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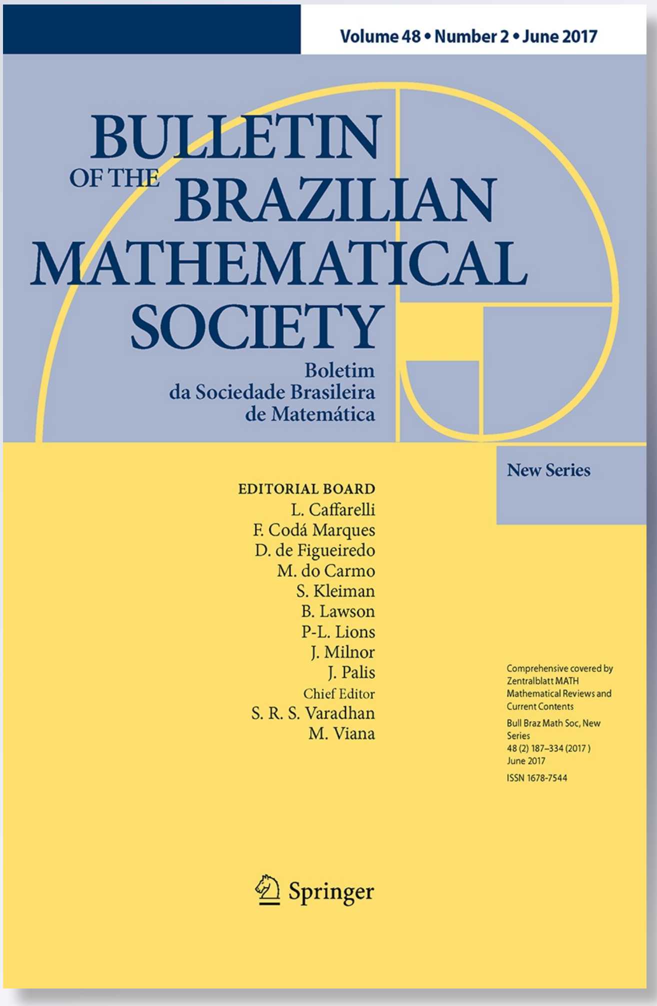

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On Leafwise Meromorphic Functions with Prescribed Poles

Aziz El Kacimi Alaoui[1]

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Abstract Let $\mathcal{F}$ be a complex foliation by Riemann surfaces defined by a trivial (in the differentiable sense) fibration $\pi : M \longrightarrow B$ but for which the complex structure on each fibre $\pi^{-1}(t)$ may depend on t. Let $\sigma : B \longrightarrow M$ be a section of π contained in a $\mathcal{F}$-relatively compact subset of M. We prove: for any $\mathcal{F}$-relatively compact open set U containing $\Sigma = \sigma(B)$ and any integer $s \geq 0$, there exists a function $U \longrightarrow \mathbb{C}$ of class C^s nonconstant on any leaf of $(U, \mathcal{F})$, meromorphic along the leaves and whose set of poles is exactly Σ.

Keywords Complex foliation · Leafwise · Dolbeault cohomology · $\mathcal{F}$-meromorphic function

1 Preliminaries

Let M be a differentiable manifold of dimension $2m + n$ endowed with a dimension $2m$ orientable foliation $\mathcal{F}$.

Definition 1.1 We say that $\mathcal{F}$ is ***complex*** if it can be defined by an open cover $\mathcal{U} = \{U_i\}$ of M and diffeomorphisms $\phi_i : \Omega_i \times \mathcal{O}_i \longrightarrow U_i$ (where Ω_i is an open polydisc in $\mathbb{C}^m$ and $\mathcal{O}_i$ is an open ball in $\mathbb{R}^n$) such that, for any pair (i, j) with $U_i \cap U_j \neq \emptyset$, the coordinate change $\phi_{ij} = \phi_j^{-1} \circ \phi_i : \phi_i^{-1}(U_i \cap U_j) \longrightarrow \phi_j^{-1}(U_i \cap U_j)$ is of the form $(z', t') = \left(\phi_{ij}^1(z, t), \phi_{ij}^2(t)\right)$ with $\phi_{ij}^1(z, t)$ holomorphic in z for t fixed.

✉ Aziz El Kacimi Alaoui
aziz.elkacimi@univ-valenciennes.fr

1 LAMAV, FR du CNRS 2956 ISTV2, Le Mont Houy, Université de Valenciennes, 59313 Valenciennes Cedex 9, France

An open set U of M like one of the cover $\mathcal{U}$ in Definition 1.1 is called *adapted* to the foliation. Any leaf of $\mathcal{F}$ is a complex manifold of dimension m. The notion of complex foliation is a natural generalization of the notion of holomorphic foliation on a complex manifold. A manifold M with a complex foliation $\mathcal{F}$ will be denoted $(M, \mathcal{F})$.

Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be two complex foliations. A *morphism* from $(M, \mathcal{F})$ to $(M', \mathcal{F}')$ is a differentiable map $f : M \longrightarrow M'$ which sends every leaf F of $\mathcal{F}$ into a leaf F' of $\mathcal{F}'$ such that the restriction map $F \xrightarrow{f} F'$ is holomorphic.

We say that a morphism $f : (M, \mathcal{F}) \longrightarrow (M', \mathcal{F}')$ is an *isomorphism* of complex foliations (*automorphism* of $(M, \mathcal{F})$ if $(M, \mathcal{F}) = (M', \mathcal{F}')$) if f is a diffeomorphism whose restriction to any leaf $F \longrightarrow F'$ [where $F' = f(F)$] is a biholomorphism. We say that two complex foliations $\mathcal{F}$ and $\mathcal{F}'$ on M are *conjugated* if there exists an isomorphism $f : (M, \mathcal{F}) \longrightarrow (M, \mathcal{F}')$. Automorphisms of $\mathcal{F}$ form a group denoted $G(\mathcal{F})$.

1.2 Examples

i) Any complex manifold M of dimension m is a complex foliation of dimension m. Its automorphism group is exactly the automorphism group of the complex manifold M.

ii) Any holomorphic foliation (on a complex manifold M) is a complex foliation.

iii) Let B be a differentiable manifold and M an open set of $\mathbb{C}^m \times B$. For $t \in B$, $M_t = \{z \in \mathbb{C}^m : (z, t) \in M\}$ is an open set of $\mathbb{C}^m$ called the *section* of M along t. The connected components of the sections of M are leaves of a complex foliation $\mathcal{F}$ of dimension m called the complex *canonical* foliation of M.

iv) Let F be a complex manifold and B a differentiable one. Any locally trivial fibration $F \hookrightarrow M \longrightarrow B$ whose cocycle takes values in the complex automorphism group $\mathrm{Aut}(F)$ of F is a complex foliation, the fibres being the leaves. If the fibration is trivial *i.e.* $M = F \times B$, we say that $\mathcal{F}$ is a *complex product foliation*. In that case all the leaves are holomorphically equivalent. Suppose that $\mathcal{F}$ is a complex foliation on $M = F \times B$ whose leaves are the factors $F \times \{t\}$ but the complex structure may depend on t; then we say that $\mathcal{F}$ is a *differentiable product*.

v) Let $\rho_1 : \mathbb{R} \longrightarrow \mathbb{R}$ and $\rho_2 : \mathbb{R}^* \longrightarrow \mathbb{R}$ be functions of class C^1 satisfying the following conditions:
   - $\rho_1(-t) = \rho_1(t)$ and $\rho_2(-t) = \rho_2(t)$;
   - $\rho_1(1) = 0$ and $\rho_1 < 0$ on $]-1, +1[$;
   - ρ_1 is strictly increasing on $[1, +\infty[$ and $\lim_{t\to+\infty} \rho_1(t) = 1$;
   - ρ_2 is strictly decreasing on $]0, +\infty[$, $\lim_{t\to+\infty} \rho_2(t) = 1$ and $\lim_{t\to+0^+} \rho_2(t) = +\infty$.

Let M be the open set of $\mathbb{C}\times\mathbb{R}$ defined by $M = \{(z, t) \in \mathbb{C} \times \mathbb{R} : \rho_1(t) < |z| < \rho_2(t)\}$ equipped with its canonical complex foliation $\mathcal{F}$. Then the leaves are: $\mathbb{C}$ if $t = 0$, open discs for $t \neq 0$ and $|t| < 1$, two punctured discs if $|t| = 1$ and the others are annulus.

On the open set $N = \{(z,t) \in M : t > 1\}$ the complex foliation $\mathcal{F}_N$ is a differentiable product. Two leaves are never isomorphic; each one has a complex structure coded by the ratio $\varepsilon(t) = \frac{\rho_2(t)}{\rho_1(t)}$. Since $\varepsilon(t) \neq \varepsilon(t')$ for $t \neq t'$, any automorphism of $\mathcal{F}_N$ must be the identity on the transversal. Then the automorphism group $G(\mathcal{F}_N)$ of $\mathcal{F}$ is generated by the group $C^\infty(]1, +\infty[, \mathbb{S}^1)$ and the map $(z,t) \longmapsto \left(\frac{\rho_1(t)\rho_2(t)}{z}, t\right)$ which preserves each annulus.

Question 1.3 *Does the odd sphere $\mathbb{S}^{2n+1}$ support a codimension one complex foliation?*

Of course, yes for $\mathbb{S}^3$ (any orientable foliation by surfaces is a complex one). In higher dimension I already asked this question in 1995 during a lecture I gave in the seminar *Géométrie dynamique* at Université de Lille 1. A construction of such foliation on the sphere $\mathbb{S}^5$ was announced by Meersseman and Verjovsky (2002). But recently they have discovered that the manifold supporting this foliation is in fact a bundle over the circle with fibre a projective Fermat surface [*cf.* Meersseman and Verjovsky (2011)]. Even the authors have failed to answer the question for $\mathbb{S}^5$ their example is highly non trivial and interesting. But the question now remains open.

