Hypoelliptic vector fields, cohomology of groups, Diophantine approximation...
a stroll through mathematics !

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ABSTRACT

The cohomology of discrete groups is defined in purely algebraic way. One might a priori think that algebra is its unique playground, but it is not always so. The purpose of our lecture is to bring the hearer to take a short walk to see how this cohomology actually crosses various branches of mathematics (analysis, geometry, dynamical systems...). We will see this on simple and significant examples in which this object appears as an effective tool to solve some difficult problems or, at least, to reformulate them in a different way to possibly attack them more easily.

0. The general problem

Let E be a Fréchet space. Mostly E will be the space $C^\infty(\mathcal{E})$ of sections of class C^∞ of a vector bundle $\mathcal{E} \longrightarrow M$ (over a compact manifold M) equipped with the C^∞-topology. For example :

i) If $\mathcal{E}$ trivial with fibre $\mathbb{C}$, E is the space of functions $M \longrightarrow \mathbb{C}$ of class C^∞.

ii) If $\mathcal{E}$ is the r^{th} exterior power of the cotangent bundle T^*M, E is the space of *differential forms* of degree r on M.

Examples of this type abound, especially in geometry !

Let $\gamma : E \longrightarrow E$ be an automorphism. Denote by E^γ the set of elements f of E invariant by γ, that is, satisfying $\gamma \cdot f = f$.

E^γ is the kernel of the (continuous) *coboundary operator* :

$$\delta : f \in E \longmapsto (f - \gamma \cdot f) \in E. \tag{1}$$

Then E^γ is a closed subspace of E and is also a Fréchet space.

Let $T : E \longrightarrow E$ be a *bounded operator commuting* with γ. We are interested by the solutions in E^γ of the equation $Tf = g$ where $g \in E^\gamma$ is given.

A natural way is to solve firstly the equation in E (forgetting that g is γ-invariant) and then correct the solution $f_0 \in E$ by adding an element h of the kernel N of T to make the new solution $f = f_0 + h$ invariant by γ, that is, satisfying the relation $\gamma \cdot (f_0 + h) = f_0 + h$ *i.e.* $h - \gamma \cdot h = \gamma \cdot f_0 - f_0$. (The element $(\gamma \cdot f_0 - f_0)$ is in N.) This gives the following problem :

Let $g \in N$. *Does there exist* $h \in N$ *s.t.* : $h - \gamma \cdot h = g$ *?*

This is the *cohomological equation* of the *dynamical system* (N, γ) : *N is a Fréchet space on which the automorphism γ acts !*

The terminology comes from the fact that the *first cohomology group* $H^1(\mathbb{Z}, N)$ of the *discrete group* $\mathbb{Z}$ with coefficients in the $\mathbb{Z}$*-module* N is exactly the *cokernel* of the operator $\delta : N \longrightarrow N$.

Problem

Let N be a Fréchet space and γ an automorphism of N. Compute the space $H^1(\mathbb{Z}, N)$.

Why this vector space is so important ?

We shall see, through the following example, some answers to this question.

1. The $\overline{\partial}$ on $\mathbb{C}^*$

Let U be an open set of $\mathbb{C}$. A point in U will be defined by its real coordinates (x, y) or its complex one $z = x + iy$. Let $C^\infty(U)$ be the space of complex functions of class C^∞ on U. We will be interested by the partial differential equation called the *Cauchy-Riemann equation* :

(CR) $$\overline{\partial} f = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = g$$

where $g \in C^\infty(U)$ is given.

The existence of a solution f to equation **(CR)** for any function g is equivalent to the triviality of the *Dolbeault cohomology group* $H^1(U, \mathcal{O})$ of the open set U where $\mathcal{O}$ is the *sheaf of germs of holomorphic functions* on U.

In the late of the 19th century mathematicians were able to solve this equation on $U = \mathbb{C}$ (by using the *Cauchy formula*) but not yet on any open set of $\mathbb{C}$. For instance, what about $U = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$?

