(y))$ where $\gamma \cdot \widehat{x}$ is the natural action of $\gamma \in \pi_1(L)$ on \widehat{L} ; then $\widehat{\mathcal{F}}$ induces a codimension n foliation \mathcal{F}_ρ on the quotient manifold: $M = \widehat{M}/(\widehat{x}, y) \sim (\gamma \cdot \widehat{x}, \rho(\gamma)(y))$. We say that \mathcal{F}_ρ is the *suspension* of the diffeomorphism group $\Gamma = \rho(\pi_1(L))$. The leaves of \mathcal{F}_ρ are transverse to the fibres of the natural fibration induced by the projection on the first factor $\widehat{L} \times W \rightarrow \widehat{L}$.

Conversely, suppose that $W \rightarrow M \xrightarrow{\pi} L$ is a fibration with compact fibre W and that M is equipped with a codimension n foliation \mathcal{F} ($n = \text{dimension of } W$) transverse to the fibres of π . Then there exists a representation $\rho : \pi_1(L) \rightarrow \text{Diff}(W)$ such that $\mathcal{F} = \mathcal{F}_\rho$.

The geometric transverse structures of the foliation \mathcal{F} are exactly the geometric structures on the manifold W invariant by the action of Γ .

4. Cohomologies

4.1. Cohomology of a discrete group

Let Γ be a (countable) discrete group and V a vector space on which Γ acts by automorphisms; this makes V a Γ -*module*.

1. For each integer $r \geq 1$, let $C^r(\Gamma, V)$ be the vector space of maps from Γ^r to V ; an element of $C^r(\Gamma, V)$ is called a r -cochain on Γ with values in V . By convention $C^0(\Gamma, V) = V$. Define the linear map $d : C^r(\Gamma, V) \longrightarrow C^{r+1}(\Gamma, V)$ by:

$$(5) \quad \begin{aligned} dc(\gamma_1, \dots, \gamma_{r+1}) &= \gamma_1 \cdot c(\gamma_2, \dots, \gamma_{r+1}) \\ &+ \sum_{i=1}^r (-1)^i c(\gamma_1, \dots, \gamma_{i-1}, \gamma_i \gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{r+1}) \\ &+ (-1)^{r+1} c(\gamma_1, \dots, \gamma_r). \end{aligned}$$

An element of the kernel $Z^r(\Gamma, V)$ of $d : C^r(\Gamma, V) \longrightarrow C^{r+1}(\Gamma, V)$ is called a *cocycle* and an element of the image $B^r(\Gamma, V)$ of $d : C^{r-1}(\Gamma, V) \longrightarrow C^r(\Gamma, V)$ is called a *coboundary*. The operator d satisfies $d^2 = 0$ and then $B^r(\Gamma, V)$ is a subspace of $Z^r(\Gamma, V)$. The quotients $H^r(\Gamma, V) = Z^r(\Gamma, V)/B^r(\Gamma, V)$ for $r \in \mathbb{N}$ are called the *cohomology spaces* of Γ with values in the Γ -module V .

2. An element c of $C^0(\Gamma, V)$ is just a vector in V and $dc(\gamma) = \gamma \cdot c - c$. So $H^0(\Gamma, V)$ is the subspace V^Γ of elements of V which are invariant under the action of Γ .

3. If $\Gamma = \mathbb{Z}$ and its action is generated by an automorphism γ of V , an easy computation shows that:

$$(6) \quad H^r(\Gamma, V) = \begin{cases} V^\Gamma & \text{if } r = 0 \\ V/\langle v - \gamma \cdot v \rangle & \text{if } r = 1 \\ 0 & \text{if } r \geq 2 \end{cases}$$

where $\langle v - \gamma \cdot v \rangle$ is the subspace of V generated by elements of the form $v - \gamma \cdot v$ with v varying in V . If, for instance, V is finite dimensional and γ does not fix any vector, the linear map $v \in V \mapsto v - \gamma \cdot v \in V$ is an isomorphism and then $H^0(\Gamma, V) = H^1(\Gamma, V) = 0$.

4.2. Sheaf cohomology

Let \mathcal{E} be a sheaf of vector spaces on M and $\mathcal{U} = \{U_i\}_{i \in I}$ a locally finite open cover of M . For any multi-index (i_0, \dots, i_q) in I , we denote $U_{i_0 \dots i_q}$ the intersection $U_{i_0} \cap \dots \cap U_{i_q}$ and Σ_q the set of multi-indices for which $U_{i_0 \dots i_q} \neq \emptyset$. Let $C^q(\mathcal{U}, \mathcal{E})$ be the set of all collections $(f_{i_0 \dots i_q})_{(i_0, \dots, i_q) \in \Sigma_q}$ where $f_{i_0 \dots i_q}$ is an element of $\mathcal{E}(U_{i_0 \dots i_q})$ (space of sections of \mathcal{E} over $U_{i_0 \dots i_q}$); it is a \mathbb{K} -vector space. An element f of $C^q(\mathcal{U}, \mathcal{E})$ is called q -cochain on \mathcal{U} with values in \mathcal{E} . Define the operator $\delta : C^q(\mathcal{U}, \mathcal{E}) \longrightarrow C^{q+1}(\mathcal{U}, \mathcal{E})$ by:

$$(7) \quad (\delta f)_{i_0 \dots i_{q+1}} = \sum_{j=0}^{q+1} (-1)^j f_{i_0 \dots \hat{i}_j \dots i_{q+1}}$$

where $f_{i_0 \dots \hat{i}_j \dots i_{q+1}}$ is the section $f_{i_0 \dots \hat{i}_j \dots i_{q+1}}$ restricted to the open set $U_{i_0 \dots i_{q+1}}$. For instance, for a 0-cochain $f = (f_i)$, the 1-cochain $\delta f = (f_{ij})$ is given by $f_{ij} = f_j - f_i$; if $q = 1$ and $f = (f_{ij})$, then $\delta f = (f_{ijk})$ with $f_{ijk} = f_{jk} - f_{ik} + f_{ij}$. One can verify that:

$$\delta \circ \delta : C^{q-1}(\mathcal{U}, \mathcal{E}) \longrightarrow C^q(\mathcal{U}, \mathcal{E}) \longrightarrow C^{q+1}(\mathcal{U}, \mathcal{E})$$

is zero. Then the kernel $Z^q(\mathcal{U}, \mathcal{E})$ of $\delta : C^q(\mathcal{U}, \mathcal{E}) \rightarrow C^{q+1}(\mathcal{U}, \mathcal{E})$ contains the image $B^q(\mathcal{U}, \mathcal{E})$ of $\delta : C^{q-1}(\mathcal{U}, \mathcal{E}) \rightarrow C^q(\mathcal{U}, \mathcal{E})$; the quotient:

$$H^q(\mathcal{U}, \mathcal{E}) = Z^q(\mathcal{U}, \mathcal{E})/B^q(\mathcal{U}, \mathcal{E})$$

is the q^{th} cohomology space of the cover \mathcal{U} with coefficients in the sheaf \mathcal{E} .