2 The $\overline{\partial}_{\mathcal{F}}$-Cohomology

Let $(M, \mathcal{F})$ be a complex foliation of dimension m. Let $A^{pq}(\mathcal{F})$ be the space of foliated differential forms of type (p,q) that is, differential forms on M which can be written in local coordinates adapted to the foliation $(z,t) = (z_1, \ldots, z_m, t_1, \ldots, t_n)$ (the foliation is defined by the differential system $dt_1 = \cdots = dt_n = 0$):

$$\alpha = \sum \alpha_{j_1 \ldots j_p k_1 \ldots k_q}(z,t) dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q}$$

where the coefficients $\alpha_{j_1 \ldots j_p k_1 \ldots k_q}$ are functions of class C^s and C^∞ along the leaves (with $s \in \mathbb{N} \cup \{\infty\}$. Let $\overline{\partial}_{\mathcal{F}} : A^{pq}(\mathcal{F}) \longrightarrow A^{p,q+1}(\mathcal{F})$ be the Cauchy-Riemann operator along the leaves defined by:

$$\overline{\partial}_{\mathcal{F}}\alpha = \sum \left(\sum_{k=1}^{m} \frac{\partial \alpha_{j_1 \ldots j_p k_1 \ldots k_q}}{\partial \overline{z}_k}(z,t) d\overline{z}_k \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q} \right)$$

where $\frac{\partial}{\partial \overline{z}_k} = \frac{1}{2}\left\{\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right\}$ with $z_k = x_k + iy_k$. It satisfies $\overline{\partial}^2_{\mathcal{F}} = 0$, hence we have a differential complex $0 \longrightarrow A^{p0}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} A^{p1}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} \cdots \xrightarrow{\overline{\partial}_{\mathcal{F}}} A^{p,m-1}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} A^{pm}(\mathcal{F}) \longrightarrow 0$ called the $\overline{\partial}_{\mathcal{F}}$-*complex* of $(M, \mathcal{F})$; its homology $H^{pq}_{\mathcal{F}}(M)$ is called the *foliated Dolbeault cohomology* (or the $\overline{\partial}_{\mathcal{F}}$-*cohomology*) of the complex foliation $(M, \mathcal{F})$. It is locally trivial *i.e.* we have a:

Lemma 2.1 *Foliated Dolbeault–Grothendieck Lemma. Let $x \in M$. Then there exists an open neighborhood U of x adapted to the foliation such that, for every $p = 0, \ldots, m$, $H^{pq}_{\mathcal{F}}(U) = 0$ for $q \geq 1$.*

The proof is a straightforward adaptation to the parametric case of the classical one.

One can describe the cohomology $H_{\mathcal{F}}^{p*}(M)$ by using a sheaf which is analogous to the sheaf of germs of holomorphic p-forms on a complex manifold. A p-form α is said to be *$\mathcal{F}$-holomorphic*, if it is foliated, of type $(p, 0)$ and satisfies $\overline{\partial}_{\mathcal{F}}\alpha = 0$. Locally, a *$\mathcal{F}$-holomorphic* p-form can be written: $\alpha = \sum \alpha_{j_1 \dots j_p}(z, t) dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ with $\alpha_{j_1 \dots j_p}$ holomorphic on z.

Let $\mathcal{H}_{\mathcal{F}}^p$ be the sheaf of germs of $\mathcal{F}$-holomorphic p-forms on M and $\mathcal{A}^{pq}(\mathcal{F})$ be the sheaf of germs of differential forms of type (p, q) on $\mathcal{F}$; $\mathcal{A}^{pq}(\mathcal{F})$ is a fine sheaf. Lemma 2.1 implies the:

Proposition 2.2 *The sequence* $0 \longrightarrow \mathcal{H}_{\mathcal{F}}^p \hookrightarrow \mathcal{A}^{p0}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} \cdots \xrightarrow{\overline{\partial}_{\mathcal{F}}} \mathcal{A}^{pm}(\mathcal{F}) \longrightarrow 0$ *is a fine resolution of* $\mathcal{H}_{\mathcal{F}}^p$. *So we have* $H^q(M, \mathcal{H}_{\mathcal{F}}^p) = H_{\mathcal{F}}^{pq}(M)$, *for* $p, q = 0, 1, \dots m$.

If $n \geq 1$, this resolution is not elliptic; it is only elliptic along the leaves. Hence the cohomology $H^*(M, \mathcal{H}_{\mathcal{F}}^p)$ is not necessarily finite dimensional even if the manifold M is compact.

Any isomorphism of complex foliations $(M, \mathcal{F}) \xrightarrow{f} (M', \mathcal{F}')$ induces an isomorphism $f^* : H^*(M', \mathcal{H}_{\mathcal{F}'}^p) \longrightarrow H^*(M, \mathcal{H}_{\mathcal{F}}^p)$. In particular $H^*(M, \mathcal{H}_{\mathcal{F}}^p)$ depends only on the complex conjugacy class of $\mathcal{F}$.

For $p = 0$, we denote $\mathcal{H}_{\mathcal{F}}$ the sheaf $\mathcal{H}_{\mathcal{F}}^0$; its sections over an open set U of M are $\mathcal{F}$-holomorphic functions on U; they form a complex vector space which we will denote by $\mathcal{H}_{\mathcal{F}}^0(U)$ and simply $\mathcal{H}(U)$ in case the codimension of $\mathcal{F}$ is zero, that is, M is a complex manifold and the foliation has just one leaf, M itself.

Let $p \in \mathbb{N}$. An open set U of M (with the induced foliation) is said to be *p-acyclic*, if $H^q(U, \mathcal{H}_{\mathcal{F}}^p) = 0$ for any $q \geq 1$. An open cover $\mathcal{U} = \{U_i\}$ is *p-acyclic* if, for any multi-index $(i_0, \dots, i_k)$ of I, the open set $U_{i_0 \dots i_k} = U_{i_0} \cap \cdots \cap U_{i_k}$ is p-acyclic. We can easily see by Lemma 2.1 that such open cover exists and, in addition, can be chosen locally finite. By Leray's Theorem (*cf.* [Gm]), $H^*(M, \mathcal{H}_{\mathcal{F}}^p) = H^*(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^p)$ for any locally finite p-acyclic open cover $\mathcal{U}$.

We have two ways for computing the $\overline{\partial}_{\mathcal{F}}$-cohomology of $\mathcal{F}$: using foliated differential forms of type (p, q) and the $\overline{\partial}_{\mathcal{F}}$ operator or a locally finite p-acyclic open cover $\mathcal{U}$ adapted to the foliation and Cěch method. Both of the two points of view will be interesting for our purpose.

Let us start with a simple example. Let F be a complex manifold of dimension m and B a differentiable manifold. We denote by $C^s(B)$ the complex vector space of complex C^s (with $s \in \mathbb{N} \cup \{\infty\}$) functions on B. The following proposition is easy to prove.

Proposition 2.3 *Suppose that the complex foliation is defined by a locally trivial fibration* $F \longrightarrow M \xrightarrow{\pi} B$ *(the cocycle is with values in the biholomorphism group of the complex manifold F). Then:*

$$H_{\mathcal{F}}^{p*}(M) = H^{p*}(F) \otimes C^s(B)$$

where $H^{p*}(F)$ is the Dolbeault cohomology of the complex manifold F. In particular, $H_{\mathcal{F}}^{p*}(M) = 0$ *for* $* \geq 1$ *if the the fibre F is a Stein manifold.*

2.4 Open Questions

Some questions inspired by the classical complex analysis are natural. Let M be a differentiable manifold with a complex foliation $\mathcal{F}$ of dimension m.

Question 1. Suppose that every leaf is closed (in the topological sense as a subset of M) and Stein that is, it can be embedded in some $\mathbb{C}^N$). Is $H^{0q}_{\mathcal{F}}(M) = 0$ for $q \geq 1$?

A weak version of this question is obtained by imposing an extra hypothesis on the foliated manifold $(M, \mathcal{F})$.

Question 2. Suppose that every leaf is closed and Stein and that $\mathcal{F}$ is a complete Riemannian foliation (the normal bundle $TM/T\mathcal{F}$ admits a Riemannian metric invariant along the leaves). Is $H^{0q}_{\mathcal{F}}(M) = 0$ for $q \geq 1$?

In fact, by a localization procedure, question 2 can be reduced essentially to the following one.

Question 3. Suppose that M is a diffentiable product $F \times B$ where F is a Stein manifold and B is a ball of $\mathbb{R}^n$; each leaf $F \times \{t\}$ is diffeomorphic to F but has a complex structure which may depend on $t \in B$ and is Stein. Is $H^{0q}_{\mathcal{F}}(M) = 0$ for $q \geq 1$?

For a study of foliated Dolbeault cohomology and its explicit calculus on some complex foliations with a more complicated dynamics see El Kacimi Alaoui and Slimène (2010). Also results on the $\overline{\partial}$-problem along the leaves can be found for instance in Gigante and Tomassini (1995).

2.5 Zeros and Poles

Suppose that the dimension of $\mathcal{F}$ is 1 that is, the leaves are Riemann surfaces. Let U be an open set of M with the induced complex foliation $\mathcal{F}$.

Let $f : U \longrightarrow \mathbb{C}$ be a $\mathcal{F}$-holomorphic function and let Z be the set of its zeros. The restriction of f to any leaf F is a holomorphic function; then, if $f : F \longrightarrow \mathbb{C}$ is not identically zero, $Z \cap F$ is a discrete set of F. So in a neighborhood of a point of $Z \cap F$ where f does not vanish identically, $Z \cap F$ is 'transverse' to F.