• Recall that we have an action of $\mathbb{Z}$ on $\mathbb{C}$ generated by the biholomorphism $\tau : \mathbb{C} \longrightarrow \mathbb{C}$ given by $\tau(z) = z + 1$; γ induces an automorphism γ on the Fréchet space $E = C^\infty(\mathbb{C})$ of C^∞-functions on $\mathbb{C}$:

$$(\gamma \cdot f)(z) = f \circ \tau(z) = f(z+1)$$

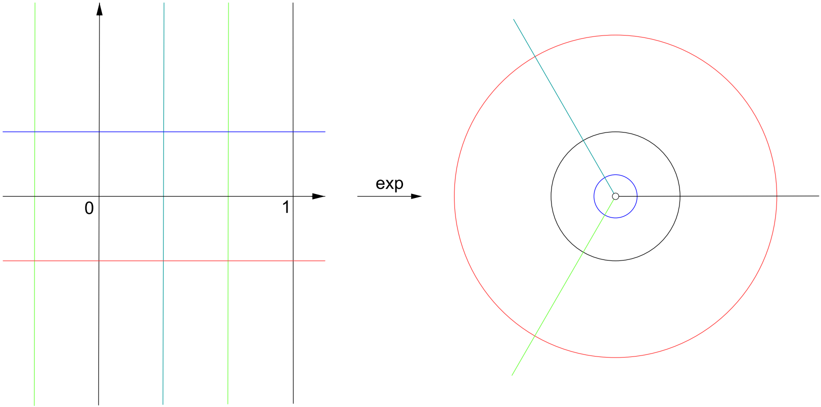
but also on the space $N = \mathcal{H}(\mathbb{C})$ of holomorphic functions which is exactly the kernel of the operator $C^\infty(\mathbb{C}) \xrightarrow{\bar\partial} C^\infty(\mathbb{C})$.

• The action of $\mathbb{Z}$ on $\mathbb{C}$ which we have defined is holomorphic, free and proper. The quotient $\mathbb{C}/\tau$ is then a Riemann surface. It can be described explicitly : it is exactly the action by translation of $\mathbb{Z}$ on the additive group $(\mathbb{C}, +)$; since $\mathbb{Z}$ is the kernel of the morphism $\mathbf{exp} : z \in \mathbb{C} \longmapsto e^{2i\pi z} \in \mathbb{C}^*$, the following sequence is exact :

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{C} \xrightarrow{\mathbf{exp}} \mathbb{C}^* \longrightarrow 1. \tag{2}$$

$$e^{2i\pi z} = e^{2i\pi(x+iy)} = e^{-2\pi y} \cdot e^{2i\pi x}.$$

This proves that the quotient $\mathbb{C}/\tau$ is a Riemann surface ; it is biholomorphically equivalent to $\mathbb{C}^*$.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exp

• **The C^∞-functions on $\mathbb{C}^*$ are identified to the C^∞-functions on $\mathbb{C}$ which are invariant by γ, that is, functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying the periodicity condition $f(z+1) = f(z)$. They constitute a closed subspace E^γ of the Fréchet space $E = C^\infty(\mathbb{C})$. Since the equation (CR) has a solution in E, we have also a solution in E^γ if the vector space $H^1(\mathbb{Z}, N)$ is trivial (this was noted in the introduction). Is it then the case ? Yes :**

Guichard's Theorem (1887)

Let $\tau : z \in \mathbb{C} \longmapsto z+1 \in \mathbb{C}$ and let γ be the automorphism of the Fréchet space $N = \mathcal{H}(\mathbb{C})$ of holomorphic functions on $\mathbb{C}$ defined by $\gamma \cdot f = f \circ \tau$. Then $H^1(\mathbb{Z}, \mathcal{H}(\mathbb{C})) = 0$.

The proof given by Guichard is purely analytic. It is very technical and long, but at that time, it answers (implicitly or explicitly) a question !

Natural question :

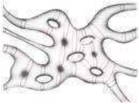
For which Riemann surface and automorphism γ we still have that kind of result ?

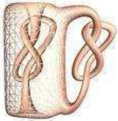
Here is a response :

Theorem (2011)

Let Σ be a noncompact Riemann surface and $\gamma : \Sigma \longrightarrow \Sigma$ an automorphism which acts on it freely and properly such that the quotient $M = \Sigma/\gamma$ is a noncompact Riemann surface. Then, for any holomorphic function $g : \Sigma \longrightarrow \mathbb{C}$ and any $\lambda \in \mathbb{C}$, there exists a holomorphic function $f : \Sigma \longrightarrow \mathbb{C}$ which is a solution of the equation $f \circ \gamma - \lambda f = g$.

The particular case $\Sigma = \mathbb{C}$, $\lambda = 1$ and $\gamma(z) = z + 1$ gives the theorem of Guichard !