If $\mathcal{U}' = \{U'_j\}_{j \in J}$ is an open cover finer than \mathcal{U} , that is, for $j \in J$, there exists $i \in I$ such that $U'_j \subset U_i$, we have a restriction morphism $\rho_q : C^q(\mathcal{U}, \mathcal{E}) \rightarrow C^{q+1}(\mathcal{U}', \mathcal{E})$ which induces a morphism $\rho_q^* : H^q(\mathcal{U}, \mathcal{E}) \rightarrow H^q(\mathcal{U}', \mathcal{E})$. The inductive limit of the system $\{\rho_q^* : H^q(\mathcal{U}, \mathcal{E}) \rightarrow H^q(\mathcal{U}', \mathcal{E})\}_{\mathcal{U}' \prec \mathcal{U}}$ is a vector space denoted $H^q(M, \mathcal{E})$ and called the q^{th} cohomology space of M with coefficients in the sheaf \mathcal{E} .

We have the following properties.

1. $H^0(M, \mathcal{E})$ is the space $\mathcal{E}(M)$ of global sections of \mathcal{E} .
2. Let M and M' be two manifolds, $f : M \rightarrow M'$ a continuous map, \mathcal{E} a sheaf on M and \mathcal{E}' its direct image by f . Then, for any integer $q \geq 0$, f induces a morphism $f^* : H^q(M', \mathcal{E}') \rightarrow H^q(M, \mathcal{E})$. If M'' is an other manifold, $g : M' \rightarrow M''$ a continuous map and \mathcal{E}'' the direct image of \mathcal{E}' by g , then $(g \circ f)^* = f^* \circ g^*$. Furthermore if $M = M'$ and f is the identity map, then f^* is the identity of $H^q(M, \mathcal{E})$.

3. We say that a sheaf \mathcal{E} is *fine* if, for any open locally finite cover $\mathcal{U} = \{U_i\}$ there are endomorphisms $h_i : \mathcal{E} \rightarrow \mathcal{E}$ such that support of h_i is a subset of U_i and $\sum_{i \in I} h_i$ is the identity of \mathcal{E} . (Recall that a *support* of a morphism $h : \mathcal{E} \rightarrow \mathcal{E}$ is the set $\text{supp}(h) = \overline{\{x \in M : h_i(\mathcal{E}_x) \neq 0\}}$ where E_x is the fibre of \mathcal{E} at x .) For instance, the sheaf \mathcal{C}^k of germs of functions of class C^k ($k \in \mathbb{N} \cup \{\infty\}$) is fine.

We have the following assertion (see [God] for the proof): *Suppose \mathcal{E} is a fine sheaf. Then $H^r(M, \mathcal{E}) = 0$ for $r \geq 1$.*

4.3. Spectral sequence of a covering

Let Γ be a countable discrete group acting freely and properly by diffeomorphisms on a manifold \widehat{M} . Then the quotient $M = \widehat{M}/\Gamma$ is a manifold and the canonical projection $\pi : \widehat{M} \rightarrow M$ is a covering. Let \mathcal{E} be a sheaf of vector spaces over M and $\pi^*(\mathcal{E}) = \widehat{\mathcal{E}}$ its pull-back by π on \widehat{M} . Then there exists a spectral sequence E_r whose E_2 term is:

$$(8) \quad E_2^{k\ell} = H^k(\Gamma, H^\ell(\widehat{M}, \widehat{\mathcal{E}}))$$

and converging to $H^*(M, \mathcal{E})$. The vector spaces $H^\ell(\widehat{M}, \widehat{\mathcal{E}})$ are viewed as Γ -modules via the action induced by the actions of Γ respectively on \widehat{M} and $\widehat{\mathcal{E}}$. This spectral sequence results from the theory of Grothendieck's derived functors [Gro] (see for instance [Bro]). If $(\widehat{M}, \widehat{\mathcal{E}})$ is acyclic, that is $H^\ell(\widehat{M}, \widehat{\mathcal{E}}) = 0$ for $\ell \geq 1$, the spectral sequence E_r converges at the E_2 term and then:

$$(9) \quad H^k(M, \mathcal{E}) = H^k(\Gamma, H^0(\widehat{M}, \widehat{\mathcal{E}})) = H^k(\Gamma, C^\infty(\widehat{\mathcal{E}})).$$

4.4. Foliated cohomologies

Let M be a connected differentiable manifold supporting a foliation \mathcal{F} of dimension m (and codimension n).

1. For any $r \in \mathbb{N}$, we denote $\Lambda^r(T^*\mathcal{F})$ the bundle of exterior algebras of degree r over $T\mathcal{F}$ (tangent bundle to \mathcal{F}). Its sections are the *foliated forms* of degree r ; they form a vector space $\Omega_{\mathcal{F}}^r(M)$. We have an operator (exterior derivative along the leaves) $d_{\mathcal{F}} : \Omega_{\mathcal{F}}^r(M) \longrightarrow \Omega_{\mathcal{F}}^{r+1}(M)$ defined (as in the classical case) by the formula:

$$\begin{aligned} d_{\mathcal{F}}\alpha(X_1, \dots, X_{r+1}) &= \sum_{i=1}^{r+1} (-1)^i X_i \cdot \alpha(X_1, \dots, \widehat{X}_i, \dots, X_{r+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{r+1}) \end{aligned}$$

where \widehat{X}_i means that the argument X_i is omitted. We easily verify that $d_{\mathcal{F}}^2 = 0$. So we obtain a differential complex (called the *de Rham foliated complex* of \mathcal{F}):

$$0 \longrightarrow \Omega_{\mathcal{F}}^0(M) \xrightarrow{d_{\mathcal{F}}} \Omega_{\mathcal{F}}^1(M) \xrightarrow{d_{\mathcal{F}}} \dots \xrightarrow{d_{\mathcal{F}}} \Omega_{\mathcal{F}}^{m-1}(M) \xrightarrow{d_{\mathcal{F}}} \Omega_{\mathcal{F}}^m(M) \longrightarrow 0.$$

Let $Z_{\mathcal{F}}^r(M)$ be the kernel of $d_{\mathcal{F}} : \Omega_{\mathcal{F}}^r(M) \longrightarrow \Omega_{\mathcal{F}}^{r+1}(M)$ and $B_{\mathcal{F}}^r(M)$ the image of $d_{\mathcal{F}} : \Omega_{\mathcal{F}}^{r-1}(M) \longrightarrow \Omega_{\mathcal{F}}^r(M)$. The quotient $H_{\mathcal{F}}^r(M) = Z_{\mathcal{F}}^r(M)/B_{\mathcal{F}}^r(M)$ is the r^{th} vector space of *foliated cohomology* of (M, \mathcal{F}) .