We say that a function $f : U \longrightarrow \mathbb{C}$ is *$\mathcal{F}$-meromorphic*, if its restriction to any leaf is a meromorphic function. Let $\mathcal{P}$ be the set of poles of f; then, similarly to the case of zeros, the intersection of $\mathcal{P}$ with any leaf is a discrete set of F [see El Kacimi Alaoui (2010)].

2.6 Statement of the Main Result

From now on $\mathcal{F}$ will be a complex foliation by Riemann surfaces on a differentiable manifold M.

Main Theorem. Suppose that $\mathcal{F}$ is defined by a differentiable trivial fibration $\pi : M \longrightarrow B$. Let $\Sigma : B \longrightarrow M$ be a section of π contained in a $\mathcal{F}$-relatively compact subset of M. Then for any relatively compact open set U containing $\Sigma = \sigma(B)$ and any integer $s \geq 0$, there exists a function $U \longrightarrow \mathbb{C}$ of class C^s nonconstant on any leaf of $(U, \mathcal{F})$, meromorphic along the leaves and whose set of poles is exactly Σ.

This result is a weak parametric version of Mittag–Leffler Theorem. A strong version was already established in El Kacimi Alaoui (2010) in case the leaves are noncompact, simply connected or with $\mathbb{Z}$ as common fundamental group.

The remaining part of the paper will be devoted to the proof of the Main Theorem stated above. It will result in a series of lemmas and propositions for which some of our proofs are inspired from methods developed in Forster (1981). The main difficulty here is the control step by step of the transverse regularity and this is far to be a trivial job.

Without lost of generality we may suppose that $n = 1$ and B is an open interval I of the real line $\mathbb{R}$ (containing the origin) or the circle $\mathbb{S}^1$. We choose to treat the case $B = I$. All leaves are noncompact and diffeomorphic. Let M_0 be a leaf of $\mathcal{F}$; there exists a diffeomorphism $\Phi : M \longrightarrow M_0 \times I$ such that, for any $t \in I$, $M_t = \Phi^{-1}(M_0 \times \{t\})$ is a leaf of $\mathcal{F}$ that is, if $M_0 \times I$ is equipped with the foliation $\mathcal{F}_0$ whose leaves are $M_0 \times \{t\}$ with $t \in I$, the two foliated manifolds $(M, \mathcal{F})$ and $(M_0 \times I, \mathcal{F}_0)$ are differentiably isomorphic.

A $\mathcal{F}$*-open set* (resp. a $\mathcal{F}$*-closed set*) of M is an open set of the type $U_0 \times I$ (resp. a closed set of the type $F_0 \times I$) where U_0 is open in M_0 (resp. F_0 is closed in M_0). A $\mathcal{F}$*-open cover* is a cover of M by $\mathcal{F}$-open sets. The unions and finite intersections of $\mathcal{F}$-open sets are also $\mathcal{F}$-open sets. A subset E of M is $\mathcal{F}$*-connected* if, for any $t \in I$, E_t is connected; it is $\mathcal{F}$*-compact* (resp. $\mathcal{F}$*-relatively compact*) if there exists a compact (resp. relatively compact) set K_0 of M_0 such that $E \subset K_0 \times I$. Let $\mathcal{U} = \{U_i\}$ and $\mathcal{V} = \{V_i\}$ two $\mathcal{F}$-open covers indexed by the same set; the notation $\mathcal{V} \ll \mathcal{U}$ means that, for any i, V_i is contained in U_i and is $\mathcal{F}$-relatively compact in this set.

Let Γ denote the common fundamental group of the leaves. The case where Γ is trivial or isomorphic to $\mathbb{Z}$ was studied in El Kacimi Alaoui (2010). So we will suppose that Γ is non Abelian; then the universal covering of each leaf is the upper half plane $\mathbb{H} = \{z = x + iy : y > 0\}$. If E is a subset of M, E_t will be its intersection with M_t.

3 Spaces of $\mathcal{F}$-Holomorphic Functions

Let $U \subset \mathbb{C} \times I$ be an $\mathcal{F}$-open set and $s \in \mathbb{N}$. Let $\mathcal{H}^s_{\mathcal{F}}(U)$ be the space of functions $U \longrightarrow \mathbb{C}$ of class C^s and $\mathcal{F}$-holomorphic; the space of basic functions (constant on the leaves) is a subspace of $\mathcal{H}^s_{\mathcal{F}}(U)$ and is canonically isomorphic to the space $C^s(I)$ of functions of class C^s on the interval I. For any function $f \in \mathcal{H}^s_{\mathcal{F}}(U)$, any measurable subset $E \subset U$, any $t \in I$ and $k \in \{0, 1, \ldots, s\}$, we set:

$$J_k(f, E_t) = \left(\int_{E_t} \left| \frac{\partial^k f}{\partial t^k} \right|^2 dz d\bar{z} \right)^{\frac{1}{2}}$$

and:

$$N_k(f, E) = \sup_{(z,t) \in E} \left| \frac{\partial^k f}{\partial t^k}(z, t) \right|.$$

We denote by $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ the space of functions $f \in \mathcal{H}^{s}_{\mathcal{F}}(U)$ such that, for any $k = 0, \ldots, s$:

$$\sup_{t\in I} J_k(f, U_t) < +\infty$$

We equip this space with the norm:

$$||f||^s_{2,U} = \max_{k=0,\ldots,s} \left\{ \sup_{t\in I} J_k(f, U_t) \right\}$$

for which it will be complete as we shall show.

Now, we consider the functions $f \in \mathcal{H}^{s}_{\mathcal{F}}(U)$ satisfying the condition:

$$N_k(f, U) < +\infty$$

for any $k \in \{0, 1, \ldots, s\}$. These functions form a vector space $\mathcal{H}^{b,s}_{\mathcal{F}}(U)$ which can be equipped with the norm:

$$||f||^s_{\infty,U} = \max_{k=0,\ldots,s} N_k(f, U).$$

By the usual methods one can easily prove that it is complete.

One can observe that, if the measures of the U_t are uniformly bounded, we have $\mathcal{H}^{b,s}_{\mathcal{F}}(U) \subset \mathcal{H}^{2,s}_{\mathcal{F}}(U)$.

The space $\mathcal{B}^s$ of basic functions of class C^s whose derivatives up to the order s are bounded equipped with the norm:

$$||\phi||^s_{\infty} = \max_{k=0,\ldots,s} \left\{ \sup_{t\in I} \left| \frac{d^k\phi}{dt^k} \right| \right\}$$

is a Banach algebra. Note that $\mathcal{B}^s$ is a subspace of $\mathcal{H}^{b,s}_{\mathcal{F}}(U)$ while it is not one of $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ except if the measures of the sections U_t (with $t \in I$) are uniformly bounded. But both of the spaces $\mathcal{H}^{b,s}_{\mathcal{F}}(U)$ and $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ are $\mathcal{B}^s$-modules. Let:

$$\mathcal{H}^b_{\mathcal{F}}(U) = \bigcap_{s\in\mathbb{N}} \mathcal{H}^{b,s}_{\mathcal{F}}(U), \quad \mathcal{H}^2_{\mathcal{F}}(U) = \bigcap_{s\in\mathbb{N}} \mathcal{H}^{2,s}_{\mathcal{F}}(U) \quad \text{and} \quad \mathcal{B} = \bigcap_{s\in\mathbb{N}} \mathcal{B}^s.$$

These are Fréchet spaces whose topologies are respectively defined by the countable families of norms considered above:

$$\left\{||\ \ ||^s_{\infty,U}\right\}_{s\in\mathbb{N}}, \quad \left\{||\ \ ||^s_{2,U}\right\}_{s\in\mathbb{N}} \quad \text{and} \quad \left\{||\ \ ||^s_{\infty}\right\}_{s\in\mathbb{N}}.$$

Now we suppose $U = B = B_0 \times I$ where B_0 is the open ball centered at a of radius $r > 0$ of the complex plane $\mathbb{C}$. Any function $f \in \mathcal{H}^s_{\mathcal{F}}(U)$ admits an expansion:

$$f(z,t) = \sum_{n=0}^{\infty} f_n(t)(z-a)^n$$

where the coefficients f_n are functions in $t \in I$ given by the integral Cauchy formula:

$$f_n(t) = \frac{1}{2i\pi} \int_{\gamma_t} \frac{f(z,t)}{(z-a)^{n+1}} dz.$$

Here $\{\gamma_t\}$ is a differentiable family of circles centered at a and $\gamma_t \subset B_t$. This shows that $f_n \in \mathcal{B}^s$ if $f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$ or $f \in \mathcal{H}^{b,s}_{\mathcal{F}}(U)$ and that the sequence (indexed by N):

$$f_N(z,t) = \sum_{n=0}^{N} f_n(t)(z-a)^n$$

converges to f both in the spaces $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ and $\mathcal{H}^{b,s}_{\mathcal{F}}(U)$. The spaces $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ and $\mathcal{H}^{b,s}_{\mathcal{F}}(U)$ are free modules over the Banach algebra $\mathcal{B}^s$ with basis $\{\phi_n(z)\}_{n\in\mathbb{N}}$ where $\phi_n(z) = (z-a)^n$.