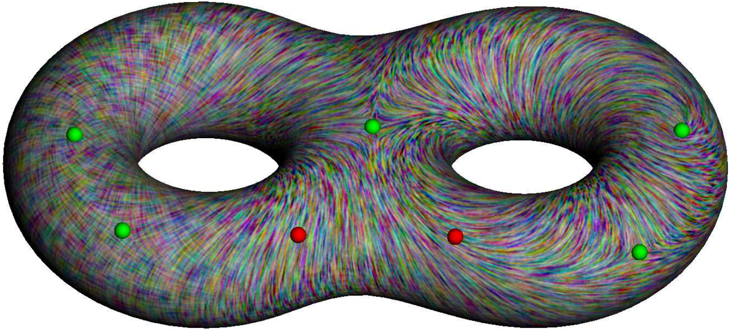




A noncompact Riemann surface
has sufficiently holomorphic functions.
A compact Riemann surface has
only the constants as holomorphic
functions and, if its genus ≥2,
its automorphism group is finite!

2. The equation $X \cdot f = g$ on a manifold

2.1. A *continuous dynamical system* (CDS for short) is a couple (M, X) where M is a manifold (compact for simplicity) and X a vector field on M.



Two CDS (M, X) and (N, Y) are said to be *conjuguated* if there exists a diffeomorphism $h : M \longrightarrow N$ such that $h_*(X) = Y$. Existence of a conjugacy implies a very important fact : *everyting that happens for one of the two dynamical systems happens also for the second one !*

2.2. Let (M, X) be a CDS. Then the vector field X defines a first order differential operator $X : C^\infty(M) \longrightarrow C^\infty(M)$ given by :

$$(X \cdot f)(x) = (d_x f)(X_x) \tag{3}$$

value of the differential $d_x f$ of f at the point x (which is a linear functional on the tangent space $T_x M$) on the vector $X_x \in T_x M$. It is natural to look for solutions of the *continuous cohomological equation* :

$$X \cdot f = g. \tag{4}$$

The operator $X : C^\infty(M) \longrightarrow C^\infty(M)$ extends naturally to an operator on the space of distributions :

$$X : T \in \mathcal{D}'(M) \longrightarrow X \cdot T \in \mathcal{D}'(M)$$

defined by $\langle X \cdot T, \varphi\rangle = -\langle T, X \cdot \varphi\rangle$. One can also be interested by the solutions of the continuous cohomological equation on distributions :

$$X \cdot T = S. \tag{4'}$$

A distribution T is *invariant* by X or X*-invariant* if it satisfies $X \cdot T = 0$, that is, it is zero on the image of the differential operator $X : C^\infty(M) \longrightarrow C^\infty(M)$. A necessary condition (which is not sufficient in general) for the equation (4) to admit a solution f is $\langle T, g\rangle = 0$ for any distribution T invariant by X.

The problem of the regularity of the solution is very important. We say that X is *globally hypoelliptic* if, for any distribution $T \in \mathcal{D}'(M)$:

$$X \cdot T \in C^\infty(M) \Longrightarrow T \in C^\infty(M).$$

In particular, if this is the case, any X-invariant distribution T is *regular* that is, there exists a function ψ of class C^∞ on M such that, for any function $f \in C^\infty(M)$ we have :

$$\langle T, f \rangle = \int_M f(x) \cdot \psi(x) dx$$

where dx is the canonical measure on M (associated to the differentiable structure).

Fundamental example

2.3. Let $n \geq 2$ be an integer. The vector space $\mathbb{R}^n$ will be equipped with its usual scalar product $\langle \ , \ \rangle$; the associated norm will be denoted $|\cdot|$. The torus $\mathbb{T}^n$ is obtained as the quotient of $\mathbb{R}^n$ by its standard lattice $\mathbb{Z}^n$. For $\mathbf{m} \in \mathbb{Z}^n$, we denote $\Theta_{\mathbf{m}}$ the function $\Theta_{\mathbf{m}}(x) = e^{2i\pi\langle \mathbf{m},x\rangle}$. A function on $\mathbb{T}^n$ is a function $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ satisfying the invariance condition $f(x + \mathbf{m}) = f(x)$ for any $x \in \mathbb{R}^n$ and any $\mathbf{m} \in \mathbb{Z}^n$.

