Some geometric aspects of polygons

by

Aziz EL KACIMI ALAOUI & Abdellatif ZEGGAR

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We describe all families of star-shaped n-polygons with prescribed perimeter and area ; they are leaves of a foliation $\mathcal{F}^{(*)}$ on the space $\mathcal{P}_n^*$ of star-shaped polygons. By the way, we study some geometric properties of convex polygons, for instance their inscriptibility in a circle and their regularity in relation with the perimeter and the area.

We denote by $\mathbb{E}$ an Euclidean vector plane equipped with its canonical affine structure and an orientation given by an orthonormal basis. The word 'isometry' means a transformation of $\mathbb{E}$ which preserves the distance; necessarily it is an affine transformation of $\mathbb{E}$. The origin of $\mathbb{E}$ is denoted O. For $A, B \in \mathbb{E}$ we denote $\overrightarrow{AB}$ the vector $B - A$. Let $n \geq 3$ be a natural integer.

0. Introduction

A *figure* of the plane is just a part of it. But usually this name is given only to a one having a certain peculiarity : we see it all (it is bounded) or at least we understand how it is made to guess its behavior when it escapes our view, like a straight line... and its outline has a little regularity. A *polygon* is an example of such figure : it is bounded and bordered by a finite number of segments called *sides* or *edges*. To each polygon, one can associate invariants, among which are two real numbers that play an important role : the *perimeter* and the *area*.

The polygons of the plane are numerous and their shapes and sizes are varied. So the question of their equivalence therefore arises naturally. But in which sense?

From the set point of view, two polygons $\wp$ and $\wp'$ are always equivalent : there is a bijection of $\mathbb{E}$ which sends one on the other. But as we are in a Euclidean plane, we would like this bijection to preserve at least one of the properties related to the affine Euclidean structure. There are several notions of equivalence; here are some of them (those that will interest us directly). We will say that $\wp$ and $\wp'$ are :

1. *isometric* if there exists an isometry $f : \mathbb{E} \longrightarrow \mathbb{E}$ such that $f(\wp) = \wp'$. We can superimpose them; and we can still do that without going out of the plan if $\wp$ and $\wp'$ are *directly isometric*, that is f preserves the orientation;
2. *similar* if there is a similarity $f : \mathbb{E} \longrightarrow \mathbb{E}$ such that $f(\wp) = \wp'$. In a way, one of them is an enlargement of the other (as for the photos);
3. *equivalent* (shortly) if they have the same area. In this case, we can always go from one to another by cutting and geometric gluing;
4. *isoperimetric* if they have the same perimeter.

The isometric equivalence is the strongest and implies all the others. So it is too rigid to be 'useful' : two isometric polygons differ only in the positions they occupy in the plane. It is rather the equivalences 3 and 4 that will occupy us here.

This paper takes its origin from the following question $(*)$: *Are there two non-isometric triangles with the same perimeter and same area*? It led us first to the construction of a foliation $\mathcal{F}^{(*)}$ on the space of triangles : each leaf consists of triangles having same perimeter and same area. Then we made a more general study for the space of star-shaped polygons; this space contains convex polygons for which some properties related to area and perimeter have also been studied. Some of them are certainly known, but little absent in the 'geometric literature', which motivated us to insert them in this text.

$(*)$ It was asked by Geoffrey Letellier to his teacher Valerio Vassallo who put it in turn to the first author. The latter has built for this purpose the foliation $\mathcal{F}$ on the space of the triangles. But just to the question asked, Geoffrey himself responded by constructing a one parameter family of isosceles triangles having the same perimeter and the same area (his example is given in subsection 5.6).

In all this paper the integer n we consider is equal or greater than 3.

1. Spaces of polygons

A non degenerate *polygon* (or n-polygon) of $\mathbb{E}$ is an element $\wp = (M_1, \cdots, M_n)$ of $\mathbb{E}^n$ such that :

(i) for $i \neq j$ the point M_i is distinct from M_j ;

(ii) for any $k \in \{1, \cdots, n\}$, the oriented angle $\widehat{M_k} = (\overrightarrow{M_kM_{k+1}}, \overrightarrow{M_kM_{k-1}})$ has its measure in $]0, 2\pi[\setminus\{\pi\}$.

By convention, we set $M_0 = M_n$ and $M_{n+1} = M_1$. The orientation of the angle $\widehat{M_k}$ is the same as the orientation of the trigonometric circle centered at the point M_k.

The points M_k and the segments $M_kM_{k+1} = [M_k, M_{k+1}]$ are respectively the *vertices* and the *sides* of the polygon $\wp$. If M_i and M_j are two non successive vertices, that is $|i-j| > 1$, we say that the segment M_iM_j is a *diagonal* of the polygon.

Recall that a polygon is said to be :

- *equilateral* if all its sides have the same length ;
- *inscribable* if all its vertices are on a same circle ;
- *regular* if it is both inscribable and equilateral.

The set of all n-polygons of the plane $\mathbb{E}$ will be denoted $\widetilde{\mathcal{P}}_n$. *Centered* elements of $\widetilde{\mathcal{P}}_n$ are polygons for which the center of gravity is the origin O of $\mathbb{E}$; their set is denoted $\widetilde{\mathcal{P}}_{n,c}$.

Let f be an isometry of the Euclidean plane $\mathbb{E}$. The image $\wp' = (M'_1, \cdots, M'_n) = (f(M_1), \cdots, f(M_n))$ of a polygon $\wp = (M_1, \cdots, M_n)$ is a polygon of $\mathbb{E}$ such that :

$$\widehat{M'_k} = \widehat{M_k} \;\text{ and }\; ||\overrightarrow{M'_kM'_\ell}|| = ||\overrightarrow{M_kM_\ell}|| \;\text{ for }\; (k,\ell) \in \{1, \cdots, n\}^2. \tag{1.1}$$

So we have a natural action :

$$\mathrm{Isom}(\mathbb{E}) \times \widetilde{\mathcal{P}}_n \longrightarrow \widetilde{\mathcal{P}}_n,\ \big(f, (M_1, \cdots, M_n)\big) \longmapsto \big(f(M_1), \cdots, f(M_n)\big) \tag{1.2}$$

where Isom($\mathbb{E}$) is the group of the affine isometries of $\mathbb{E}$. The quotient space of this action will be denoted :

$$\mathcal{P}_n = \widetilde{\mathcal{P}}_n/\mathrm{Isom}(\mathbb{E}). \tag{1.3}$$

1.1. Definition. *The elements of $\mathcal{P}_n$ are called* ***geometric polygons*** *of $\mathbb{E}$.*

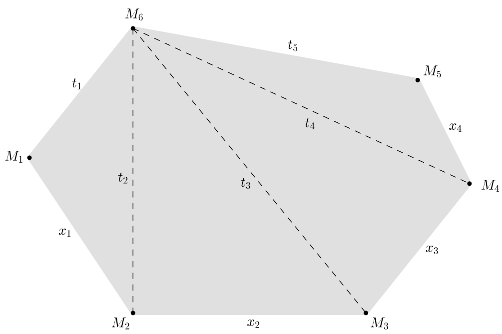
A geometric polygon of $\mathbb{E}$ is said to be *equilateral* (resp. *inscribable*) if it admits a representative which is equilateral (resp. inscribable).