A simple computation shows that the L^2-norm $||\phi_n||_2$ of ϕ_n (considered as a function in the ball B_0) is:

$$||\phi_n||_2 = \frac{\sqrt{\pi} r^{n+1}}{\sqrt{n+1}}$$

from which we deduce that, for any $s \in \mathbb{N}$, we have:

$$||f_n \phi_n||^s_{2,B} = ||f_n||^s_\infty \frac{\sqrt{\pi} r^{n+1}}{\sqrt{n+1}}.$$

Then:

$$||f||^s_{2,B} \leq \sum_{n=0}^{\infty} ||f_n||^s_\infty \frac{\sqrt{\pi} r^{n+1}}{\sqrt{n+1}}.$$

Theorem 3.1 *Let $D \subset \mathbb{C}$ be an open set and $r > 0$. We set $D_r = \{z \in \mathbb{C} : B_0(z,r) \subset D\}$, $U = D \times I$ and $U_r = D_r \times I$. (Here $B_0(z,r)$ is the open ball centered at z with radius r in $\mathbb{C}$.) Then, for any function $f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$, we have:*

$$||f||^s_{\infty,U_r} \leq \frac{1}{\sqrt{\pi} r} ||f||^s_{2,U}.$$

Proof Let $(a,t) \in U_r$. Then, on $B = B_0 \times I$, we have $f(z,t) = \sum_{n=0}^{\infty} f_n(t)(z-a)^n$. So $f(a,t) = f_0(t)$ and then, for any $k = 0, \ldots, s$, we have:

$$\left|\frac{d^k f}{dt^k}(a,t)\right|^2 = \left|\frac{d^k f_0}{dt^k}(t)\right|^2 \leq \frac{1}{\sqrt{\pi} r} J_k(f, B_t)^2 \leq \frac{1}{\sqrt{\pi} r} J_k(f, D_t)^2.$$

Taking the upper bound of this quantity over $t \in I$ and the maximum on $k \in \{0, 1, \ldots, s\}$, we obtain the following relations:

$$||f||^s_{\infty, U_r} = \sup_{U_r} \left|\frac{d^k f}{dt^k}(a,t)\right| \leq \frac{1}{\sqrt{\pi} r} ||f||^s_{2,U}$$

which are exactly the desired inequalities. ◇

Corollary 3.2 *The space $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ equipped with the norm $||\ \ ||^s_{2,U}$ is complete.*

4 Proof of the Main Theorem

For brevity, we will agree to the following definition: a submodule A of the $\mathcal{B}^s$-module $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ is of *finite cotype* [(FC)-*submodule* for short] if the quotient $\mathcal{B}^s$-module $\mathcal{H}^{2,s}_{\mathcal{F}}(U)/A$ is finitely generated (or of *finite type*).

Lemma 4.1 *Let D_0 and D'_0 be two open sets of $\mathbb{C}$ such that D'_0 is contained an relatively compact in D_0. We set $U = D_0 \times I$ and $U' = D'_0 \times I$. Let $s \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists a closed (FC)-submodule A of the $\mathcal{B}^s$-module $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ such that:*

$$||f||^s_{2,U'} \leq \varepsilon ||f||^s_{2,U} \quad \text{for any function} \quad f \in A.$$

Proof Since $\overline{U}'$ is $\mathcal{F}$-compact in U, there exist $r > 0$ and finitely many points $a_1, \ldots, a_u$ in D such that:

(i) $B_0(a_j, r) \times I \subset U$ for $j = 1, \ldots, u$. ($B_0(a_j, r)$ is the ball of radius r centered at a_j.)
(ii) $U' \subset \bigcup_{j=1}^{u} B_0\left(a_j, \frac{r}{2}\right) \times I$.

Let n be an integer such that $u \leq 2^{n+1}\varepsilon$. Let A be the set of functions $f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$ whose restriction to any transversal $\{a_j\} \times I$ is zero up to the order n. Then A is a closed (FC)-submodule of $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$; the number of generators of the quotient $\mathcal{B}^s$-module $\mathcal{H}^{2,s}_{\mathcal{F}}(U)/A$ is less or equal to $n \cdot u$. Let $f \in A$; in a neighborhood of $\{a_j\} \times I$ we have:

$$f(z,t) = \sum_{\ell=n}^{\infty} f_\ell (z - a_j)^\ell.$$

Let $\rho \le r$; for any $k = 0, \ldots, s$:

$$J_k(f, B_0(a_j, \rho))^2 = \sum_{\ell=n}^{\infty} \frac{\pi \rho^{2\ell+2}}{\ell+1} \left| \frac{d^k f_\ell}{dt^k} \right|^2$$

thus:

$$\begin{aligned} J_k\left(f, B_0\left(a_j, \frac{r}{2}\right)\right)^2 &= \sum_{\ell=n}^{\infty} \frac{\pi r^{2\ell+2}}{r^{2\ell+2}(\ell+1)} \left| \frac{d^k f_\ell}{dt^k} \right|^2 \\ &\le 2^{-2(n+1)} \sum_{\ell=n}^{\infty} \frac{\pi r^{2\ell+2}}{\ell+1} \left| \frac{d^k f_\ell}{dt^k} \right|^2 \\ &\le 2^{-2(n+1)} (J_k(f, B_0(a_j, r))^2. \end{aligned}$$

So, using the properties (i) and (ii):

$$\begin{aligned} J_k(f, U_t) &\le \sum_{j=1}^{u} J_k(f, B_0(a_j, r)) \\ &\le u \cdot 2^{-n-1} J_k(f, U_t) \\ &\le \varepsilon J_k(f, U_t) \end{aligned}$$

Taking the upper bound on $t \in I$ and the maximum on $k = 0, 1, \ldots, s$ of the two sides of this inequality, we obtain:

$$||f||^s_{2,U'} \le \varepsilon ||f||^s_{2,U},$$

which gives the desired inequality. $\diamond$

We have already observed that $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ is a module over the Banach algebra $\mathcal{B}^s$. Let $f, g \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$; for any $t \in I$, we set:

$$\langle f, g \rangle_t = \sum_{k=0}^{s} \int_{U_t} \frac{\partial^k f}{\partial t^k} \cdot \frac{\partial^k \overline{g}}{\partial t^k} d\mu_t.$$

For fixed $t \in I$, $\langle \ , \ \rangle_t$ is a Hermitian product on $\mathcal{H}^{2,s}_{\mathcal{F}}(U_t)$ for which it is a Hilbert space. The following lemma is almost immediate to establish.

Lemma 4.2 *Let $s \in \mathbb{N}$ and f and g be two functions in $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$. The function $\lambda : I \longrightarrow \mathbb{C}$ which associates to each $t \in I$ the complex number $\langle f, g \rangle_t$ belongs to $\mathcal{B}^0$. Moreover there exists a positive constant C such that, for $f, g \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$, we have:*

$$||\lambda||_\infty \le C ||f||^s_{2,U} \cdot ||g||^s_{2,U}$$

that is, the family of Hermitian forms $(f, g) \longmapsto \langle f, g \rangle_t$ is continuous.

We say that two functions $f, g \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$ are *orthogonal* if $\langle f, g\rangle_t = 0$ for any $t \in I$. Of course, any orthonormal system is a free system over the ring $\mathcal{B}^s$. Let $A \subset \mathcal{H}^{2,s}_{\mathcal{F}}(U)$; the *orthogonal* of A is the subset:

$$A^{\perp} = \left\{ f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U) : \langle f, g\rangle_t = 0 \quad \text{for any} \quad g \in \mathcal{H}^{2,s}_{\mathcal{F}}(U) \quad \text{and any} \quad t \in I \right\}.$$

Since for any fixed $g \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$ the map $f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U) \longmapsto \langle f, g\rangle_t \in \mathcal{B}^0$ is $\mathcal{B}^s$-linear and continuous, $A^{\perp}$ is a closed submodule of the $\mathcal{B}^s$-module $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$.

4.3 Orthogonal projections in $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$

Let V be a closed $\mathcal{B}^s$-submodule of $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$. Then there exists a continuous $\mathcal{B}^s$-linear map $P : \mathcal{H}^{2,s}_{\mathcal{F}}(U) \longrightarrow V$ such that:

(i) For any $g \in V$, $||f - P(f)||^s_{2,U} \leq ||f - g||^s_{2,U}$ that is, $P(f)$ realizes the minimal "distance" from f to V.
(ii) For any $f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$ and any $v \in V$ we have: $\langle f - P(f), v\rangle_t = 0$ for any $t \in I$.
(iii) If V is non trivial the norm $|||P|||^s_{2,U}$ of P is equal to 1.
The map P is called the *orthogonal projection* from $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ on V. The proof of its existence is a slight adaptation of the classical one on a Hilbert space.

Proof (i) Let ε be a positive real number and $f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$. Let:

$$\delta_s = \inf_{v \in V} ||f - v||^s_{2,U}.$$

Then there exists a sequence (v_n) in V such that $\lim ||f - v_n||^s_{2,U} = 0$ that is:

$$\left(||f - v_n||^s_{2,U}\right)^2 \leq \delta^2 + \varepsilon^2$$

for n sufficiently large and also $J_k(f - v_n, U_t)^2 \leq \delta^2 + \varepsilon^2$ for any $t \in I$ and any $k = 0, 1, \ldots, s$. Let $t \in I$. By the parallelogram identity we have:

$$J_k((f - v_n) - (f - v_p), U_t)^2 + J_k((f - v_n) + (f - v_p), U_t)^2$$
$$= 2\{J_k((f - v_n), U_t)^2 + J_k((f - v_p), U_t)^2\}.$$

Thus

$$J_k(v_n - v_p, U_t)^2 = 2\left\{ J_k(f - v_n, U_t)^2 + J_k(f - v_p, U_t)^2 - 2J_k\left(f - \frac{v_n + v_p}{2}, U_t\right)^2 \right\}.$$

Since:

$$J_k\left(f - \frac{v_n + v_p}{2}, U_t\right)^2 \geq \delta^2$$

we obtain the inequality $J_k(v_n - v_p, U_t)^2 \leq \varepsilon^2$. Taking the upper bound over all $t \in I$, the maximum over $k = 0, \ldots, s$, and the square roots we get $||v_n - v_p||^s_{2,U} \leq \varepsilon$ which shows that (v_n) is a Cauchy sequence in V with respect to the norm $||\ \ ||^s_{2,U}$; since V is complete, this sequence converges to an element $v \in V$.