Foliated cohomology was substantially used in the study of the parametric rigidity of some Lie group actions. (See [MM] and [Asa] for an account on the subject).

2. Let $\tau : E \longrightarrow M$ be a \mathcal{F} -foliated vector bundle. Then $d_{\mathcal{F}}$ extends to the space $\Omega_{\mathcal{F}}^*(M, E)$ of foliated forms with values in E and gives rise to a differential complex:

$$(10) \quad 0 \longrightarrow \Omega_{\mathcal{F}}^0(M, E) \xrightarrow{d_{\mathcal{F}}} \Omega_{\mathcal{F}}^1(M, E) \xrightarrow{d_{\mathcal{F}}} \dots \xrightarrow{d_{\mathcal{F}}} \Omega_{\mathcal{F}}^{m-1}(M, E) \xrightarrow{d_{\mathcal{F}}} \Omega_{\mathcal{F}}^m(M, E) \longrightarrow 0$$

whose homology $H_{\mathcal{F}}^*(M, E)$ is called the *foliated cohomology* of the foliated manifold (M, \mathcal{F}) with values in E . If \mathcal{E}_b is the sheaf of germs of basic sections of E and $\widetilde{\Omega}_{\mathcal{F}}^r(E)$ the sheaf of germs of foliated r -forms, we have a fine resolution:

$$(11) \quad 0 \longrightarrow \mathcal{E}_b \hookrightarrow \widetilde{\Omega}_{\mathcal{F}}^0(E) \xrightarrow{d_{\mathcal{F}}} \widetilde{\Omega}_{\mathcal{F}}^1(E) \xrightarrow{d_{\mathcal{F}}} \dots \xrightarrow{d_{\mathcal{F}}} \widetilde{\Omega}_{\mathcal{F}}^{m-1}(E) \xrightarrow{d_{\mathcal{F}}} \widetilde{\Omega}_{\mathcal{F}}^m(E) \longrightarrow 0$$

first established by Vaisman in [Vai]. Then $H_{\mathcal{F}}^*(M, E) = H^*(M, \mathcal{E}_b)$.

3. If E is the trivial vector bundle of rank 1, \mathcal{E}_b is the sheaf of germs of basic functions and so $H^r(M, \mathcal{E}_b) = H_{\mathcal{F}}^r(M)$ for $r \in \mathbb{N}$.

4. Suppose that the foliation \mathcal{F} is a fibration $M \xrightarrow{\pi} W$ with fibre L . If \overline{E} is a vector bundle over W and E its pull-back by π to M , then:

$$(12) \quad H_{\mathcal{F}}^*(M, E) = H^*(L) \otimes C^\infty(\overline{E})$$

where $H^*(L)$ is the usual de Rham cohomology of L and $C^\infty(\overline{E})$ the Fréchet space of C^∞ -sections of \overline{E} .

4.5. Integrable homotopy

This property is for foliated cohomology of a foliation what the ordinary homotopy is for de de Rham cohomology of a manifold.

Let (M, \mathcal{F}) and (M', \mathcal{F}') be two foliated manifolds. We equip $M \times \mathbb{R}$ with the foliation \mathfrak{F} whose leaves are $L \times \mathbb{R}$ where L is a leaf of \mathcal{F} . Let $f, g : M \rightarrow M'$ be two differentiable maps. An *integrable homotopy* from f to g is a differentiable foliated map $H : (M \times \mathbb{R}, \mathfrak{F}) \rightarrow (M', \mathcal{F}')$ such that:

$$H(z, t) = \begin{cases} f(z) & \text{for } t \leq 0 \\ g(z) & \text{for } t \geq 1. \end{cases}$$

We have the following assertions (see [Ek1]) established in the case of foliated cohomology but easily generalized in the case where it is valued in a foliated vector bundle.

1. *If there exists an integrable homotopy from f to g then the induced morphisms $f^*, g^* : H_{\mathcal{F}'}^*(M') \rightarrow H_{\mathcal{F}}^*(M)$ are the same.*

2. *Suppose that the two foliated manifolds (M, \mathcal{F}) and (M', \mathcal{F}') have the same integrable homotopy type. Then their foliated cohomologies $H_{\mathcal{F}}^*(M)$ and $H_{\mathcal{F}'}^*(M')$ are isomorphic.*

5. Infinitesimal deformations of developable foliations

In all this section, M will be a connected compact manifold of dimension $d = m + n$. As usual, all the objects we will consider are supposed to be of class C^∞ .

5.1. Preliminaries

All that we are going to expose in this subsection is taken from the paper [Ham].

For any $x \in M$, we denote $G_x(M, m)$ the Grassmannian of m -planes of $T_x M$. We obtain a locally trivial bundle $\mathcal{G}(M, m) \rightarrow M$ whose typical fibre is the Grassmannian $G(d, m)$ of the vector space \mathbb{R}^d . A C^∞ -*field* (or just a *field*) of m -planes on M is nothing but a section of $\mathcal{G}(M, m) \rightarrow M$. Let τ be a field of m -planes and (τ_1, \dots, τ_m) a basis of local sections of τ . If $X = \sum_{i=1}^m a_i \tau_i$ and $Y = \sum_{j=1}^m b_j \tau_j$ are two local sections of τ , we have:

$$[X, Y] = \sum_{i,j=1}^m a_i b_j [\tau_i, \tau_j] + \sum_{i,j=1}^m \{a_i(\tau_i \cdot b_j) \tau_j - b_j(\tau_j \cdot a_i) \tau_i\}.$$

In the quotient $\mathcal{V} = TM/\tau$, the value of $[X, Y]$ at a point $x \in M$ depends only on the values of X and Y at this point and not on the values of their derivatives. This gives rise to a 2-form $Q(\tau) : \tau \times \tau \rightarrow \mathcal{V}$ whose value at X_x and Y_x is the class in the quotient $\mathcal{V}_x = T_x M/\tau_x$ of the vector $[X, Y]_x$.