For an element $\wp = (M_1, \cdots, M_n)$ of $\widetilde{\mathcal{P}}_n$, we will use the following notations :

$$\begin{cases} < M_1, \cdots, M_n > \text{ is the equivalence class of } (M_1, \cdots, M_n) \text{ in } \widetilde{\mathcal{P}}_n, \\ x_k = M_kM_{k+1} = ||\overrightarrow{M_kM_{k+1}}|| \text{ for } k \in \{1, \cdots, n-2\}, \\ t_k = M_nM_k = ||\overrightarrow{M_nM_k}|| \text{ for } k \in \{1, \cdots, n-1\}. \end{cases} \tag{1.4}$$

The positive numbers $t_1, x_1, \cdots, x_{n-2}, t_{n-1}$ are the lengths of the sides and $t_2, t_3, \cdots, t_{n-2}$ are the lengths of the diagonals from the vertex M_n (see the picture bellow for the case of the hexagon). We have $(n-2)$ lengths of type x_k and $(n-1)$ lengths of type t_k.

Since an isometry preserves the distance in the Euclidean plane $\mathbb{E}$, $(t_1, x_1, t_2, x_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1})$ does not depend on the choice of the representative $(M_1, \cdots, M_n)$ of the geometric polygon $< M_1, \cdots, M_n >$.



For any triangle (M_k, M_n, M_{k+1}) with $1 \leq k \leq n-2$, the lengths of the sides t_k, x_k, t_{k+1} satisfy the inequalities :

$$(1.5) \qquad \begin{cases} 0 < t_k < x_k + t_{k+1} \\ 0 < x_k < t_k + t_{k+1} \\ 0 < t_{k+1} < x_k + t_k \end{cases}$$

that is, the triplet (t_k, x_k, t_{k+1}) is an element of the open set $\mathcal{V}_3$ of $\mathbb{R}^3$ consisting of the triplets (x, y, z) satisfying :

$$\begin{cases} 0 < x < y + z \\ 0 < y < x + z \\ 0 < z < x + y. \end{cases}$$

Moreover, one can verify by induction that, for the lengths of the sides of a polygon, each length is strictly smaller than the sum of the others.

For instance, we have the following nice :

1.2. Theorem [Pen]. *For any natural integer $n \geq 3$ and any n-tuple $u = (u_1, \cdots, u_n)$ of positive real numbers, there exists a unique inscribable geometric polygon $< M_1, \cdots, M_n >$ such that $M_k M_{k+1} = u_k$ for $k \in \{1, \cdots, n\}$ if and only if, for any $j \in \{1, \cdots, n\}$, we have the inequality :*

$$u_j < \sum_{k \neq j} u_k.$$

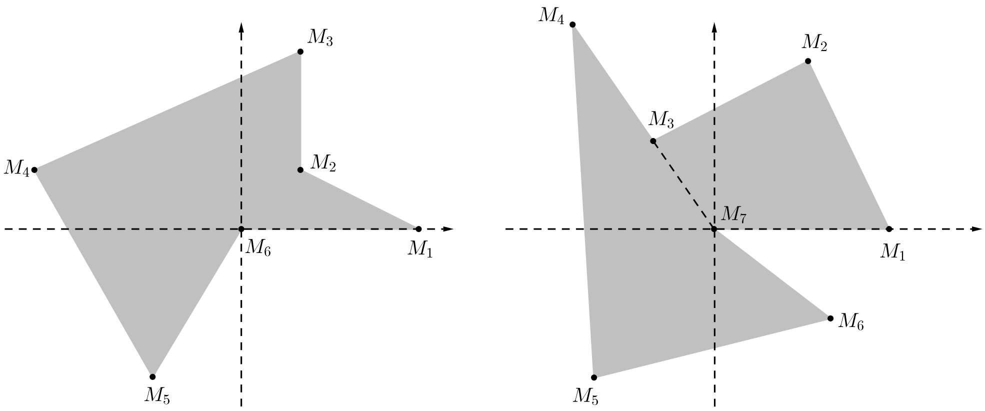
1.3. Definition. *A n-polygon $\wp = (M_1, \cdots, M_n)$ of $\mathbb{E}$ is **convex** if, for any $k \in \{1, \cdots, n\}$, the oriented angle $\widehat{M_k} = (\overrightarrow{M_k M_{k+1}}, \overrightarrow{M_k M_{k-1}})$ has its measure in $]0, \pi[$. It is called **star-shaped polygon** with respect to the vertex M if, for any vertex $N \in \{M_1, \cdots, M_n\}$ the segment $[MN]$ is contained in $\wp$ and for any $k \in \{1, \cdots n-2\}$ the angle $M_k \widehat{M_n} M_{k+1}$ has measure in $]0, \pi[$. Of course, any convex polygon is a star-shaped polygon with respect to any of its vertices.*

Star-shaped polygons form a subspace $\widetilde{\mathcal{P}}_n^*$ of $\widetilde{\mathcal{P}}_n$, invariant under the action of Isom($\mathbb{E}$) (on $\widetilde{\mathcal{P}}_n$).

One can easily see that any element of $\widetilde{\mathcal{P}}_n^*$ is isometric to a unique star-shaped polygon $\wp = (M_1, \cdots, M_n)$ with respect to M_n such that :

$$(1.6) \quad \begin{cases} M_n = M_0 \text{ is the origin of } \mathbb{E}, \\ M_1 \text{ has coordinates } (t_1, 0) \text{ with } t_1 > 0, \\ \text{for any } k \in \{1, \cdots, n-2\}, \text{ the measure of the angle } M_k \widehat{M_n} M_{k+1} = (\overrightarrow{M_n M_k}, \overrightarrow{M_n M_{k+1}}) \text{ is in }]0, \pi[. \end{cases}$$

$\mathcal{P}_n^*$ will denote the space of the star-shaped polygons satisfying these three conditions.