The uniqueness is a consequence of the fact that the construction of the projection is unique for each fixed $t \in I$ in the Hilbert space $\mathcal{H}^{2,s}_{\mathcal{F}}(U_t)$. We set $P(f) = v$. For the same reason we deduce the property (ii) and the inequality $||P(f)||^s_{2,U_t} \leq 1$ which implies assertion (iii) because P is the identity on V. $\diamondsuit$

4.4 Orthogonal Decomposition

This is an important consequence of the existence of orthogonal projections: *for any closed submodule* V of $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ *we have an orthogonal decomposition*: $\mathcal{H}^{2,s}_{\mathcal{F}}(U) = V \oplus V^{\perp}$.

A $\mathcal{F}$-*presheaf* of vector spaces $\mathcal{E}$ on M is a presheaf which associates to any $\mathcal{F}$-open set U a vector space $\mathcal{E}(U)$ with the same conditions of restriction as for a presheaf in the usual sense. It is a $\mathcal{F}$-*sheaf* if, in addition, it possesses the property of gluing local sections on $\mathcal{F}$-open covers to global ones. A $\mathcal{F}$-*sheaf* is *fine* if it is fine in the usual sense for $\mathcal{F}$-open covers. Let us give some:

4.5 Examples

(i) For any $\mathcal{F}$-open set U, we associate the space $\mathcal{H}^{b,s}_{\mathcal{F}}(U)$ of functions on U of class C^s which are $\mathcal{F}$-holomorphic with locally bounded derivatives $\frac{\partial^k f}{\partial t^k}$ up to the order s (locally bounded means bounded on subsets $K_0 \times I$ where K_0 is a compact set of M_0). Then we obtain a $\mathcal{F}$-sheaf $\mathcal{H}^{b,s}_{\mathcal{F}}$ on M.

(ii) For any $\mathcal{F}$-open set U, we associate the space $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ of functions on U which are of class C^s, $\mathcal{F}$-holomorphic and such that the quantities $J_k(f, U_t)$ previously defined are bounded for $k = 0, \ldots, s$. So we obtain a $\mathcal{F}$-presheaf $\mathcal{H}^{2,s}_{\mathcal{F}}$ on M.

(iii) For any $\mathcal{F}$-open set U, we associate the space $A^0_{b,s}(U)$ (resp. $A^1_{b,s}(U)$) of functions f [resp. foliated (0, 1)-forms] on U which are of class C^s, C^∞ along the leaves and whose transverse derivatives $\frac{\partial^s f}{\partial t^s}$ up to the order s are locally bounded. Then we obtain a $\mathcal{F}$-sheaf $\mathcal{A}^0_{b,s}$ (resp. $\mathcal{A}^1_{b,s}$).

Let $\mathcal{U} = \{U_i\}$ be a locally finite $\mathcal{F}$-open cover of $M = M_0 \times I$. Let $p : M_0 \times I \longrightarrow M_0$ be the first projection and denote by $\overline{U}_i$ the open set $p(U_i)$. Let $\overline{\rho}_i$ be a C^∞-partition of 1 on M_0 associated to the open cover $\overline{\mathcal{U}} = \{\overline{U}_i\}$ and set $\rho_i = \overline{\rho}_i \circ p$. For any i and any $s \in \mathbb{N}$, the function $\overline{\rho}_i$ is an element of $A^0_{b,s}(M)$ and the family $\{\rho_i\}$ is a

C^∞-partition of unity associated to the $\mathcal{F}$-open cover $\mathcal{U} = \{U_i\}$. Using this partition of unity $\{\rho_i\}$ one can easily prove that the two $\mathcal{F}$-sheaves $\mathcal{A}^0_{b,s}$ and $\mathcal{A}^1_{b,s}$ are fine.

4.6 Cohomology with Values in a $\mathcal{F}$-Sheaf

The definition is the same as in the classical case. Let $\mathcal{E}$ be a $\mathcal{F}$-sheaf on M and $\mathcal{U} = \{U_i\}$ a $\mathcal{F}$-open cover. For any integer $q \in \mathbb{N}$ we denote by $C^q(\mathcal{U}, \mathcal{E})$ the vector space of families $\{c_{i_0 \ldots i_q}\}$ where $c_{i_0 \ldots i_q}$ is an element of $\mathcal{E}(U_{i_0} \cap \cdots \cap U_{i_q})$ (which, by convention, is zero if the intersection $U_{i_0} \cap \cdots \cap U_{i_q}$ is empty). As usual, we define a linear operator $\delta : C^q(\mathcal{U}, \mathcal{E}) \longrightarrow C^{q+1}(\mathcal{U}, \mathcal{E})$ by:

$$(\delta c)_{i_0 \ldots i_{q+1}} = \sum_{i=0}^{q+1} (-1)^i c_{i_0 \ldots \hat{\imath} \ldots i_{q+1}}.$$

This operator satisfies the relation $\delta^2 = 0$; thus we obtain a differential complex:

$$0 \longrightarrow C^0(\mathcal{U}, \mathcal{E}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^q(\mathcal{U}, \mathcal{E}) \xrightarrow{\delta} C^{q+1}(\mathcal{U}, \mathcal{E}) \xrightarrow{\delta} \cdots$$

whose cohomology is denoted $H^*(\mathcal{U}, \mathcal{E})$. If $\mathcal{U}'$ is a $\mathcal{F}$-open cover finer than $\mathcal{U}$ we have a morphism induced by restriction $H^*(\mathcal{U}, \mathcal{E}) \longrightarrow H^*(\mathcal{U}', \mathcal{E})$. The cohomology $H^*(M, \mathcal{E})$ of M with values in the $\mathcal{F}$-sheaf will be, by definition, the inductive limit of $H^*(\mathcal{U}, \mathcal{E})$ over $\mathcal{F}$-open covers. The cohomology $H^*(M, \mathcal{E})$ satisfies all the usual known properties, for instance we have the:

4.7 Leray's Theorem

Let $\mathcal{U} = \{U_i\}$ be an acyclic $\mathcal{F}$-open cover that is, for any intersection $U_{i_0} \cap \cdots \cap U_{i_q}$, we have $H^q(U_{i_0} \cap \cdots \cap U_{i_q}, \mathcal{E}) = 0$ for any $q \geq 1$. Then $H^*(M, \mathcal{E}) = H^*(\mathcal{U}, \mathcal{E})$.

For the proof see Godement (1959). We can easily show that, if $\mathcal{E}$ is $\mathcal{F}$-fine, then $H^q(M, \mathcal{F}) = 0$ for $q \geq 1$. We have also the:

4.8 Abstract de Rham Theorem

Let $\mathcal{E}$ a $\mathcal{F}$. Suppose that F admits a resolution : $0 \longrightarrow \mathcal{E} \hookrightarrow \mathcal{E}^0 \xrightarrow{D_0} \mathcal{E}^1 \xrightarrow{D_1} \cdots$ where each $\mathcal{E}^q$ is a fine $\mathcal{F}$-sheaf. Then the cohomology $H^*(M, \mathcal{E})$ is naturally isomorphic to the cohomology of the differential complex: $0 \longrightarrow \mathcal{E}^0(M) \xrightarrow{D_0} \mathcal{E}^1(M) \xrightarrow{D_1} \cdots$ where, for each q, $\mathcal{E}^q(M)$ is the space of global sections of the $\mathcal{F}$-sheaf $\mathcal{E}^q$.

These two theorems will be of interest for our purpose. We can first remark that the $\mathcal{F}$-sheaf $\mathcal{H}^{b,s}_{\mathcal{F}}$ admits a fine resolution:

$$0 \longrightarrow \mathcal{H}_{\mathcal{F}}^{b,s} \hookrightarrow \mathcal{A}_{b,s}^{0} \xrightarrow{\overline{\partial}_{\mathcal{F}}} \mathcal{A}_{b,s}^{1} \longrightarrow 0.$$

Hence:

$$H^*(M, \mathcal{H}_{\mathcal{F}}^{b,s}) = A_{b,s}^1(M)/\mathrm{Im}\left(A_{b,s}^0(M) \xrightarrow{\overline{\partial}_{\mathcal{F}}} A_{b,s}^1(M)\right)$$

where $A_{b,s}^0(M)$ and $A_{b,s}^1(M)$ are the spaces of global sections respectively of the $\mathcal{F}$-sheaves $\mathcal{A}_{b,s}^0$ and $\mathcal{A}_{b,s}^1$.

Let $\mathcal{U}^* = \{U_i^*\}_{i=1,\dots,n}$ be a finite family of $\mathcal{F}$-open sets such that each one of them is equivalent to $\mathbb{D} \times I$ (where $\mathbb{D}$ is the open unit disc in $\mathbb{C}$) by a trivialization $\varphi_i : \mathbb{D} \times I \longrightarrow U_i^*$ of the foliation $\mathcal{F}$ restricted to U_i^*. Let $\mathcal{U} = \{U_i\}_{i=1,\dots,n}$ be an other family of $\mathcal{F}$-open sets such that $U_i \subset U_i^*$ for any $i = 1, \dots, n$. We shall introduce norms on the spaces $C^*(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s})$. Let $\eta = \{f_i\} \in C^0(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s})$ and $\zeta = \{f_{ij}\} =\in C^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s})$; we set:

$$||\eta||_{2,\mathcal{U}}^s = \sum_{i=1}^{n} ||f_i||_{2,U_i}^s \quad \text{and} \quad ||\zeta||_{2,\mathcal{U}}^s = \sum_{i,j}^{n} ||f_{ij}||_{2,U_{ij}}^s.$$

The 0-cocyles and the 1-cocycles constitute closed spaces $Z^0(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s})$ and $Z^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s})$ respectively of $C^0(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s})$ and $C^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s})$.