If $f : \mathbb{T}^n \longrightarrow \mathbb{C}$ is integrable, it admits a *Fourier series* expansion :

$$f(x) = \sum_{\mathbf{m}\in\mathbb{Z}^n} f_{\mathbf{m}}\Theta_{\mathbf{m}}(x) \tag{5}$$

where $f_{\mathbf{m}}$ are the *Fourier coefficients* of f given by the integral formulaes :

$$f_{\mathbf{m}} = \int_{\mathbb{T}^n} f(x)e^{-2i\pi\langle \mathbf{m},x\rangle}dx. \tag{6}$$

If, in addition, the function f is square integrable, the coefficients $f_{\mathbf{m}}$ satisfy the convergence condition :

$$\sum_{\mathbf{m}\in\mathbb{Z}^n} |f_{\mathbf{m}}|^2 < +\infty.$$

In the same way, any distribution T on the torus $\mathbb{T}^n$ (viewed as a $\mathbb{Z}^n$-periodic distribution on $\mathbb{R}^n$) can be written :

$$T = \sum_{\mathbf{m}\in\mathbb{Z}^n} T_{\mathbf{m}}\Theta_{\mathbf{m}}$$

where the family of complex numbers $T_{\mathbf{m}}$ (indexed by $\mathbf{m} \in \mathbb{Z}^n$) is at most of *polynomial growth*, that is, there exist an integer $r \in \mathbb{N}$ and a constant $C > 0$ such that $|T_{\mathbf{m}}| \leq C|\mathbf{m}|^r$ for any $\mathbf{m} \in \mathbb{Z}^n$.

2.4. For any $r \in \mathbb{N}$, we denote $W^{1,r}$ the space of functions f on the torus $\mathbb{T}^n$ given by their Fourier coefficients $(f_{\mathbf{m}})_{\mathbf{m}\in\mathbb{Z}^n}$ and satisfying the condition $\sum_{\mathbf{m}\in\mathbb{Z}^n} |\mathbf{m}|^r |f_{\mathbf{m}}| < +\infty$. Similarly, $W^{2,r}$ will be the space of functions f on the torus $\mathbb{T}^n$ given by their Fourier coefficients $(f_{\mathbf{m}})_{\mathbf{m}\in\mathbb{Z}^n}$ and satisfying the condition $\sum_{\mathbf{m}\in\mathbb{Z}^n} |\mathbf{m}|^{2r} |f_{\mathbf{m}}|^2 < +\infty$. These spaces are complete with respect to the norms :

$$||f||_{1,r} = |f_0| + \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{0\}} |\mathbf{m}|^r |f_{\mathbf{m}}| \quad \text{for } f \in W^{1,r}$$

and :

$$||f||_{2,r} = \sqrt{|f_0|^2 + \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{0\}} |\mathbf{m}|^{2r} |f_{\mathbf{m}}|^2} \quad \text{for } f \in W^{2,r}$$

$W^{2,r}$ is the r^{th} Sobolev space on $\mathbb{T}^n$; it has a Hilbert structure given by the Hermtian product :

$$\langle f, g\rangle_r = f_{\mathbf{0}}\overline{g}_{\mathbf{0}} + \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} |\mathbf{m}|^{2r} f_{\mathbf{m}}\overline{g}_{\mathbf{m}}.$$

We have natural inclusions :

$$C^\infty(\mathbb{T}^n) \subset \cdots \subset W^{1,r+1} \subset W^{1,r} \subset \cdots \subset W^{1,0}$$

and :

$$C^\infty(\mathbb{T}^n) \subset \cdots \subset W^{2,r+1} \subset W^{2,r} \subset \cdots \subset W^{2,0} = L^2(\mathbb{T}^n).$$

The following proposition is easy to establish :

Proposition

Let $T = \sum_{\mathbf{m}\in\mathbb{Z}^n} T_{\mathbf{m}}\Theta_{\mathbf{m}}$ *a series (the* $T_{\mathbf{m}}$ *are complex numbers).*

The following assertions i), ii) are iii) are equivalent :

i) T *is a regular distribution, that is,* T *is a* C^∞*-function.*

ii) for any $r \in \mathbb{N}$*, the series* $\sum_{\mathbf{m}\in\mathbb{Z}^n} |\mathbf{m}|^{2r}|T_{\mathbf{m}}|^2$ *converges.*

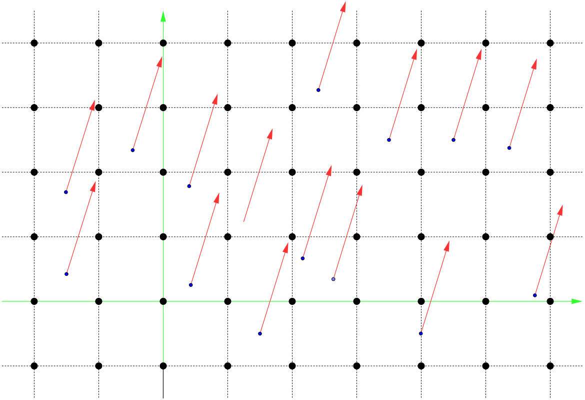
iii) for any $r \in \mathbb{N}$*, the series* $\sum_{\mathbf{m}\in\mathbb{Z}^n} |\mathbf{m}|^{r}|T_{\mathbf{m}}|$ *converges.*