Allowed

The measure of the angle $\widehat{M_3 M_7 M_4}$ is 0. So it is not allowed.

Now consider the open set $\mathcal{V}_n$ of $\mathbb{R}^n$ and the open set Ω_n of $\mathbb{R}^{2n-3}$ given as follow :

$$(1.7) \qquad \begin{cases} \mathcal{V}_n = \Big\{(u_1, \cdots, u_n) \in]0, +\infty[^n : \forall j \in \{1, \cdots, n\},\ u_j < \sum_{k \neq j} u_k \Big\} \\ \Omega_n = \big\{(t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \in \mathbb{R}^{2n-3} : k \in \{1, \cdots, n-2\},\ (t_k, x_k, t_{k+1}) \in \mathcal{V}_3 \big\} \end{cases}$$

It is easy to see that : (i) $\Omega_3 = \mathcal{V}_3$; (ii) for any natural integer $n \geq 3$, the open sets $\mathcal{V}_n$ and Ω_n are convex cones respectively in $\mathbb{R}^n$ and $\mathbb{R}^{2n-3}$.

Let $\mathcal{L}_n : \mathcal{P}_n^* \longrightarrow \Omega_n$ be the map defined by :

$$\mathcal{L}_n(\wp) = \mathcal{L}_n(< M_1, \cdots, M_n >) = (t_1, x_1, t_2, x_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1})$$

where the real numbers $t_1, x_1, t_2, x_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1}$ are given by the formulaes (1.4). The proof of the following proposition is almost immediate (it is a simple consequence of the definition of the map $\mathcal{L}_n$).

1.4. Proposition. *The map $\mathcal{L}_n$ is a bijection and identifies the elements of $\mathcal{P}_n^*$ to the elements of Ω_n.*

From now on we will identify the geometric star-shaped polygons to the points of the convex cone Ω_n by using the bijection $\mathcal{L}_n$.

This identification $\mathcal{P}_n^* \simeq \Omega_n$ will enables one to study easily some properties of the space of geometric star-shaped polygons.

Let $\varphi : X \to Y$ be any map where X and Y are nonempty sets. Any nonempty subset of X of the form $\varphi^{-1}(\{y\})$ will be called the *level set* (level line, level surface, level manifold...) of φ at level $y \in Y$.

2. The geometric inscribable polygons for $n \geq 4$

We denote by Γ_n the set of geometric inscribable polygons of $\mathbb{E}$ which we will see as a subset of the open set Ω_n. (Recall that any regular geometric polygon is inscribable.)

2.1. Remark. The fact that a polygon $(M_1, \cdots, M_n)$ is inscribable is equivalent to the fact that each one of the $n-3$ quadrilaterals : $Q_1 = (M_n, M_1, M_2, M_3), \ \cdots, Q_{n-3} = (M_n, M_{n-3}, M_{n-2}, M_{n-1})$ is inscribable.

2.2. Lemma. *A convex quadrilateral (A, B, C, D) is inscribable if and only if the distances $a = AB, \ b = BC, \ c = CD, \ d = DA$ and $e = BD$ satisfy the following relation* :

$$ad(b^2 + c^2 - e^2) + bc(a^2 + d^2 - e^2) = 0. \tag{2.1}$$

Proof. Let α and β denote the measures respectively of the angles $\widehat{A}$ and $\widehat{C}$. Then (by the cosine's law of Al-Kashi) :

$$0 < \alpha < \pi, \ 0 < \beta < \pi \quad \text{and} \quad a^2 + d^2 - 2ad \cos\alpha = e^2 = b^2 + c^2 - 2bc \cos\beta.$$

We deduce :

$$\begin{cases} \cos\alpha = \frac{a^2+d^2-e^2}{2ad} \\ \cos\beta = \frac{b^2+c^2-e^2}{2bc}. \end{cases}$$

On the other hand, the quadrilateral (A, B, C, D) is inscribable if and only if $\alpha + \beta = \pi$ or, equivalently, if $\cos\beta = -\cos\alpha$. Hence :

$$(A, B, C, D) \text{ inscribable} \Longleftrightarrow \frac{b^2 + c^2 - e^2}{2bc} = -\frac{a^2 + d^2 - e^2}{2ad}$$

which is also equivalent to $ad(b^2 + c^2 - e^2) + bc(a^2 + d^2 - e^2) = 0$.

Remark 2.1 and Lemma 2.2 make possible to realize Γ_n as a level set of a differentiable map. More precisely, consider the maps $\Theta : \mathbb{R}^5 \longrightarrow \mathbb{R}$, $\gamma_k : \Omega_n \longrightarrow \mathbb{R}$ and $\gamma : \Omega_n \longrightarrow \mathbb{R}^{n-3}$ defined by :

$$\Theta(u) = u_1 u_2 (u_4^2 + u_5^2 - u_3^2) + u_4 u_5 (u_1^2 + u_2^2 - u_3^2) \ \text{ for } \ u = (u_1, u_2, u_3, u_4, u_5) \in \mathbb{R}^5,$$

$$\gamma_k(\omega) = \Theta(t_k, x_k, t_{k+1}, x_{k+1}, t_{k+2}) \ \text{ for } \ k \in \{1, \cdots, n-3\}$$

and :

$$\gamma(\omega) = (\gamma_1(\omega), \cdots, \gamma_{n-3}(\omega)) \quad \text{where} \quad \omega = (t_1, x_1, t_2, \cdots, x_{n-2}, t_{n-1}) \in \Omega_n.$$

2.3. Proposition. *The set Γ_n of geometric inscribable polygons is given by $\Gamma_n = \gamma^{-1}(\{0\})$. Moreover, this set is a differentiable submanifold of dimension n of the Euclidean space $\mathbb{R}^{2n-3}$.*

Proof. Let $\omega = (t_1, x_1, ..., t_{n-2}, x_{n-2}, t_{n-1})$ be an element of Ω_n represented by a polygon $(M_1, \cdots, M_n)$. Then we have the following equivalences :

$$\begin{aligned}
\omega \in \Gamma_n &\Longleftrightarrow (M_n, M_k, M_{k+1}, M_{k+2}) \text{ is inscribable for } 1 \le k \le n-3 \\
&\Longleftrightarrow \Theta(t_k, x_k, t_{k+1}, x_{k+1}, t_{k+2}) = 0 \ \text{ for } 1 \le k \le n-3 \\
&\Longleftrightarrow \gamma_k(\omega) = 0 \ \text{ for } 1 \le k \le n-3 \\
&\Longleftrightarrow \gamma(\omega) = 0 \\
&\Longleftrightarrow \omega \in \gamma^{-1}(\{0\}).
\end{aligned}$$

Now we will prove, by induction on $n \ge 4$, that the map γ is a submersion at each point ω of Ω_n.