For each $i = 1, \dots, n$ let V_i be a relatively compact $\mathcal{F}$-open set of U_i and denote by $\mathcal{V}$ the $\mathcal{F}$-open cover $\{V_i\}_{i=1,\dots,n}$. For any $\zeta \in C^q(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{b,s})$, we have $||\zeta||_{2,\mathcal{V}}^s < +\infty$. Applying Lemma 4.1, we easily prove that, for any $\varepsilon > 0$, there exists a closed (FC)-submodule A in the $\mathcal{B}^s$-module $Z^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s})$ such that:

$$||\zeta||_{2,\mathcal{V}}^s \le \varepsilon ||\zeta||_{2,\mathcal{U}}^s \quad \text{for any} \quad \zeta \in A.$$

Lemma 4.9 *We take the same $\mathcal{F}$-open covers $\mathcal{U}^*$, $\mathcal{U}$ and $\mathcal{V}$ as before and we consider a fourth one $\mathcal{W} = \{W_i\}_{i=1,\dots,n}$. We suppose that $\mathcal{W} \ll \mathcal{V} \ll \mathcal{U} \ll \mathcal{U}^*$. Then, for any $s \in \mathbb{N}$, there exists a constant $C_s > 0$ such that, for any $\xi \in Z^1(\mathcal{V}, \mathcal{H}_{\mathcal{F}}^{2,s})$, there exists $\zeta \in Z^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s})$ and $\eta \in C^0(\mathcal{W}, \mathcal{H}_{\mathcal{F}}^{2,s})$ with $\zeta = \xi + \delta\eta$ on $\mathcal{W}$ and:*

$$\max\left(||\zeta||_{2,\mathcal{V}}^s, ||\eta||_{2,\mathcal{W}}^s\right) \le C_s ||\xi||_{2,\mathcal{U}}^s.$$

Proof Let $\xi = \{f_{ij}\} \in Z^1(\mathcal{V}, \mathcal{H}_{\mathcal{F}}^{2,s})$. Then, using a same method proving Theorem 3.1, we easily establish that $\xi \in Z^1(\mathcal{V}, \mathcal{A}_{b,s}^0)$. We have observed in Sect. 4.5. iii) that the $\mathcal{F}$-sheaf $\mathcal{A}_{b,s}^0$ of functions of class C^s whose transverse derivatives up to the order s are locally bounded is fine; hence $H^1(\mathcal{V}, \mathcal{A}_{b,s}^0) = 0$. Then there exists $\{g_i\} \in C^0(\mathcal{V}, \mathcal{A}_{b,s}^0)$ such that:

$$f_{ij} = g_j - g_i \quad \text{on} \quad V_i \cap V_j.$$

Since $\overline{\partial}_{\mathcal{F}} f_{ij} = 0$, we have $\overline{\partial}_{\mathcal{F}} g_i = \overline{\partial}_{\mathcal{F}} g_j$ on $V_i \cap V_j$; hence the collection of (0, 1)-forms $\{\overline{\partial}_{\mathcal{F}} g_i\}$ defines a foliated (0, 1)-form ω on the union $|\mathcal{V}| = V_1 \cup \cdots \cup V_n$ such that $\omega_{|V_i} = \overline{\partial}_{\mathcal{F}} g_i$. Because $|\mathcal{W}|$ is $\mathcal{F}$-relatively compact in $|\mathcal{V}|$, there exists a function $\psi \in A^0_{b,s}(M)$, basic for the fibration $(z, t) \in M = M_0 \times I \longmapsto z \in M_0$ and such that:

$$\operatorname{supp}(\psi) \subset |\mathcal{V}| \quad \text{and} \quad \psi_{||\mathcal{W}|} = 1.$$

Then $\psi\omega$ can be considered as an element of $A^1_{b,s}(|\mathcal{U}^*|)$. Since, by Ahlfors-Bers Theorem Ahlfors and Bers (1960), for each $i \in \{1, \ldots, n\}$, the $\mathcal{F}$-open set U_i^* is isomorphic to the product $\mathbb{D} \times I$, by Proposition 2.3 there exists a function $h_i \in A^0_{b,s}(U_i^*)$ such that $\overline{\partial}_{\mathcal{F}} h_i = \psi\omega_{|U_i^*}$. But:

$$\overline{\partial}_{\mathcal{F}} h_i = \overline{\partial}_{\mathcal{F}} h_j \quad \text{on} \quad U_i^* \cap U_j^*;$$

thus $F_{ij} = h_j - h_i \in \mathcal{H}^{b,s}_{\mathcal{F}}(U_i^* \cap U_j^*)$. Denote by ζ the cocycle $\{F_{ij}\}$; since $\mathcal{U} \ll \mathcal{U}^*$, $\zeta \in Z^1(\mathcal{U}, \mathcal{H}^{2,s}_{\mathcal{F}})$. On W_i we have $\overline{\partial}_{\mathcal{F}} h_i = \psi\omega = \omega = \overline{\partial}_{\mathcal{F}} g_i$, $h_i - g_i \in \mathcal{H}^{b,s}_{\mathcal{F}}(W_i)$ and also $h_i - g_i \in \mathcal{H}^{2,s}_{\mathcal{F}}(W_i)$ that is, the 0-cochain $\eta = \{h_i - g_i\}$ is in $C^0(\mathcal{W}, \mathcal{H}^{2,s}_{\mathcal{F}})$. So we easily see that, on $W_i \cap W_j$:

$$F_{ij} - f_{ij} = (h_j - g_j) - (h_i - g_i) \quad i.e. \quad \zeta - \xi = \delta\eta \text{ on } \mathcal{W}$$

which is exactly the desired relation.

Now, let:

$$E = Z^1(\mathcal{U}, \mathcal{H}^{2,s}_{\mathcal{F}}) \times Z^1(\mathcal{V}, \mathcal{H}^{2,s}_{\mathcal{F}}) \times C^0(\mathcal{W}, \mathcal{H}^{2,s}_{\mathcal{F}}).$$

Equipped with norm $||(\zeta, \xi, \eta)||^s_E = ||\zeta||^s_{2,\mathcal{U}} + ||\xi||^s_{2,\mathcal{V}} + ||\eta||^s_{2,\mathcal{W}}$, E is a Banach space. The subspace:

$$L = \{(\zeta, \xi, \eta) \in E : \zeta = \xi + \delta\eta \quad \text{on} \quad \mathcal{W}\}$$

is closed and then is a Banach space. Since the map:

$$\pi : (\zeta, \xi, \eta) \in L \longmapsto \xi \in Z^1(\mathcal{V}, \mathcal{H}^{2,s}_{\mathcal{F}})$$

is continuous and surjective, it is also open (by the Open Map Theorem). Hence there exists a constant $C_s > 0$ such that, for any $\xi \in Z^1(\mathcal{V}, \mathcal{H}^{2,s}_{\mathcal{F}})$, there exists $(\zeta, \xi, \eta) \in L$ satisfying $\pi((\zeta, \xi, \eta)) = \xi$ and:

$$||(\zeta, \xi, \eta)||^s_E \le C_s ||\xi||^s_{2,\mathcal{V}}.$$

The constant C_s satisfies the desired inequality. ◇

Lemma 4.10 *Let the hypotheses be like in Lemma 4.9. There exists a finitely generated $\mathcal{B}^s$-submodule $S \subset Z^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{b,s})$ satisfying the following property: for any $\xi \in Z^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{b,s})$, there exists $\sigma \in S$ and $\eta \in C^0(\mathcal{W}, \mathcal{H}_{\mathcal{F}}^{b,s})$ such that:*

$$\sigma = \xi + \delta\eta \quad on\ \mathcal{W}.$$

This means that the image of the natural $\mathcal{B}^s$-linear map $H^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{b,s}) \longrightarrow H^1(\mathcal{W}, \mathcal{H}_{\mathcal{F}}^{b,s})$ is a finitely generated $\mathcal{B}^s$-submodule.

Proof Let C_s be the constant given in Lemma 4.9 and set $\varepsilon = \frac{1}{2C_s}$. By the Lemma 4.1 there exists a closed (FC)-submodule $A \subset Z^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s})$ such that:

$$||\xi||_{2,\mathcal{V}}^s \leq \varepsilon ||\xi||_{2,\mathcal{U}}^s \quad \text{for any} \quad \xi \in \text{A}.$$

Let S be the orthogonal of A in $Z^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s})$ [which is a finitely generated $\mathcal{B}^s$-submodule because A is a (FC)-submodule] that is:

$$Z^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s}) = A \oplus S.$$

Let $\xi \in Z^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{b,s})$. Since $\mathcal{V} \ll \mathcal{U}$, ξ is in fact in $Z^1(\mathcal{V}, \mathcal{H}_{\mathcal{F}}^{2,s})$; let $\tau = ||\xi||_{2,\mathcal{V}}^s < +\infty$. By Lemma 4.9, there exists $\zeta_0 \in Z^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s})$ and $\eta_0 \in C^0(\mathcal{W}, \mathcal{H}_{\mathcal{F}}^{2,s})$ such that:

$$\zeta_0 = \xi + \delta\eta_0 \quad \text{on} \quad \mathcal{W}$$

with the inequalities $||\zeta_0||_{2,\mathcal{V}}^s \leq C_s\tau$ and $||\eta_0||_{2,\mathcal{V}}^s \leq C_s\tau$. On the other hand, ζ_0 decomposes into a sum:

$$\zeta_0 = \xi_0 + \sigma_0 \quad \text{with} \quad \xi_0 \in A \quad \text{and} \quad \sigma_0 \in S.$$

Now we shall construct elements:

$$\zeta_k \in Z^1(\mathcal{U}, \mathcal{H}_{\mathcal{F}}^{2,s}), \quad \eta_k \in C^0(\mathcal{W}, \mathcal{H}_{\mathcal{F}}^{2,s}), \quad \xi_k \in A \quad \text{and} \quad \sigma_k \in S$$

with:

(i) $\zeta_k = \xi_{k-1} + \delta\eta_k$ on $\mathcal{W}$;
(ii) $\zeta_k = \xi_k + \sigma_k$ (orthogonal decomposition);
(iii) $||\zeta_k||_{2,\mathcal{U}}^s \leq \frac{C_s\tau}{2^k}$ and $||\eta_k||_{2,\mathcal{U}}^s \leq \frac{C_s\tau}{2^k}$.