The 2-form $Q(\tau)$ is an element of the space $C^\infty(\Lambda^2(\mathcal{G}(TM, m)))$ of the C^∞ -sections of the bundle $\Lambda^2(\mathcal{G}(TM, m)) \rightarrow \mathcal{G}(TM, m)$ whose fibre over a m -plane τ of TM is the

space of 2-skew-symmetric forms $\Lambda^2(\tau, TM/\tau)$. By Frobenius theorem, τ is integrable if and only if the $Q(\tau)$ is identically zero. In this case, τ defines a dimension m foliation \mathcal{F} on M . The C^∞ -map $Q : \tau \in C^\infty(\mathcal{G}(M, m)) \mapsto Q(\tau) \in C^\infty(\Lambda^2(\mathcal{G}(TM, m)))$ plays a fundamental role in the theory of foliations.

The space $C^\infty(\mathcal{G}(M, m))$ of C^∞ -sections of the bundle $\mathcal{G}(M, m)$ equipped with its natural C^∞ -topology is a Fréchet manifold [Ham]. The subset $\mathfrak{F}(M, m)$ of foliations of dimension m on M (“zero set of Q ”) is closed; we equip it with the induced topology.

Let \mathcal{F} be a foliation of dimension m , that is, \mathcal{F} is an integrable field of m -planes on M . For any diffeomorphism φ of M , the pull-back $\varphi^*(\mathcal{F})$ of \mathcal{F} by φ is a dimension m foliation on M *i.e.* an element of $\mathcal{G}(M, m)$ such that its associated 2-form $Q(\mathcal{F})$ is identically 0. Then we have a C^∞ -map $\Theta_{\mathcal{F}} : \varphi \in \text{Diff}(M) \mapsto \varphi^*(\mathcal{F}) \in C^\infty(\mathcal{G}(M, m))$ (with values in fact in $\mathfrak{F}(M, m)$) and a sequence:

$$(13) \quad \text{Diff}(M) \xrightarrow{\Theta_{\mathcal{F}}} C^\infty(\mathcal{G}(M, m)) \xrightarrow{Q} C^\infty(\Lambda^2(\mathcal{G}(TM, m)))$$

satisfying $Q \circ \Theta_{\mathcal{F}} = 0$. This sequence is called the *non-linear deformation complex* of the foliation \mathcal{F} . We say that \mathcal{F} is *C^∞ -stable* if there exist an open neighborhood \mathfrak{U} of \mathcal{F} in $C^\infty(\mathcal{G}(M, m))$ and an open neighborhood \mathfrak{W} of the identity in $\text{Diff}(M)$ with $\Theta_{\mathcal{F}}(\mathfrak{W}) \subset \mathfrak{U}$ and such that the complex:

$$(14) \quad \mathfrak{W} \xrightarrow{\Theta_{\mathcal{F}}} \mathfrak{U} \xrightarrow{Q} C^\infty(\Lambda^2(\mathcal{G}(TM, m)))$$

is exact. This means that any foliation \mathcal{F}' of dimension m on M sufficiently close (in the C^∞ -topology) to \mathcal{F} is conjugate to \mathcal{F} by a diffeomorphism φ of M close to the identity *i.e.* $\varphi^*(\mathcal{F}') = \mathcal{F}$.

The study of the stability of \mathcal{F} requires the linearization of the nonlinear complex of its deformations. But this one is exactly the foliated de Rham complex (up to the degree 2) with values in the normal bundle (cf. [Ham] for calculations):

$$(15) \quad 0 \longrightarrow \Omega_{\mathcal{F}}^0(M, \mathcal{V}) \xrightarrow{d_{\mathcal{F}}} \Omega_{\mathcal{F}}^1(M, \mathcal{V}) \xrightarrow{d_{\mathcal{F}}} \Omega_{\mathcal{F}}^2(M, \mathcal{V}) \dots$$

An element of $H_{\mathcal{F}}^1(M, \mathcal{V})$ (cohomology at degree 1 of the complex (15)) is called an *infinitesimal deformation* of \mathcal{F} . So in all the following sections we will focus our attention mostly on the vector spaces $H_{\mathcal{F}}^*(M, \mathcal{V})$.

Now the foliation \mathcal{F} we will consider on our manifold M will be developable. Let $\pi : \widehat{M} \rightarrow M$ be its Γ -covering and $D : \widehat{M} \rightarrow W$ its developing map whose fibres are the leaves \widehat{L} of the the pull-back $\widehat{\mathcal{F}} = \pi^*(\mathcal{F})$.

Let $E \rightarrow M$ be a \mathcal{F} -foliated vector bundle and $\widehat{E} \rightarrow \widehat{M}$ its pull-back to \widehat{M} . Then $\widehat{E} \rightarrow \widehat{M}$ is a $\widehat{\mathcal{F}}$ -foliated vector bundle; moreover Γ acts on \widehat{E} by automorphisms. We denote \mathcal{E}_b and $\widehat{\mathcal{E}}_b$ the sheaves of germs of basic sections respectively of the bundles E and \widehat{E} .

5.2. Theorem. *Suppose that \widehat{E} is the pull-back of a vector bundle \overline{E} on W . Then:*

1. *If the de Rham cohomology $H^*(\widehat{L})$ of the leaf \widehat{L} is trivial for $* \geq 1$, the foliated cohomology $H_{\mathcal{F}}^*(M, E)$ is isomorphic to the cohomology $H^*(\Gamma, C^\infty(\overline{E}))$ of the discrete group Γ with values in the space $C^\infty(\overline{E})$ of C^∞ -sections of \overline{E} (which is a Γ -module).*

2. *If $H^1(\widehat{L})$ is trivial, the space $H_{\mathcal{F}}^1(M, \mathcal{V})$ of infinitesimal deformations of \mathcal{F} is isomorphic to the cohomology space $H^1(\Gamma, \mathfrak{X}(W))$ where $\mathfrak{X}(W)$ is the Γ -module of vector fields on W .*

Proof. It is almost immediate since we have prepared in the preceding sections all the material for this purpose. Let \mathcal{E}_b and $\widehat{\mathcal{E}}_b$ denote the sheaves of germs of basic sections respectively of the bundles E and \widehat{E} . The group Γ acts also on these sheaves.