- **The case** $n = 4$**.** We have $\gamma = \gamma_1$ and the map :

$$\gamma : \Omega_4 \longrightarrow \mathbb{R},\ \omega = (t_1, x_1, t_2, x_2, t_3) \longmapsto \gamma(\omega) = t_1 x_1 (t_3^2 + x_2^2 - t_2^2) + t_3 x_2 (t_1^2 + x_1^2 - t_2^2)$$

is a submersion at each point $\omega \in \Omega_4$. Indeed, $\dfrac{\partial \gamma}{\partial t_2}(\omega) = -2t_2(t_1 x_1 + t_3 x_2) \neq 0$. So $\Gamma_4 = \gamma^{-1}(\{0\})$ is a codimension 1 submanifold of the open set Ω_4 and then a submanifold of dimension 4 of $\mathbb{R}^5$.

- **Heredity.** Suppose that, for a fixed integer $n \ge 4$, the map γ is a submersion at each point of Ω_n. If to each element $\omega = (t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1}, x_{n-1}, t_n) \in \Omega_{n+1}$ we associate :

$$\begin{cases} \omega' = (t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_n \\ \\ \omega'' = (t_{n-2}, x_{n-2}, t_{n-1}, x_{n-1}, t_n) \in \Omega_4 \end{cases}$$

then we can write $\omega = (\omega', x_{n-1}, t_n)$ and $\gamma(\omega) = (\gamma(\omega'), \gamma(\omega''))$. Note that here we are using the same notation γ for three different maps. The Jacobian matrix of the map $\gamma : \Omega_{n+1} \subset \mathbb{R}^{2n-1} \to \mathbb{R}^{n-2}$ at each point $\omega \in \Omega_{n+1}$ is given by :

$$\begin{pmatrix} A(\omega') & 0 \\ B(\omega'') & C(\omega'') \end{pmatrix} \tag{2.2}$$

where :

$$A(\omega') = \begin{pmatrix}
\frac{\partial \gamma_1}{\partial t_1}(\omega') & \frac{\partial \gamma_1}{\partial x_1}(\omega') & \cdots & \frac{\partial \gamma_1}{\partial t_{n-2}}(\omega') & \frac{\partial \gamma_1}{\partial x_{n-2}}(\omega') & \frac{\partial \gamma_1}{\partial t_{n-1}}(\omega') \\
\frac{\partial \gamma_2}{\partial t_1}(\omega') & \frac{\partial \gamma_2}{\partial x_1}(\omega') & \cdots & \frac{\partial \gamma_2}{\partial t_{n-2}}(\omega') & \frac{\partial \gamma_2}{\partial x_{n-2}}(\omega') & \frac{\partial \gamma_2}{\partial t_{n-1}}(\omega') \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial \gamma_{n-3}}{\partial t_1}(\omega') & \frac{\partial \gamma_{n-3}}{\partial x_1}(\omega') & \cdots & \frac{\partial \gamma_{n-3}}{\partial t_{n-2}}(\omega') & \frac{\partial \gamma_{n-3}}{\partial x_{n-2}}(\omega') & \frac{\partial \gamma_{n-3}}{\partial t_{n-1}}(\omega')
\end{pmatrix}$$

which is the Jacobian matrix of the map $\gamma : \Omega_n \subset \mathbb{R}^{2n-3} \to \mathbb{R}^{n-3}$ at the point ω',

$$B(\omega'') = \left(0, \cdots, 0, \frac{\partial \Theta}{\partial u_1}(\omega''), \frac{\partial \Theta}{\partial u_2}(\omega''), \frac{\partial \Theta}{\partial u_3}(\omega'')\right) \ \text{ and } \ C(\omega'') = \left(\frac{\partial \Theta}{\partial u_4}(\omega''), \frac{\partial \Theta}{\partial u_5}(\omega'')\right).$$

By the induction hypothesis, the matrix $A(\omega')$ has rank $n-3$. On the other hand, the two partial derivatives :

$$\begin{cases} \frac{\partial \Theta}{\partial u_4}(\omega'') = 2x_{n-1} t_{n-2} x_{n-2} + t_n \left(x_{n-2}^2 + t_{n-2}^2 - t_{n-1}^2\right) \\ \\ \frac{\partial \Theta}{\partial u_5}(\omega'') = 2t_n t_{n-2} x_{n-2} + x_{n-1} \left(x_{n-2}^2 + t_{n-2}^2 - t_{n-1}^2\right) \end{cases}$$

do not vanish simultaneously since otherwise we will have :

$$0 = x_{n-1} \frac{\partial \Theta}{\partial u_4}(\omega'') - t_n \frac{\partial \Theta}{\partial u_5}(\omega'') = 2x_{n-1} t_{n-2} (x_{n-1}^2 - t_n^2)$$

which implies $x_{n-1} = t_n$. Taking into account this equality in the relation $\frac{\partial \Theta}{\partial u_4}(\omega'') = 0$ we obtain :

$$0 = t_n \left[(x_{n-2} + t_{n-2})^2 - t_{n-1}^2\right]$$

and then $x_{n-2} + t_{n-2} = t_{n-1}$. This contradicts the inequality $t_{n-1} < x_{n-2} + t_{n-2}$ since these are the lengths of the sides of the non degenerate triangle $(M_{n+1}, M_{n-2}, M_{n-1})$.

The Jacobian matrix of the map $\gamma : \Omega_{n+1} \subset \mathbb{R}^{2n-1} \to \mathbb{R}^{n-2}$ is then of rank $n-2$. The proof by induction is then over.

We deduce that, for any $n \ge 4$, the map $\gamma : \Omega_n \subset \mathbb{R}^{2n-3} \to \mathbb{R}^{n-3}$ is a submersion and then the nonempty set $\Gamma_n = \gamma^{-1}(\{0\})$ is a codimension $(n-3)$ submanifold of the open set Ω_n of $\mathbb{R}^{2n-3}$. This implies that Γ_n is a submanifold of dimension n of $\mathbb{R}^{2n-3}$.