For any $r \in \mathbb{N}$*, the injections* $j_{1,r} : W^{1,r+1} \hookrightarrow W^{1,r}$ *and* $j_{2,r} : W^{2,r+1} \hookrightarrow W^{2,r}$ *are compact operators.*

The three first points mean :

$$\bigcap_{r\in\mathbb{N}} W^{1,r} = \bigcap_{r\in\mathbb{N}} W^{2,r} = C^\infty(\mathbb{T}^n).$$

2.5. **Now we consider the linear vector field** $X = \sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial x_k}$ **on the torus** $\mathbb{T}^n$ **where** $\alpha = (\alpha_1, \cdots, \alpha_n)$ **is a vector of** $\mathbb{R}^n$**.**



A linear vector field on $\mathbb{R}^2$
The lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$

We suppose that the numbers $\alpha_1, \cdots, \alpha_n$ are $\mathbb{Q}$-linearly independent; this implies in particular that the orbits of X are all dense and that the number $\alpha_1 + \cdots + \alpha_n$ is non equal to 0.

The 1-form $\frac{1}{\alpha_1+\cdots+\alpha_n}\sum_{i=1}^n dx_i$ will be denoted χ; its value on X is 1 and its kernel ν is the hyperplane whose equation is $\alpha_1 z_1 + \cdots + \alpha_n z_n = 0$. The vector field X defines a first order differential operator and the associated continuous cohomological equation is :

$$\sum_{k=1}^{n} \alpha_k \frac{\partial f}{\partial x_k} = g. \tag{7}$$

To solve this equation we will use the Fourier expansion of functions on the torus $\mathbb{T}^n$. We have :

$$f(x) = \sum_{\mathbf{m} \in \mathbb{Z}^n} f_{\mathbf{m}} e^{2i\pi\langle \mathbf{m}, x\rangle} \quad \text{and} \quad g(x) = \sum_{\mathbf{m} \in \mathbb{Z}^n} g_{\mathbf{m}} e^{2i\pi\langle \mathbf{m}, x\rangle}.$$

In terms of Fourier coefficients, equation (7) is equivalent to the system :

$$(8) \qquad 2i\pi\langle \mathbf{m}, \alpha\rangle f_{\mathbf{m}} = g_{\mathbf{m}} \quad \text{with} \quad \mathbf{m} \in \mathbb{Z}^n$$

This gives a formal solution :

$$(9) \qquad f_{\mathbf{m}} = \begin{cases} 0 & \text{if } \mathbf{m} = \mathbf{0} \\ \frac{g_{\mathbf{m}}}{2i\pi\langle \mathbf{m}, \alpha\rangle} & \text{otherwise} \end{cases}$$

***Problem* : The quantity $2i\pi\langle \mathbf{m}, \alpha\rangle$ may tends to 0 more quickly than the $g_{\mathbf{m}}$! This may prevent the Fourier series to converge! This brings us to the notion of *Diopantine approximation*.**

Any vector $\alpha \in \mathbb{R}^n$ defines a linear functional on $\mathbb{R}^n$: $x \in \mathbb{R}^n \longmapsto \langle \alpha, x\rangle \in \mathbb{R}$ and then on the lattice $\mathbb{Z}^n$.

Definition

i) We say that the vector α is Diophantine if there exist real $A > 0$ and $\delta > 0$ such that :

$$|\langle \alpha, \mathbf{m} \rangle| \geq \frac{A}{|\mathbf{m}|^{1+\delta}}$$

for any $\mathbf{m} = (m_1, ..., m_n) \in \mathbb{Z}^n$ different from $\mathbf{0}$. In this case, we say that X is a Diophantine vector field.

ii) We say that α is a Liouville vector if there exists $A > 0$ such that, for any $\delta > 0$, there exists $\mathbf{m}_\delta \in \mathbb{Z}^n$ satisfying :

$$|\langle \alpha, \mathbf{m}_\delta \rangle| \leq \frac{A}{|\mathbf{m}_\delta|^{\delta}}.$$

We say that X is a Liouville vector field.