1. By 4.3. we have a spectral sequence E_r converging to $H^*(M, \mathcal{E}_b)$ and whose E_2 term is:

$$(16) \quad E_2^{pq} = H^p(\Gamma, H_{\mathcal{F}}^q(\widehat{M}, \widehat{\mathcal{E}}_b)).$$

Since the foliation $(\widehat{M}, \widehat{\mathcal{F}})$ is the fibration $D : \widehat{M} \rightarrow W$ and \widehat{E} is the pull-back by D of \overline{E} , $H_{\mathcal{F}}^q(\widehat{M}, \widehat{\mathcal{E}}_b) = H^q(\widehat{L}) \otimes C^\infty(\overline{E})$ (following equality (12)). Replacing in (16), we get:

$$E_2^{pq} = H^p(\Gamma, H^q(\widehat{L}) \otimes C^\infty(\overline{E})).$$

Taking into account the fact that, by hypothesis, $H^*(\widehat{L})$ is trivial for $* \geq 1$, we obtain:

$$H^p(M, \mathcal{E}_b) = E_2^{p0} = H^p(\Gamma, C^\infty(\overline{E})).$$

But as it was observed in subsection 4.4. 2), $H_{\mathcal{F}}^p(M, E) = H^p(M, \mathcal{E}_b)$. This implies:

$$H_{\mathcal{F}}^p(M, E) = E_2^{p0} = H^p(\Gamma, C^\infty(\overline{E})).$$

2. Let us take again the equality $E_2^{pq} = H^p(\Gamma, H^q(\widehat{L}) \otimes C^\infty(\overline{E}))$. The spectral sequence E_r converges to $H^*(M, \mathcal{E}_b) = H_{\mathcal{F}}^*(M, E)$, so $H_{\mathcal{F}}^1(M, E) = E_\infty^{01} \oplus E_\infty^{10}$. Since (by hypothesis) the space $H^1(\widehat{L})$ is trivial, we have in fact:

$$E_\infty^{01} \oplus E_\infty^{10} = E_\infty^{10} = E_2^{10} = H^1(\Gamma, C^\infty(\overline{E})).$$

Finally:

$$H_{\mathcal{F}}^1(M, E) = H^1(\Gamma, C^\infty(\overline{E})).$$

Now, applying this to the (foliated) normal bundle $\mathcal{V} = TM/T\mathcal{F}$, for which $\widehat{\mathcal{V}}$ is the pull-back by D of TW (the tangent bundle of W), we obtain:

$$(17) \quad H_{\mathcal{F}}^1(M, \mathcal{V}) = H^1(\Gamma, \mathfrak{X}(W)).$$

This completes the proof of the theorem. □

6. Examples and explicit computations

Some of them are well known and have already been studied under a continuous aspect (using vector fields and differential forms). We take them back by applying the different methods of cohomology of discrete groups we have just exposed.

6.1. Example 1

Let M be the torus $\mathbb{T}^{m+n} = \mathbb{R}^{m+n}/\mathbb{Z}^{m+n}$ where m and n are positive integers. We can view it as the product $\mathbb{T}^m \times \mathbb{T}^n$ and then as the quotient of $\widehat{M} = \mathbb{R}^m \times \mathbb{T}^n$ by the free and proper action of the group $\Gamma = \mathbb{Z}^m$:

$$(18) \quad (\mathbf{k}, (x, y)) \in \mathbb{Z}^m \times (\mathbb{R}^m \times \mathbb{T}^n) \mapsto (x + \mathbf{k}, y) \in \mathbb{R}^m \times \mathbb{T}^n.$$

The canonical projection $\pi : \widehat{M} \rightarrow M$ is a Γ -covering. Denote by (e_1, \dots, e_m) the canonical basis of \mathbb{R}^m , that is e_i (for $i = 1, \dots, m$) has its i^{th} component equal to 1 and all the others equal to 0. This basis is also a system of generators of Γ .

Let $\rho : \Gamma \rightarrow \text{Diff}(\mathbb{T}^n)$ be the representation which associates to each generator e_i the translation on \mathbb{T}^n by a vector $\alpha_i = (\alpha_i^1, \dots, \alpha_i^n)$. The suspension of ρ gives rise to a foliation \mathcal{F} on M whose associated lifted foliation $\widehat{\mathcal{F}}$ to \widehat{M} is just the one defined by the second projection $\widehat{M} = \mathbb{R}^m \times \mathbb{T}^n \rightarrow \mathbb{T}^n = W$. The action of Γ on W is given by the representation ρ .

The normal bundle \mathcal{V} is identified to the tangent bundle of $W = \mathbb{T}^n$ which is trivial as a foliated bundle: any linear vector field Y on W is basic. So \mathcal{V} is trivial as foliated vector bundle and also as a Γ -bundle. So:

$$(19) \quad H_{\mathcal{F}}^1(M, \mathcal{V}) = H^1(\Gamma, C^\infty(\mathbb{T}^n)) \otimes \mathbb{R}^n.$$

Then to determine $H_{\mathcal{F}}^1(M, \mathcal{V})$ it is sufficient to determine the space $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ which is the ordinary foliated cohomology $H_{\mathcal{F}}^1(M)$ of the manifold (M, \mathcal{F}) .

Let us compute the vector space $H^1(\Gamma, C^\infty(\mathbb{T}^n))$. We shall do it by induction on the integer m . Suppose that $m = 1$, that is, Γ is reduced to its first factor $\Gamma_1 = \mathbb{Z}$. Then by formulas (6), $H^r(\Gamma_1, C^\infty(\mathbb{T}^n)) = 0$ for $r \geq 2$, $H^0(\Gamma_1, C^\infty(\mathbb{T}^n))$ is the subspace of $C^\infty(\mathbb{T}^n)$ consisting of functions invariant by τ_1 and $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ is the cokernel of the operator:

$$\delta : f \in C^\infty(\mathbb{T}^n) \mapsto (f - f \circ \tau_1) \in C^\infty(\mathbb{T}^n)$$

where τ_1 is the translation by the vector $\alpha_1 = (\alpha_1^1, \dots, \alpha_1^n)$. Then we have to solve the cohomological equation:

$$(20) \quad f(y) - f(y + \alpha_1) = g(y) \quad \text{for all } y \in \mathbb{T}^n$$

where g is given in $C^\infty(\mathbb{T}^n)$.