3. Equilateral and articulated polygons

3.1. Equilateral geometric polygons

A polygon $< M_1, \cdots, M_n > \simeq (t_1, x_1, t_2, x_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_n$ is equilateral if and only if we have $M_1M_2 = M_nM_1$, $M_2M_3 = M_nM_1$,...,$M_{n-2}M_{n-1} = M_nM_1$ and $M_{n-1}M_n = M_nM_1$ which we can express also by the equalities $x_1 = t_1$, $x_2 = t_1$,..., $x_{n-2} = t_1$ and $t_{n-1} = t_1$.

Then one can see the set $\mathcal{E}_n$ of equilateral geometric polygons as the level set $\lambda_n^{-1}(\{0\})$ of the map $\lambda_n : \Omega_n \longrightarrow \mathbb{R}^{n-1}$ defined by :

$$\lambda_n(t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) = (x_1 - t_1, x_2 - t_1, ..., x_{n-2} - t_1, t_{n-1} - t_1). \tag{3.1}$$

Proposition. *The space $\mathcal{E}_n$ of equilateral geometric polygons is a convex differentiable submanifold of dimension $n-2$ of $\mathbb{R}^{2n-3}$.*

Proof. λ_n is the restriction to Ω_n of the surjective linear map $\widetilde{\lambda}_n : \mathbb{R}^{2n-3} \longrightarrow \mathbb{R}^{n-1}$ defined as follows : $\widetilde{\lambda}_n(t_1, x_1, t_2, x_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) = (x_1 - t_1, x_2 - t_1, \cdots, x_{n-2} - t_1, t_{n-1} - t_1)$. Then it is a submersion at each point of Ω_n. This shows that the level set :

$$\lambda_n^{-1}(\{0\}) = \Omega_n \cap \widetilde{\lambda}_n^{-1}(\{0\}) \tag{3.2}$$

is a closed submanifold of dimension $n-2$ of Ω_n. It is convex as intersection of convex sets.

3.2. Articulated geometric polygons

An *articulated polygon* is a mechanical system consisting of n rigid rods $[M_1, M_2]$, $[M_1, M_2]$,...,$[M_{n-1}, M_n]$, $[M_n, M_1]$ articulated at their extremities $M_1, \cdots, M_n$. The lengths of the rods $u_1 = M_1M_2, \cdots, u_n = M_nM_1$ define an element $u = (u_1, \cdots, u_n)$ of the open set $\mathcal{V}_n$. Then the set of articulations is the level set $q_n^{-1}(\{u\})$ of the map $\Omega_n \subset \mathbb{R}^{2n-3} \xrightarrow{q_n} \mathcal{V}_n$ given by :

$$q_n(t_1, x_1, t_2, x_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) = (x_1, x_2, \cdots, x_{n-2}, t_{n-1}, t_1). \tag{3.3}$$

Proposition. *The map q_n is a surjective submersion whose each level set is a convex differentiable submanifold of dimension $n-3$ of $\mathbb{R}^{2n-3}$. These level sets are leaves of a foliation $\mathcal{F}_q$ of dimension $n-3$ on the space Ω_n whose leaf space is $\mathcal{V}_n$.*

Proof. The map q_n is the restriction to the space Ω_n of the linear projection $\mathbb{R}^{2n-3} \xrightarrow{\widetilde{q}_n} \mathbb{R}^n$ defined by :

$$\widetilde{q}_n(t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) = (x_1, x_2, \cdots, x_{n-2}, t_{n-1}, t_1).$$

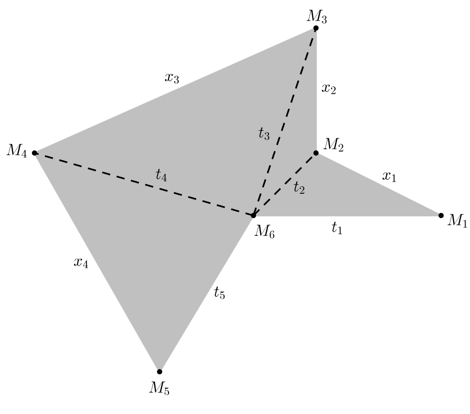
It is a submersion at each point of Ω_n. The fact that q_n is surjective is a consequence of Theorem 1.2. Thus we have a codimension n foliation $\mathcal{F}_q$ (and of dimension $n-3$) on the space Ω_n ; its leaves are the level sets of the map q_n. The open set $\mathcal{V}_n$ is the leaf space of this foliation.

For any $u \in \mathcal{V}_n$, the set $q_n^{-1}(\{u\}) = \Omega_n \cap \widetilde{q}_n^{-1}(\{u\})$ is a closed submanifold of dimension $n-3$ of Ω_n. It is convex as intersection of convex sets.

4. The area and perimeter foliations

4.1. We define the maps $p : \Omega_n \longrightarrow \mathbb{R}$, $\mathcal{A} : \Omega_n \longrightarrow \mathbb{R}$ and $\Psi : \Omega_n \longrightarrow \mathbb{R}^2$ by :

$$\begin{cases} p(\omega) = \text{perimeter of } \omega \\ \mathcal{A}(\omega) = \text{area of } \omega \\ \Psi(\omega) = (p(\omega), \mathcal{A}(\omega)). \end{cases} \tag{4.1}$$



For any $\omega = (t_1, x_1, t_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_n$, we have :

$$(4.2) \qquad \begin{cases} p(\omega) = t_1 + x_1 + x_2 + \cdots + x_{n-2} + t_{n-1} \\ \mathcal{A}(\omega) = \frac{1}{4}\sqrt{f(\omega_1)} + \cdots + \frac{1}{4}\sqrt{f(\omega_{n-2})} \end{cases}$$

with :

$$\begin{cases} \omega_k = (t_k, x_k, t_{k+1}) \in \mathcal{V} \text{ for } k \in \{1, ..., n-2\} \\ \text{area of } \omega_k = \frac{1}{4}\sqrt{f(\omega_k)} \text{ for } k \in \{1, \cdots, n-2\} \text{ (Héron's formula)} \\ f(x,y,z) = (x+y+z)(-x+y+z)(x-y+z)(x+y-z) \text{ for } (x,y,z) \in \mathcal{V}. \end{cases}$$