For example : *Any vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ whose components are algebraic numbers which are $\mathbb{Q}$-linearly independent is a Diophantine vector*.
Indeed, by multiplying the components by a common denominator, one may suppose that the α_i are algebraic integers. For $i = 1, \cdots, n$, let σ_i be the different embeddings of the field numbers $\mathbb{Q}[\alpha_1, \cdots, \alpha_n]$ in $\overline{\mathbb{Q}}$ and G the Galois group of an algebraic extension of this field. For any n-uple $\mathbf{m}$ of non zero integers, the product $\prod_j \sigma_j(\langle \alpha, \mathbf{m}\rangle)$ is a non zero algebraic integer invariant by G, then it is a non zero integer. This implies $\left|\prod_j \sigma_j(\langle \alpha, \mathbf{m}\rangle)\right| \geq 1$, and then, if $\sigma_1 = \mathrm{Id}$:

$$|\langle \alpha, \mathbf{m}\rangle| \geq \frac{1}{\left|\prod_{j\geq 2} \sigma_j(\langle \alpha, \mathbf{m}\rangle)\right|} \geq \frac{C}{|\mathbf{m}|^{d-1}}$$

where d is the degree of the extension $\mathbb{Q}[\alpha_1, \cdots, \alpha_n]$ and C is a real positive constant.

To understand what is exactly the Diophantine approximation, we will take $n = 2$. We have $\alpha = (\alpha_1, \alpha_2)$ which we can normalize $\alpha = (1, \theta)$ and $\mathbf{m} = (m_1, m_2)$. So $\langle \alpha, \mathbf{m} \rangle = m_1 + m_2\theta$ and then :

- θ is ***Diophantine*** if there exist constants $A > 0$ and $\delta > 0$ such that, for any $m_1 \in \mathbb{Z}$ and any $m_2 \in \mathbb{Z}^*$, we have :

$$\left| \theta - \frac{m_1}{m_2} \right| \geq \frac{A}{|m_2|^{2+\delta}}$$

that is, θ is not **well approximated** by rational numbers.

- θ is ***Liouville*** if there exists a constant $A > 0$ such that, for any $s \in \mathbb{N}$, there exist integers $m_{1,s} \in \mathbb{Z}$ and $m_{2,s} \in \mathbb{Z}^*$ satisfying :

$$\left| \alpha - \frac{m_{1,s}}{m_{2,s}} \right| \leq \frac{A}{|m_{2,s}|^s}.$$

Liouville numbers are irrational numbers which are "well approximated" by rationals. One can construct such numbers by sum of series of rapidly decreasing, for instance $\theta = \sum_{s=1}^{\infty} 2^{-s!}$ **(for which Liouville proved the property of *transcendence*).**

The principal theorem

i) Suppose that X is Diophantine. Then the equation $X \cdot f = g_\delta$ has a solution $f \in C^\infty(\mathbb{T}^n)$ if, and only if $\int_{\mathbb{T}^n} g(x)dx = 0$.

ii) If X is Liouville, there exists an infinite family of linearly independent functions $(g_\delta)_{\delta \in \mathbb{N}}$ satisfying the condition $\int_{\mathbb{T}^n} g_\delta(x)dx = 0$ and such that equation $X \cdot f = g_\delta$ has no solution ; furthermore the image of the operator X is not closed for the C^∞-topology on $C^\infty(\mathbb{T}^n)$.

iii) In the two cases, the space $\mathcal{D}'_X(\mathbb{T}^n)$ of X-invariant distributions has dimension 1 and is generated by the Haar measure $dx_1 \otimes \cdots \otimes dx_n$ on the Lie group $\mathbb{T}^n$.

Finally one remark that the differential operator X :

- is not *globally hypoelliptic* if α is a *Liouville vector*.
- is *globally hypoelliptiqc* if α est *Diophantine vector*.

The only one known example of a globally hypoelliptic operator is a *Diophantine linear vector field on* $\mathbb{T}^n$. This brings the :

Conjecture de Greenfield-Wallach (1973)

Let M be an oriented compact manifold of dimension n and X a non singular vector field which preserve a C^∞-volume on M. Suppose that X is globally hypoelliptic. Then M is diffeomorphic to the torus $\mathbb{T}^n$ and X is conjugated to a Diophantine linear vector field.

This conjecture was proved dimension $n = 3$ by different authors. But it is still open in dimension $n \geq 4$ even if we suppose that M is a homogeneous space G/Γ and the vector field X induced by an un element of the Lie algebra $\mathcal{G}$ of the Lie group G !

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