One can find the resolution of this equation in several works. We will quickly resume the one given in [EH] and focus on the case where the components of the vector

$\alpha_1 = (\alpha_1^1, \dots, \alpha_1^n)$ are linearly independent over \mathbb{Q} . We denote $\langle \cdot, \cdot \rangle$ the usual scalar product in \mathbb{R}^n .

i) We say that α_1 is **Diophantine** if there exist real numbers $C > 0$ and $\eta > 0$ such that $|\langle \mathbf{m}, \alpha_1 \rangle| \geq \frac{C}{|\mathbf{m}|^\eta}$ for all $\mathbf{m} \in \mathbb{Z}^n$ different from 0.

ii) We say that α_1 is **Liouville** if there exists a real number $C > 0$ such that for any $\eta > 0$, there exists $\mathbf{m}_\eta \in \mathbb{Z}^n$ satisfying the equality $|\langle \mathbf{m}_\eta, \alpha_1 \rangle| \leq \frac{C}{|\mathbf{m}_\eta|^\eta}$.

Any vector $\alpha_1 \in \mathbb{R}^n$ whose components $\alpha_1^1, \dots, \alpha_1^n$ are algebraic numbers and linearly independent over \mathbb{Q} is Diophantine (see [EH] for the proof).

Let $\mathcal{L} : C^\infty(\mathbb{T}^n) \longrightarrow \mathbb{C}$ be the functional defined by $\mathcal{L}(g) = \int_{\mathbb{T}^n} g(x) dx$. Its kernel \mathcal{N} is a closed codimension one subspace of $C^\infty(\mathbb{T}^n)$. We have the following assertions:

iii) $H^0(\Gamma, C^\infty(\mathbb{T}^n))$ consists of constant functions: it is isomorphic to \mathbb{R} .

iv) If the vector α_1 is Diophantine, then the image of the operator δ is equal to \mathcal{N} . In this case, $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ is a Hausdorff topological vector space of dimension 1 generated by the constant function equal to 1.

v) If the vector α_1 is a Liouville then $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ is a non Hausdorff infinite dimensional topological vector space.

Suppose now that $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_1 and Γ_2 are copies of \mathbb{Z} . By Künneth formula, the space $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ is:

$$H^0(\Gamma_1, C^\infty(\mathbb{T}^n)) \otimes H^1(\Gamma_2, C^\infty(\mathbb{T}^n)) \oplus H^1(\Gamma_1, C^\infty(\mathbb{T}^n)) \otimes H^0(\Gamma_2, C^\infty(\mathbb{T}^n)).$$

But $H^0(\Gamma_1, C^\infty(\mathbb{T}^n))$ and $H^0(\Gamma_2, C^\infty(\mathbb{T}^n))$ are isomorphic to \mathbb{R} ; then:

$$H^1(\Gamma, C^\infty(\mathbb{T}^n)) = H^1(\Gamma_1, C^\infty(\mathbb{T}^n)) \oplus H^1(\Gamma_2, C^\infty(\mathbb{T}^n)).$$

So $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ is Hausdorff and isomorphic to \mathbb{R}^2 if both the vectors α_1 and α_2 are Diophantine and it is a non Hausdorff infinite dimensional topological vector space if one of these vector is Liouville.

A repeated application of Künneth's formula gives the following result for the case where the group Γ is \mathbb{Z}^m :

vi) If all the vectors $\alpha_1, \dots, \alpha_m$ are Diophantine, $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ is Hausdorff and isomorphic to \mathbb{R}^m . Then:

$$(21) \quad H_{\mathcal{F}}^1(M, \mathcal{V}) = H^1(\Gamma, C^\infty(\mathbb{T}^n)) \otimes \mathbb{R}^n = \mathbb{R}^m \otimes \mathbb{R}^n.$$

In fact one can easily prove that:

$$(22) \quad H^q(\Gamma, C^\infty(\mathbb{T}^n)) = \mathbb{R}^{C_m^q}$$

for $q = 0, 1, \dots, m$ where $C_m^q = \frac{m!}{q!(m-q)!}$.

vii) If at least one of the vectors $\alpha_1, \dots, \alpha_m$ is Liouville, $H^1(\Gamma, C^\infty(\mathbb{T}^n))$ is a non Hausdorff infinite dimensional topological vector space. Therefore it is the same for the vector space $H_{\mathcal{F}}^1(M, \mathcal{V})$.

6.2. Example 2

The construction of this example is given in [EN] where only the vector space $H_{\mathcal{F}}^1(M, \mathcal{V})$ was computed. We will take it back and calculate explicitly all spaces of its foliated cohomologies.

1. Let A be a square matrix with integer coefficients and determinant 1. We suppose that A is diagonalizable on the field of the real numbers and having all its eigenvalues $\mu_1, \dots, \mu_m, \lambda_1, \dots, \lambda_n$ such that: $0 < \mu_1 \dots, \mu_m < 1 < \lambda_1, \dots, \lambda_n$. Because the product of these positive numbers is 1, the integers m and n are all such that $m \geq 1$ and $n \geq 1$.

We can think of A as a diffeomorphism of \mathbb{T}^{m+n} . Let $u_1, \dots, u_m, v_1, \dots, v_n$ be the eigenvectors corresponding respectively to the eigenvalues $\mu_1, \dots, \mu_m, \lambda_1, \dots, \lambda_n$ and $X_1, \dots, X_m, Y_1, \dots, Y_n$ be linear vector fields on \mathbb{T}^{m+n} whose directions are given respectively by $u_1, \dots, u_m, v_1, \dots, v_n$. We have:

$$(23) \quad A_* X_i = \mu_i X_i, \quad A_* Y_j = \lambda_j Y_j \quad \text{for } i = 1, \dots, m \quad \text{and} \quad j = 1, \dots, n.$$