Setting $s(x,y,z) = x+y+z$, we obtain for any $v = (x,y,z) \in \mathcal{V}$:

$$f(v) = s(v)\,(s(v)-2x)\,(s(v)-2y)\,(s(v)-2z)\,.$$

On the other hand, the maps p, $\mathcal{A}$ and Ψ are clearly differentiable with gradient vectors $\nabla p(\omega)$ and $\nabla \mathcal{A}(\omega)$ given by :

$$\begin{cases} \nabla p(\omega) = \left(\frac{\partial p}{\partial t_1}(\omega), \frac{\partial p}{\partial x_1}(\omega), \cdots, \frac{\partial p}{\partial x_{n-2}}(\omega), \frac{\partial p}{\partial t_{n-1}}(\omega)\right) = (1,1,0,1,\cdots,0,1,1) \\ \nabla \mathcal{A}(\omega) = \left(\frac{\nabla f(\omega_1)}{8\sqrt{f(\omega_1)}}, \cdots, \frac{\nabla f(\omega_{n-2})}{8\sqrt{f(\omega_{n-2})}}\right) = \left(\frac{\sqrt{f(\omega_1)}}{8}\cdot\frac{\nabla f(\omega_1)}{f(\omega_1)}, \cdots, \frac{\sqrt{f(\omega_{n-2})}}{8}\cdot\frac{\nabla f(\omega_{n-2})}{f(\omega_{n-2})}\right) \end{cases}$$

where the logarithmic derivative $\frac{\nabla f}{f}$ is given at each point $v = (x,y,z) \in \mathcal{V}$ by :

$$\frac{\nabla f(v)}{f(v)} = \frac{\nabla s(v)}{s(v)} + \frac{\nabla\,(s-2x)\,(v)}{s(v)-2x} + \frac{\nabla\,(s-2y)\,(v)}{s(v)-2y} + \frac{\nabla\,(s-2z)\,(v)}{s(v)-2z}.$$

We then deduce :

$$\begin{cases} \frac{1}{f(v)}\frac{\partial f}{\partial x}(v) = \frac{1}{s(v)} - \frac{1}{s(v)-2x} + \frac{1}{s(v)-2y} + \frac{1}{s(v)-2z} \\ \frac{1}{f(v)}\frac{\partial f}{\partial y}(v) = \frac{1}{s(v)} + \frac{1}{s(v)-2x} - \frac{1}{s(v)-2y} + \frac{1}{s(v)-2z} \\ \frac{1}{f(v)}\frac{\partial f}{\partial z}(v) = \frac{1}{s(v)} + \frac{1}{s(v)-2x} + \frac{1}{s(v)-2y} - \frac{1}{s(v)-2z}. \end{cases}$$

This gives the partial derivatives of $\mathcal{A}$:

$$\begin{cases} \frac{\partial \mathcal{A}}{\partial t_1}(\omega) = \frac{\sqrt{f(\omega_1)}}{8}\left(\frac{1}{s(\omega_1)} - \frac{1}{s(\omega_1)-2t_1} + \frac{1}{s(\omega_1)-2x_1} + \frac{1}{s(\omega_1)-2t_2}\right) \\ \frac{\partial \mathcal{A}}{\partial x_1}(\omega) = \frac{\sqrt{f(\omega_1)}}{8}\left(\frac{1}{s(\omega_1)} + \frac{1}{s(\omega_1)-2t_1} - \frac{1}{s(\omega_1)-2x_1} + \frac{1}{s(\omega_1)-2t_2}\right) \\ \frac{\partial \mathcal{A}}{\partial t_{n-1}}(\omega) = \frac{\sqrt{f(\omega_{n-2})}}{8}\left(\frac{1}{s(\omega_{n-2})} + \frac{1}{s(\omega_{n-2})-2t_{n-2}} + \frac{1}{s(\omega_{n-2})-2x_{n-2}} - \frac{1}{s(\omega_{n-2})-2t_{n-1}}\right) \end{cases}$$

and for $k \in \{2, \cdots, n-2\}$:

$$\begin{cases} \frac{\partial \mathcal{A}}{\partial t_k}(\omega) = \frac{\sqrt{f(\omega_{k-1})}}{8}\left(\frac{1}{s(\omega_{k-1})} + \frac{1}{s(\omega_{k-1})-2t_{k-1}} + \frac{1}{s(\omega_{k-1})-2x_{k-1}} - \frac{1}{s(\omega_{k-1})-2t_k}\right) \\ \qquad\qquad + \frac{\sqrt{f(\omega_k)}}{8}\left(\frac{1}{s(\omega_k)} + \frac{1}{s(\omega_k)-2x_k} + \frac{1}{s(\omega_k)-2t_{k+1}} - \frac{1}{s(\omega_k)-2t_k}\right) \\ \frac{\partial \mathcal{A}}{\partial x_k}(\omega) = \frac{\sqrt{f(\omega_k)}}{8}\left(\frac{1}{s(\omega_k)} + \frac{1}{s(\omega_k)-2t_k} - \frac{1}{s(\omega_k)-2x_k} + \frac{1}{s(\omega_k)-2t_{k+1}}\right). \end{cases}$$

4.2. Theorem. *We have the following assertions.*

(1) *The perimeter function p and the area function $\mathcal{A}$ are submersions on Ω_n. Then the level sets of p (resp. of $\mathcal{A}$) are leaves of a codimension 1 foliation $\mathcal{F}_p$ (resp. $\mathcal{F}_a$) on Ω_n.*

(2) *For $\omega \in \Omega_n$, the differential $d\Psi(\omega)$ is of rank 2 if ω is not a regular polygon and of rank 1 if ω is a regular polygon. Then the map Ψ defines a codimension 2 foliation $\mathcal{F}$ on the open set Ω_n of $\mathbb{R}^{2n-3}$ which consists of non regular polygons ω of Ω_n.*

Proof. Let $\omega = (t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1})$ be an element of Ω_n and $(M_1, \cdots, M_n)$ one of its representatives as a satr-shaped polygon.

Point (1)

($\star$) For any $\omega \in \Omega_n$, $dp(\omega) \neq 0$ since $\frac{\partial p}{\partial t_1}(\omega) = 1 \neq 0$. Then p is a submersion on Ω_n.

($\star$) For any $\omega \in \Omega_n$, $\frac{\partial \mathcal{A}}{\partial t_1}(\omega) \neq 0$ or $\frac{\partial \mathcal{A}}{\partial x_1}(\omega) \neq 0$. Indeed we have the implications :

$$\begin{cases} \frac{\partial \mathcal{A}}{\partial t_1}(\omega) = 0 \\ \text{et} \\ \frac{\partial \mathcal{A}}{\partial x_1}(\omega) = 0 \end{cases} \Longrightarrow \frac{\partial \mathcal{A}}{\partial t_1}(\omega) + \frac{\partial \mathcal{A}}{\partial x_1}(\omega) = 0 \Longrightarrow \frac{2}{s(\omega_1)} + \frac{2}{s(\omega_1) - 2t_2} = 0 \Longrightarrow t_1 + x_1 = 0.$$

But the equality $t_1 + x_1 = 0$ can not be satisfied. Then $d\mathcal{A}(\omega) \neq 0$. This proves that $\mathcal{A}$ is a submersion on Ω_n.