Suppose that these elements are constructed up to the rank k. Since $\zeta_k = \xi_k + \sigma_k$ we have by the orthogonal decomposition:

$$||\xi_k||_{2,\mathcal{U}}^s \leq ||\zeta_k||_{2,\mathcal{U}}^s \leq \frac{C_s\tau}{2^k}.$$

This gives:

$$||\xi_k||^s_{2,\mathcal{V}} \leq \varepsilon ||\xi_k||^s_{2,\mathcal{U}} \leq \frac{\varepsilon C_s \tau}{2^k} \leq \frac{\tau}{2^{k+1}}.$$

By Lemma 4.9 there exists $\zeta_{k+1} \in Z^1(\mathcal{U}, \mathcal{H}^{2,s}_{\mathcal{F}})$ and $\eta_{k+1} \in C^0(\mathcal{W}, \mathcal{H}^{2,s}_{\mathcal{F}})$ such that:

$$\zeta_{k+1} = \xi_k + \delta\eta_{k+1} \quad \text{on} \quad \mathcal{W}.$$

and

$$\max\left(||\zeta_{k+1}||^s_{2,\mathcal{U}}, ||\eta_{k+1}||^s_{2,\mathcal{W}}\right) \leq \frac{C_s \tau}{2^{k+1}}.$$

The element ζ_{k+1} admits an orthogonal decomposition:

$$\zeta_{k+1} = \xi_{k+1} + \sigma_{k+1} \;\text{ with }\; \xi_{k+1} \in A \text{ and } \sigma_{k+1} \in S.$$

Then we have constructed, up to the rank $k+1$, the sequences (ξ_k), (ζ_k), (η_k) and (σ_k) with the desired properties. By $\zeta_0 = \xi + \delta\eta_0$ and the points (i) and (ii) we have (up to rank k):

$$(*) \qquad \xi_k + \sum_{\ell=0}^{k} \sigma_\ell = \xi + \delta\left(\sum_{\ell=0}^{k} \eta_\ell\right) \text{ on } \mathcal{W}.$$

From (ii) and (iii) we deduce that:

$$\max\left(||\xi_\ell||^s_{2,\mathcal{U}}, ||\sigma_\ell||^s_{2,\mathcal{U}}, ||\eta_\ell||^s_{2,\mathcal{W}}\right) \leq \frac{C_s \tau}{2^k}.$$

Thus $\lim_k \xi_k = 0$ and the series $\sum_{k=0}^{\infty} \sigma_k$ and $\sum_{k=0}^{\infty} \eta_k$ converge respectively to elements $\sigma \in S$ and $\eta \in C^0(\mathcal{W}, \mathcal{H}^{2,s}_{\mathcal{F}})$. In fact, by Theorem 3.1:

$$\sigma \in S \cap Z^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}}) \text{ and } \eta \in C^0(\mathcal{W}, \mathcal{H}^{b,s}_{\mathcal{F}}).$$

From $(*)$ we obtain $\sigma = \xi + \delta\eta$ on $\mathcal{W}$. This ends the proof of the lemma. ◇

Theorem 4.11 *Suppose that $M' = M'_0 \times I$ and $M'' = M''_0 \times I$ are $\mathcal{F}$-open sets of M an that M'_0 is contained and relatively compact in M''_0. Then the image of the natural morphism $H^1(M'', \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(M', \mathcal{H}^{b,s}_{\mathcal{F}})$ induced by the restriction is a finitely generated $\mathcal{B}^s$-module.*

Proof Let $\mathcal{U}^*, \mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ four $\mathcal{F}$-open covers as in Lemma 4.10 and such that:

(i) $M' \subset \bigcup_{i=1}^n W_i =: M_1$ and is relatively compact in $M_2 := \bigcup_{i=1}^n U_i \subset M''$;
(ii) the U_i^*, U_i and W_i are isomorphic to $\mathbb{D} \times I$.

By Lemma 4.10, the image of the morphism $H^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(\mathcal{W}, \mathcal{H}^{b,s}_{\mathcal{F}})$ is a finitely generated $\mathcal{B}^s$-module. On the other hand, the $\mathcal{F}$-open covers $\mathcal{U}$ and $\mathcal{W}$ are acyclic; then by Leray's theorem, $H^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}}) = H^1(M_2, \mathcal{H}^{b,s}_{\mathcal{F}})$ and $H^1(\mathcal{W}, \mathcal{H}^{b,s}_{\mathcal{F}}) = H^1(M_1, \mathcal{H}^{b,s}_{\mathcal{F}})$. Then the result follows from the canonical factorization:

$$H^1(M'', \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(M_2, \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(M_1, \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(M', \mathcal{H}^{b,s}_{\mathcal{F}}).$$

The theorem is then proved. $\diamond$

Corollary 4.12 *Suppose that all leaves are compact (they are all diffeomorphic to a compact Riemann surface of genus $g \geq 2$). Then $H^1(M, \mathcal{H}^{b,s}_{\mathcal{F}})$ is a finitely generated $\mathcal{B}^s$-module.*

Question 4.13 *Suppose that all leaves are compact. Is the $\mathcal{B}^s$-module $H^1(M, \mathcal{H}^{b,s}_{\mathcal{F}})$ free? If this is the case, is its dimension equal g?*

Now we have amassed all that is necessary to prove the Main Theorem. It is immediate to see that it follows from the following one.

Theorem 4.14 *Let $M' = M'_0 \times I$ be a $\mathcal{F}$-open set of M with M'_0 relatively compact and strictly contained in M_0. Then for any $a \in M'_0$, there exists a $\mathcal{F}$-meromorphic function $f : M' \longrightarrow \mathcal{C}$ nonconstant on any leaf, with set of poles $\{a\} \times I$ and $\mathcal{F}$-holomorphic on $M' \backslash \{a\} \times I$.*

Let us first recall a result on modules of finite type illustrated in the following lemma. Its proof can be found for instance in Atiyah and McDonald (1969) (Proposition 2.4 p. 21).

Lemma 4.15 *Let E be a finitely generated module over a ring R, $\mathfrak{a}$ an ideal of R and θ an R-endomorphism of E such that $\theta(E) \subset \mathfrak{a}E$. Then there exists $a_1, \ldots, a_n \in \mathfrak{a}$ such that $\theta^n + a_1\theta^{n-1} + \cdots + a_{n-1}\theta + a_n = 0$.*

Proof of Theorem 4.14 Let U_1 be a $\mathcal{F}$-open neighborhood of $\{a\} \times I$ isomorphic to $\mathbb{D} \times I$ by a diffeomorphism (which is a biholomorphism on the leaves) $\varphi_1 : U_1 \longrightarrow \mathbb{D} \times I$ sending $\{a\} \times I$ on $\{0\} \times I$; by restricting the open set U_1 if necessary we can assume that the transverse derivatives (up to the order s) of φ_1 are bounded. Denote by U_2 the open set $M \backslash \{a\} \times I$. Then $\mathcal{U} = \{U_1, U_2\}$ is a $\mathcal{F}$-open cover of M. For each $j \in \mathbb{N}^*$, the function $\frac{1}{\varphi(\cdot,0)^j}$ is in $\mathcal{H}^{b,s}_{\mathcal{F}}(U_1 \cap U_2)$ and represents a cocycle $\zeta_j \in Z^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}})$. By Lemma 4.10 the image of the $\mathcal{B}^s$-morphism:

$$H^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(\mathcal{U} \cap M', \mathcal{H}^{b,s}_{\mathcal{F}})$$

is a finitely generated $\mathcal{B}^s$-module E. Applying Lemma 4.15 to the $\mathcal{B}^s$-module E, the ring $R = \mathcal{B}^s$, its ideal $\mathfrak{a} = \mathcal{B}^s$ and the morphism $\theta : g \in E \longmapsto \zeta_1 g \in E$ one can find elements $a_1, \ldots, a_n \in \mathcal{B}^s$ such that: $\theta^n + a_1\theta^{n-1} + \cdots + a_{n-1}\theta + a_n = 0$. The value of the morphism $\Theta = \theta^n + a_1\theta^{n-1} + \cdots + a_{n-1}\theta + a_n$ at the constant function $\chi = 1$ gives a cocyle:

$$\Theta(\chi) = \theta^n(\chi) + a_1\theta^{n-1}(\chi) + \cdots + a_{n-1}\theta(\chi) + a_n(\chi)$$

which is cohomologous to zero. This means that there exist elements $c_1, \ldots, c_{n+1} \in \mathcal{B}^s$ and a 0-cochain $\eta = \{f_1, f_2\} \in C^0(\mathcal{U} \cap M', \mathcal{H}_{\mathcal{F}}^{b,s})$ such that:

$$c_1\zeta_1 + \cdots + c_{n+1}\zeta_{n+1} = \delta\eta \quad \text{on} \quad \mathcal{U} \cap M'$$

that is:

$$c_1\zeta_1 + \cdots + c_{n+1}\zeta_{n+1} = f_2 - f_1 \quad \text{on} \quad U_1 \cap U_2 \cap M'.$$

The desired $\mathcal{F}$-meromorphic function on M' is defined by: $c_1\zeta_1 + \cdots + c_{n+1}\zeta_{n+1} + f_1$ on $U_1 \cap M'$ and f_2 on $U_2 \cap M'$. $\diamondsuit$

Corollary 4.16 *We take the same hypotheses as previously and suppose that M' is not the whole manifold M. Then there exists a $\mathcal{F}$-holomorphic function $f : M' \longrightarrow \mathbb{C}$ which is not constant on any leaf of any connected component of M'.*

Proof Let $M'' = M_0'' \times I$ be a $\mathcal{F}$-open set of M where M_0'' is a relatively compact open set of M_0 containing M_0' in which the latter is relatively compact. We apply then the preceding theorem by taking $a \in M_0'' \backslash M_0'$. $\diamondsuit$

5 Examples

In this section we give examples of differentiably trivial fibrations but far to be even locally trivial in the complex sense.