2. Denote by \mathcal{F}_0 the foliation on \mathbb{T}^{m+n} defined by the vector fields X_1, \dots, X_m . The product of \mathcal{F}_0 by \mathbb{R} gives a codimension n foliation $\widehat{\mathcal{F}}$ on $\widehat{M} = \mathbb{T}^{m+n} \times \mathbb{R}$ which is invariant by the diffeomorphism ϕ of $\mathbb{T}^{m+n} \times \mathbb{R}$ sending (z, t) to $(A(z), t + 1)$. So it induces a codimension n foliation \mathcal{F} on the quotient manifold $\mathbb{T}_A^{m+n+1} = \mathbb{T}^{m+n} \times \mathbb{R} / \phi$. Notice that \mathbb{T}_A^{m+n+1} is a flat bundle over the circle \mathbb{S}^1 with fibre \mathbb{T}^{m+n} . In fact the manifold $M = \mathbb{T}_A^{m+n+1}$ is the homogeneous space G/Γ where G is the semi-direct product $\mathbb{R}^{m+n} \rtimes \mathbb{R}$ given by the action:

$$(t, z) \in \mathbb{R} \times \mathbb{R}^{m+n} \longmapsto A^t z \in \mathbb{R}^{m+n}$$

and Γ is the discrete subgroup $\{(\mathbf{k}, s) \in G \mid \mathbf{k} \in \mathbb{Z}^{m+n}, s \in \mathbb{Z}\}$. The subgroup:

$$H = \left\{ \left(\sum_{i=1}^m a_i u_i, b \right) \in G \mid a_1, \dots, a_m, b \in \mathbb{R} \right\}$$

is isomorphic to the semi-direct product $\mathbb{R}^m \rtimes \mathbb{R}_+^*$ where \mathbb{R}_+^* acts on \mathbb{R}^m by homothety in each eigendirection. The action of H on \mathbb{T}_A^{m+n+1} induced by this identification is a locally free action whose orbits define the foliation \mathcal{F} .

3. From now on we shall assume that the matrix A fulfills the following condition.

(C) The basis of \mathbb{R}^{m+n} given by the eigenvectors $u_1, \dots, u_m, v_1, \dots, v_n$ is such that the coordinates w^1, \dots, w^{m+n} of any of its vectors w are linearly independent over \mathbb{Q} .

Under this assumption any of the the vectors u_1, \dots, u_m fulfills the Diophantine condition (cf. [Sch]) we gave in the Example 5.1.

4. It is well known that if the eigenvalues of A are all different then condition (C) is fulfilled if and only if the characteristic polynomial $\chi_A(t)$ of A is irreducible over \mathbb{Q} . The following example shows the existence of matrices fulfilling the above properties.

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 0 & \dots & 0 \\ 1 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & d_{m+n} \end{pmatrix}$$

where the diagonal elements d_1, \dots, d_{m+n} of A are inductively defined by $d_1 = 1$ and $d_{\ell+1} = 1 + d_1 \cdot d_2 \cdot \dots \cdot d_\ell$ for $\ell = 1, \dots, m+n-1$. In this case the matrix A has one eigenvalue ν_1 in the interval $]0, 1[$ and exactly one eigenvalue ν_2, \dots, ν_{m+n} in each of the intervals $]d_2, d_3[$, $]d_3, d_4[$, \dots , $]d_{m+n}, \infty[$. In particular all the eigenvalues are different and there is only one eigenvalue smaller than 1. This implies that $\chi_A(t)$ is irreducible over \mathbb{Z} and thus over \mathbb{Q} because of the Gauss Lemma. Therefore the matrix A fulfills condition (C).

5. To compute the two cohomologies $H_{\mathcal{F}}^*(M)$ and $H_{\mathcal{F}}^*(M, \mathcal{V})$ we shall use the spectral sequence (8) of the foliated \mathbb{Z} -covering $\pi : (\widehat{M}, \widehat{\mathcal{F}}) \longrightarrow (M, \mathcal{F})$. (Recall that \mathbb{Z} acts on $\widehat{M} = \mathbb{T}^{m+n} \times \mathbb{R}$ by its generator $\phi(z, t) = (A(z), t+1)$.) We will do the calculation just for $H_{\mathcal{F}}^*(M, \mathcal{V})$, it is practically the same for $H_{\mathcal{F}}^*(M)$. The spectral sequence E_r in our situation has its term E_2 :

$$(24) \quad E_2^{pq} = H^p(\mathbb{Z}, H_{\mathcal{F}}^q(\widehat{M}, \widehat{\mathcal{V}}))$$

and converges to the cohomology $H_{\mathcal{F}}^*(M, \mathcal{V})$. (Here $\widehat{\mathcal{V}}$ denotes the normal bundle of the foliation $\widehat{\mathcal{F}}$.) Since the retraction of $\widehat{M} = \mathbb{T}^{m+n} \times \mathbb{R}$ along the second factor is an integrable homotopy from the foliated manifold $(\widehat{M}, \widehat{\mathcal{F}})$ to the foliated manifold $(\mathbb{T}^{m+n}, \mathcal{F}_0)$ (see point 2 of subsection 4.5), we have:

$$H_{\widehat{\mathcal{F}}}^q(\widehat{M}, \widehat{\mathcal{V}}) = H_{\mathcal{F}_0}^q(\mathbb{T}^{m+n}, \mathcal{V}_0)$$

where \mathcal{V}_0 is the normal bundle of the foliation \mathcal{F}_0 . Since all the vectors u_1, \dots, u_m are diophantine:

$$(25) \quad H_{\mathcal{F}_0}^q(\mathbb{T}^{m+n}, \mathcal{V}_0) = \mathbb{R}^{C_m^q} \otimes \mathbb{R}^n.$$

6. Now we will describe the action of the diffeomorphism ϕ of \widehat{M} on these two spaces. For this let us take again the vector fields $X_1, \dots, X_m, Y_1, \dots, Y_n$ satisfying the relations (23). They constitute a basis of the module $\mathfrak{X}(\mathbb{T}^{m+n})$ of tangent vector fields to \mathbb{T}^{m+n} ; the first ones X_1, \dots, X_m are those who are tangent to the foliation \mathcal{F}_0 . Let $\theta_1, \dots, \theta_m$ be the 1-forms (with constant coefficients) such that, for $i, k = 1, \dots, m$:

$$\theta_i(X_k) = \delta_i^k \quad \text{and} \quad \theta_i(Y_j) = 0 \quad \text{for} \quad j = 1, \dots, n$$

where δ_i^k is the Kronecker symbol. The family $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_q} : 1 \leq i_1 < \cdots < i_q \leq m\}$ is a basis of $H_{\mathcal{F}_0}^q(\mathbb{T}^{m+n})$ for $q \geq 1$; $H_{\mathcal{F}_0}^0(\mathbb{T}^{m+n}) = \mathbb{R}$ is generated by the constant function equal to 1. Then one can take:

$$\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_q} \otimes Y_j : 1 \leq i_1 < \cdots < i_q \leq m \text{ and } j = 1, \dots, n\}$$