Point (2)

($\star$) If the sides of the polygon are not all equal, there exist two successive sides having M as common point and different lengths. One can suppose that $M = M_1$. In these conditions, the lengths $t_1 = M_nM_1$ and $x_1 = M_1M_2$ are different. This implies $\dfrac{\partial \mathcal{A}}{\partial t_1}(\omega) \neq \dfrac{\partial \mathcal{A}}{\partial x_1}(\omega)$ since we have the implication $\frac{\partial \mathcal{A}}{\partial t_1}(\omega) = \frac{\partial \mathcal{A}}{\partial x_1}(\omega) \Longrightarrow t_1 = x_1$. The Jacobian matrix of the map Ψ at the point ω is :

$$\mathcal{J}(\Psi, \omega) = \begin{pmatrix} 1 & 1 & 0 & 1 & \cdots & 0 & 1 & 1 \\ \frac{\partial \mathcal{A}}{\partial t_1}(\omega) & \frac{\partial \mathcal{A}}{\partial x_1}(\omega) & \frac{\partial \mathcal{A}}{\partial t_2}(\omega) & \frac{\partial \mathcal{A}}{\partial x_2}(\omega) & \cdots & \frac{\partial \mathcal{A}}{\partial tn_2}(\omega) & \frac{\partial \mathcal{A}}{\partial x_{n-2}}(\omega) & \frac{\partial \mathcal{A}}{\partial t_{n-1}}(\omega) \end{pmatrix}$$

and then it is of rank 2 since the 2×2-matrix of $\mathcal{J}(\Psi, \omega)$ consisting of the two first columns is invertible.

($\star$) Suppose the polygon is equilateral. We consider two cases :

- The condition (C) below is satisfied.

$$\text{(C)} \qquad \begin{cases} \text{The partial derivatives } \frac{\partial \mathcal{A}}{\partial t_1}(\omega), \frac{\partial \mathcal{A}}{\partial t_{n-1}}(\omega), \frac{\partial \mathcal{A}}{\partial x_k}(\omega) \\ \text{ are not equal, } k \in \{1, \cdots, n-2\} \\ \text{or} \\ \text{at least one of the derivatives } \frac{\partial \mathcal{A}}{\partial t_k}(\omega),\ k \in \{1, \cdots, n-2\} \text{ is not zero.} \end{cases}$$

The matrix $\mathcal{J}(\Psi, \omega)$ has rank 2 since it admits a matrix of order 2 which is reversible.

- The condition (C) is not satisfied. In this case the following condition non(C) is satisfied :

$$\text{non(C)} \qquad \begin{cases} \text{The partial derivatives } \frac{\partial \mathcal{A}}{\partial t_1}(\omega), \frac{\partial \mathcal{A}}{\partial t_{n-1}}(\omega), \frac{\partial \mathcal{A}}{\partial x_k}(\omega) \\ \text{ are all equal for } k \in \{1, \cdots, n-2\} \\ \text{and} \\ \frac{\partial \mathcal{A}}{\partial t_k}(\omega),\ k \in \{1, \cdots, n-2\} \text{ are all zero} \end{cases}$$

and we have :

$$\mathcal{J}(\Psi, \omega) = \begin{pmatrix} 1 & 1 & 0 & 1 & \cdots & 0 & 1 & 1 \\ \frac{\partial \mathcal{A}}{\partial t_1}(\omega) & \frac{\partial \mathcal{A}}{\partial t_1}(\omega) & 0 & \frac{\partial \mathcal{A}}{\partial t_1}(\omega) & \cdots & 0 & \frac{\partial \mathcal{A}}{\partial t_1}(\omega) & \frac{\partial \mathcal{A}}{\partial t_1}(\omega) \end{pmatrix} \tag{4.3}$$

This implies that $\mathcal{J}(\Psi, \omega)$ is of rank 1. We shall prove that, in this case, ω is regular polygon.

For $k \in \{2, \cdots, n-2\}$, we have $\dfrac{\partial \mathcal{A}}{\partial x_{k-1}}(\omega) = \dfrac{\partial \mathcal{A}}{\partial x_k}(\omega)$ which implies :

$$\left(\frac{\partial \mathcal{A}}{\partial x_{k-1}}(\omega)\right)^2 - \left(\frac{\partial \mathcal{A}}{\partial x_k}(\omega)\right)^2 = 0.$$

Thus, taking into account the fact that the sides x_k are all equal in this case, we obtain by factorization :

$$\frac{64x_k^2t_k^2(t_{k-1} - t_{k+1})(t_{k-1} + t_{k+1})(t_{k-1}t_{k+1} - x_k^2 + t_k^2)(t_{k-1}t_{k+1} + x_k^2 - t_k^2)}{\Delta} = 0$$

where :

$$\begin{aligned} \Delta = \quad & (x_k - t_k - t_{k+1})(x_k - t_k + t_{k+1})(x_k + t_k - t_{k+1})(x_k + t_k + t_{k+1}) \\ & (t_{k-1} - x_k - t_k)(t_{k-1} - x_k + t_k)(t_{k-1} + x_k - t_k)(t_{k-1} + x_k + t_k). \end{aligned}$$

Thus $(t_{k-1}-t_{k+1})(t_{k-1}t_{k+1}-x_k^2+t_k^2)(t_{k-1}t_{k+1}+x_k^2-t_k^2)=0$.

$(\star)$ If $t_{k-1}-t_{k+1}=0$, taking into account the relation $\frac{\partial \mathcal{A}}{\partial t_k}(\omega)=0$, we obtain :

$$t_k^2 = x_k^2 + t_{k+1}^2.$$

This implies that the two triangles (M_n, M_{k-1}, M_k) and (M_n, M_{k+1}, M_k) are rectangular respectively at M_{k-1} and M_{k+1}. Consequently the quadrilateral $(M_n, M_{k-1}, M_k, M_{k+1})$ is inscribable ($\widehat{M}_{k-1}=\frac{\pi}{2}=\widehat{M}_{k+1}$).