5.1 Leaves are Simply Connected

Let $\pi : M \longrightarrow B$ be a differentiably trivial fibration whose fibers are holomorphically equivalent to the unit disc $\mathbb{D}$ (or the half plane $\mathbb{H}$). Then, by Ahlfors–Bers Theorem (Ahlfors and Bers 1960) it is isomorphic to the product $\mathbb{D} \times B$ as a complex foliation. As this case is not interesting for us here we shall give an example with parabolic leaves that is, each leaf is individually isomorphic to $\mathbb{C}$ but the complex foliation we obtain is not equivalent to a complex product.

Denote by $P^1(\mathbb{C})$ the complex projective space of dimension one. Let $I =]0, 1[$ and $\phi : I \longrightarrow P^1(\mathbb{C})$ be a C^k-map which is not C^{k+1}. Let $M = P^1(\mathbb{C}) \times I \backslash \mathcal{G}$ where $\mathcal{G}$ is the graph of ϕ. This is a differentiable trivial fibration over I all of whose fibers are isomorphic to $\mathbb{C}$ and the complex structure on the fibers varies in a C^∞ way in the transverse direction. We obtain a complex foliation $\mathcal{F}$ whose leaves are the fibers of the trivial fibration $\pi : M \longrightarrow I$ where π is the restriction to M of the second projection $(z, t) \in P^1(\mathbb{C}) \times I \longmapsto t \in I$.

The complex foliation $\mathcal{F}$ on M constructed above is not C^{k+1}-equivalent to the complex product $\mathbb{C} \times I$.

Indeed, suppose that there exists a C^{k+1}-diffeomorphism $\Psi : \mathbb{C} \times I \longrightarrow M$ which is holomorphic between the fibers. In the coordinates on M given by its inclusion in $P^1(\mathbb{C}) \times I$, this map Ψ is necessarily of the form:

$$\Psi(z, t) = \left(t, \frac{a(t)z + b(t)}{c(t)z + d(t)}\right)$$

(where, for each fixed $t \in I$, $\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$ is a matrix in $\mathrm{SL}(2, \mathbb{C})$) because any holomorphic embedding of $\mathbb{C}$ in $P^1(\mathbb{C})$ is given by a Moëbius map. From the fact that Ψ is C^{k+1} we see that the functions a, b, c and d are C^{k+1} on t. Then ϕ is also C^{k+1} because:

$$\phi(t) = \Psi(\infty, t) = \frac{a(t)}{b(t)}.$$

5.2 Leaves are Annulus

The following example is more or less the one given in Sect. 1.2 for the open set N (Example v). Let $I =]0, +\infty[$ and $\widetilde{M} = \mathbb{H} \times I$. We have an action Φ of $\mathbb{Z}$ on $\widetilde{M}$ given by :

$$\Phi(k, (z, t)) = (t^k z, t).$$

This action is free and proper; moreover it is holomorphic on each leaf $\mathbb{H} \times \{t\}$ of the product complex foliation $\widetilde{\mathcal{F}}$. Then it defines a complex foliation $\mathcal{F}$ on the quotient $M = \widetilde{M}/\Phi$ which is differentiably isomorphic to the product $(\mathbb{H}/\Phi_t) \times I$ where Φ_t is the loxodromy $\Phi_t(z) = tz$. The complex foliation $\mathcal{F}$ is not a locally trivial fibration. Indeed, because each leaf $(\mathbb{H}/\Phi_t) \times \{t\}$ is an annulus whose complex structure is coded by the ratio t, two different leaves $\mathbb{H}/\Phi_t \times \{t\}$ and $\mathbb{H}/\Phi_{t'} \times \{t'\}$ with $t \neq t'$) cannot be holomorphically equivalent.

5.3 Remark

In Example 5.1 the leaves are simply connected (all parabolic) and in Example 5.2 they have $\mathbb{Z}$ as common fundamental group. In these two cases it was proved in El Kacimi Alaoui (2010) that the first foliated Dolbeault cohomology group $H^{01}_{\mathcal{F}}(M)$ is trivial. This permits to give a more stronger foliated version of Mittag–Leffler Theorem.

5.4 Fundamental Group of Leaves is Non Abelian

Let $\widetilde{M} = \mathbb{H} \times I$. Fon any $t \in I$ let ϕ_t be the Moëbius transformation of $\mathbb{H}$ defined by:

$$\phi_t(z) = \frac{z+1}{tz + (1+t)}.$$

The family of matrices Θ_t (indexed by t) in the group $\mathrm{SL}(2, \mathbb{R})$ corresponding to the family ϕ_t is:

$$\Theta_t = \begin{pmatrix} 1 & 1 \\ t & (1+t) \end{pmatrix}.$$

Easy calculations show that, on the interval $I =]-\frac{1}{2}, 0[$:

- The matrices Θ_t and $\Theta_{t'}$ have different eigenvalues for $t \neq t'$; then Θ_t and $\Theta_{t'}$ are not conjugated in $\mathrm{SL}(2, \mathbb{R})$.
- Each ϕ_t has a unique fixed point $z_0(t)$ in $\mathbb{H}$.
- The family $\{z_0(t)\}_{t \in I}$ is the graph $\mathcal{G}$ in $\mathbb{H} \times I$ of a C^∞-function $\alpha : I \longrightarrow \mathbb{H}$.

Let a be a point in $\mathbb{H}$ different from $z_0(t)$ for any $t \in I$. For each $t \in I$, let $\mathcal{O}_t(a)$ be the orbit of a under the action of ϕ_t. Let:

$$\widetilde{M} = \mathbb{H} \times I \setminus \left\{ \mathcal{G} \cup \left(\bigcup_{t \in I} \mathcal{O}_t(a) \right) \right\}.$$

For each $t \in I$, $\widetilde{M}$ is a ϕ_t-invariant open set of $\mathbb{H} \times I$ and then it supports the action Ψ of $\mathbb{Z}$ defined by:

$$\Psi(k, (z, t)) = (\phi_t^k(z), t).$$

This action is free and proper; so the quotient $M = \widetilde{M}/\Psi$ is a manifold diffeomorphic to a product $M_0 \times I$ (where M_0 is a noncompact Riemann surface) with a complex foliation $\mathcal{F}$. Each leaf L_t of $\mathcal{F}$ is the quotient of:

$$\mathbb{H} \setminus (\mathcal{O}_t(a) \cup \{z_0(t)\})$$

by the automorphism ϕ_t. Because the matrices Θ_t and $\Theta_{t'}$ are not conjugated for $t \neq t'$, the two leaves L_t and $L_{t'}$ are not holomorphically equivalent. Then the foliation is not a locally trivial fibration in the complex sense.

In this example all leaves are diffeomorphic. Then they have the same fundamental group: the free non Abelain group generated by a countable infinite set.

5.5 All Leaves are Compact

Let $\varphi : \omega \longmapsto \omega' = \frac{a\omega+b}{c\omega+d}$ be a non trivial biholomorphism of $\mathbb{H}$ [where $\binom{a\ b}{c\ d}$ is an element of $\mathrm{SL}(2, \mathbb{R})$]. The map:

$$\Phi : (p, (z, \omega)) \in \mathbb{Z} \times \mathbb{C}^* \times \mathbb{H} \longmapsto (e^{-ip\varphi(\omega)} z, \omega) \in \mathbb{C}^* \times \mathbb{H}$$

is a free and proper holomorphic action of $\mathbb{Z}$ on $\widehat{M} = \mathbb{C}^* \times \mathbb{H}$. It preserves the foliation $\widehat{\mathcal{F}}$ whose leaves are the factors $\mathbb{C}^* \times \{\omega\}$ (in fact the action Φ preserves each leaf individually). The quotient space $M = \mathbb{C}^* \times \mathbb{H}/\Phi$ is a complex manifold of dimension 2. The induced complex foliation $\mathcal{F}$ on M has dimension 1 and all its leaves are elliptic curves $\mathbb{T}_\omega$; the complex structure of each $\mathbb{T}_\omega$ depends on $\omega \in \mathbb{H}$. Two leaves $\mathbb{T}_\omega$ and $\mathbb{T}_{\omega'}$ are isomorphic if, and only if, there exists a matrix $B \in \mathrm{SL}(2, \mathbb{Z})$ such that $\varphi(\omega') = B\varphi(\omega)$. The complex equivalence class of a leaf is then a countable set. Hence this foliation is not a locally trivial complex fibration even if it is a differentiable product.

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