as a basis of $H_{\mathcal{F}_0}^q(\mathbb{T}^{m+n}, \mathcal{V}_0)$. The action of the matrix A on this space is given on the elements of this basis by:

$$\begin{aligned} A^*(\theta_{i_1} \wedge \cdots \wedge \theta_{i_q} \otimes Y_j) &= (A^{-1})^*(\theta_{i_1}) \wedge \cdots \wedge (A^{-1})^*(\theta_{i_q}) \otimes A_*(Y_j) \\ &= \left(\mu_{i_1}^{-1} \cdots \mu_{i_q}^{-1}\right) (\theta_{i_1} \wedge \cdots \wedge \theta_{i_q}) \otimes (\lambda_j Y_j) \\ &= (\mu_{i_1} \cdots \mu_{i_q})^{-1} \lambda_j (\theta_{i_1} \wedge \cdots \wedge \theta_{i_q} \otimes Y_j). \end{aligned}$$

Since $0 < \mu_1, \dots, \mu_m < 1$ and $1 < \lambda_1, \dots, \lambda_n$, all the products $(\mu_{i_1} \cdots \mu_{i_q})^{-1} \lambda_j$ are greater than 1; then no one of the elements $\theta_{i_1} \wedge \cdots \wedge \theta_{i_q} \otimes v_j$ is invariant under the action of A . Therefore, for any $q \in \{0, 1, \dots, m\}$:

$$E_2^{0q} = H^0(\mathbb{Z}, H_{\mathcal{F}_0}^q(\mathbb{T}^{m+n}, \mathcal{V}_0)) = 0.$$

7. Recall that if V is a vector space on which \mathbb{Z} acts freely $H^p(\mathbb{Z}, V) = 0$ for $p \geq 2$. Applying this to our situation, we get, for $p \geq 2$:

$$E_2^{pq} = H^p(\mathbb{Z}, H_{\mathcal{F}_0}^q(\mathbb{T}^{m+n}, \mathcal{V}_0)) = 0.$$

Then the second differential $d_2 : E_2^{pq} \longrightarrow E_2^{p+2, q-1}$ is zero. This implies the convergence of the spectral sequence at the E_2 term, and then $H_{\mathcal{F}}^r(M, \mathcal{V}) = E_2^{0r} \oplus E_2^{1, r-1}$. But $E_2^{0r} = 0$, so:

$$(26) \quad H_{\mathcal{F}}^r(M, \mathcal{V}) = E_2^{1, r-1} = H^1(\mathbb{Z}, H_{\mathcal{F}_0}^{r-1}(\mathbb{T}^{m+n}, \mathcal{V}_0)) = H^1(\mathbb{Z}, \mathbb{R}^d \otimes \mathbb{R}^n)$$

where $d = C_m^{r-1} = \frac{n!}{(r-1)!(m-r+1)!}$. Because the action of \mathbb{Z} is without fixed vector on the space $\mathbb{R}^d \otimes \mathbb{R}^n$ (which is finite dimensional), $H^1(\mathbb{Z}, \mathbb{R}^d \otimes \mathbb{R}^n)$ is trivial and then so is $H_{\mathcal{F}}^r(M, \mathcal{V})$ for any $r = 0, 1, \dots, m$.

8. We just have showed that $H_{\mathcal{F}}^r(M, \mathcal{V}) = 0$ for any $r = 0, 1, \dots, m$, in particular $H_{\mathcal{F}}^1(M, \mathcal{V}) = 0$. The foliation \mathcal{F} is then infinitesimally stable. This result is not new: the C^∞ -stability (which is more stronger) was already proved in [EN].

9. In a similar way one can prove that the two spaces $H_{\mathcal{F}}^0(M)$ and $H_{\mathcal{F}}^1(M)$ are equal to \mathbb{R} and that $H_{\mathcal{F}}^r(M) = 0$ for $r \geq 2$.

6.3. Example 3

Let M be a compact manifold equipped with a Lie G -foliation \mathcal{F} . Let $\pi : \widehat{M} \longrightarrow M$ be the corresponding Γ -covering and $D : \widehat{M} \longrightarrow G$ its developing map. As it was

mentioned in the subsection 3.3, there exists an injective representation $\rho : \Gamma \hookrightarrow G$ such that, for any $\gamma \in \Gamma$, the following diagram is commutative:

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\gamma} & \widehat{M} \\ D \downarrow & & \downarrow D \\ G & \xrightarrow{\rho(\gamma)} & G. \end{array}$$

As a subgroup of G , Γ acts on it by left translations. By Theorem 5.2, the space $H_{\mathcal{F}}^1(M, \mathcal{V})$ of infinitesimal deformations of \mathcal{F} is identified to the space $H^1(\Gamma, \mathfrak{X}(G))$. But $\mathfrak{X}(G) = C^\infty(G) \otimes \mathcal{G}$ where \mathcal{G} is the Lie algebra of the group G . Because the action of Γ on \mathcal{G} is trivial, we immediately have:

$$(27) \quad H_{\mathcal{F}}^1(M, \mathcal{V}) = H^1(\Gamma, C^\infty(G)) \otimes \mathcal{G}$$

where the action of Γ on G is by left translations and by the induced usual one on the Fréchet space $C^\infty(G)$.

The following problem is of interest independently of that which it could have in deformation theory of Lie foliations.

Problem. *Let G be a connected Lie group and Γ a countable subgroup. Compute the cohomology $H^*(\Gamma, C^\infty(G))$.*

For Γ finite or discrete, we have (see [Ek2]) $H^r(\Gamma, C^\infty(G)) = 0$ for $r \geq 1$. The case $\Gamma \simeq \mathbb{Z}$ was studied in [EH]. The one where G is a torus and $\Gamma \simeq \mathbb{Z}^m$ is more or less our example 6.1.

When Γ is infinite and not closed, the quotient $Q = G/\Gamma$ is a Q -manifold in the sense of [Bar]. Then one can ask: *What relationship is there between $H^*(\Gamma, C^\infty(G))$ and the cohomology of Q -manifolds defined in [Bar]?*

In the general case of a transversely homogeneous G/H -foliation \mathcal{F} , the vector space $H_{\mathcal{F}}^1(M, \mathcal{V})$ is $H_{\mathcal{F}}^1(\Gamma, \mathfrak{X}(G/H))$ where the action of Γ on G/H is induced by the its left action on G .

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