$(\star)$ If $t_{k-1}t_{k+1}-x_k^2+t_k^2=0$, the convex quadrilateral $(M_n, M_{k-1}, M_k, M_{k+1})$ has the following properties :

$$\tag{4.4} \begin{cases} M_{k-1}M_k = x_k = M_kM_{k+1} \quad \text{(The polygon is equilateral in this case.)} \\ t_k^2 = x_k^2 - t_{k-1}t_{k+1} \\ \cos\widehat{M_{k-1}} = \frac{t_k^2-t_{k-1}^2-x_k^2}{2t_{k-1}x_k} = \frac{x_k^2-t_{k-1}t_{k+1}-t_{k-1}^2-x_k^2}{2t_{k-1}x_k} = -\frac{t_{k+1}+t_{k-1}}{2x_k} \\ \cos\widehat{M_{k+1}} = \frac{t_k^2-t_{k+1}^2-x_k^2}{2t_{k+1}x_k}\frac{x_k^2-t_{k-1}t_{k+1}-t_{k+1}^2-x_k^2}{2t_{k+1}x_k} = -\frac{t_{k-1}+t_{k+1}}{2x_k} \\ \widehat{M_{k-1}} = \widehat{M_{k+1}} \text{ (by the equality of the cosines).} \end{cases}$$

The triangle (M_{k-1}, M_n, M_{k+1}) is then isosceles at the M_n and $t_{k-1}=t_{k+1}$. This implies, like in the preceding case, that the quadrilateral $(M_n, M_{k-1}, M_k, M_{k+1})$ is inscribable.

$(\star)$ If $t_{k-1}t_{k+1}+x_k^2-t_k^2=0$, then the convex quadrilateral $(M_n, M_{k-1}, M_k, M_{k+1})$ is still inscribable since it satisfies the relation $\cos\widehat{M_{k-1}} = -\cos\widehat{M_{k+1}}$. Indeed, we have :

$$\tag{4.5} \begin{cases} t_k^2 = t_{k-1}t_{k+1} + x_k^2 \\ \cos\widehat{M_{k-1}} = \frac{t_k^2-t_{k-1}^2-x_k^2}{2t_{k-1}x_k} = \frac{x_k^2+t_{k-1}t_{k+1}-t_{k-1}^2-x_k^2}{2t_{k-1}x_k} = t_{k+1}-t_{k-1}2x_k \\ \cos\widehat{M_{k+1}} = \frac{t_k^2-t_{k+1}^2-x_k^2}{2t_{k+1}x_k}\frac{x_k^2+t_{k-1}t_{k+1}-t_{k+1}^2-x_k^2}{2t_{k+1}x_k} = \frac{t_{k-1}-t_{k+1}}{2x_k}. \end{cases}$$

We have proved that, in all cases, the quadrilateral $(M_n, M_{k-1}, M_k, M_{k+1})$ is inscribable for any $k \in \{2, \cdots, n-2\}$. Then the polygon $(M_1, \cdots, M_n)$ is inscribable. Since the latter is equilateral, it is necessarily regular.

Finally the singular points of the map Ψ are the regular polygons and this map induces a submersion on Ω_n^* whose level sets are leaves of a foliation $\mathcal{F}$.

5. The example of triangles

It is the situation where we see things more concretely and where the drawings are more visible. The way to treat the topic in this section will be slightly different from the others.

5.1. The space of non degenerate triangles

To give oneself a non degenerate triangle (in any Euclidean finite dimensional space) is to give oneself three real numbers $x>0$, $y>0$ and $z>0$ such that :

$$\tag{5.1} \begin{cases} x < y+z \\ y < z+x \\ z < x+y \end{cases}$$

which represent the lengths of the sides. Exceptionally in this section, we shall denote a triangle by $\langle xyz\rangle$ instead of (X,Y,Z) where the points X, Y and Z are the vertices. Indeed it is well known that $\langle xyz\rangle$ is isometric to $\langle x'y'z'\rangle$ if $x=x'$, $y=y'$ and $z=z'$. (For the moment we will make the difference between a triangle and another obtained by permutation of the three numbers representing it even if, geometrically, they are the same!) From now on, λ will be the half perimeter $\lambda=\frac{x+y+z}{2}$.

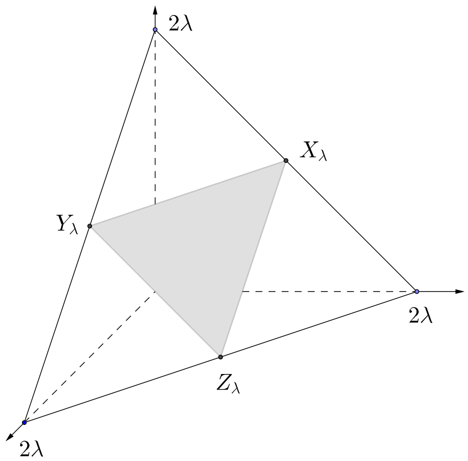
The set of non degenerate triangles is thus the open set $\Omega_3 \subset (\mathbb{R}_+^*)^3$ given by (1.7). We will describe it explicitly. To inequalities (5.1) are associated three equations defining respectively three planes :

$$\tag{5.2} \begin{cases} \Sigma_1 = \{x = y+z\} \\ \Sigma_2 = \{y = z+x\} \\ \Sigma_3 = \{z = x+y\} \end{cases}.$$

In the slice $\{x+y+z=2\lambda\}$ of $\mathbb{R}_+^3$, Σ_1, Σ_2 and Σ_3 are the sides of an equilateral triangle in $\mathbb{R}_+^3$ whose vertices are $X_\lambda=(0,\lambda,\lambda)$, $Y_\lambda=(\lambda,0,\lambda)$ and $Z_\lambda=(\lambda,\lambda,0)$ (see the picture bellow) ; the interior P_λ of the convex hull of these three points represents the space of triangles $\langle xyz\rangle$ whose perimeter is 2λ.

When λ varies to λ', we obtain another $P_{\lambda'}$, image of P_λ by the homothety centered at the origin and with ratio $k=\frac{\lambda'}{\lambda}$. Thus, the space Ω_3 is *foliated* by these P_λ ; Ω_3 is in fact the open cone with vertex the origin and basis anyone of these plaques P_λ, for instance P_1 :

$$\Omega_3=\bigcup_{\lambda\in\mathbb{R}_+^*}\lambda P_1=\left\{\lambda X:\ X\in P_1 \text{ et } \lambda\in\mathbb{R}_+^*\right\}. \tag{5.3}$$



For a particular situation which will appear thereafter, we recall the following result which we have already established in the general case of polygons.

For a given family of triangles with prescribed perimeter, the maximum of the area is realized by the equilateral triangle.

5.2. The perimeter foliation $\mathcal{F}_p$

Each P_λ (where $\lambda\in\mathbb{R}_+^*$) is the level set $p(x,y,z)=2\lambda$ where p is the perimeter function $p(x,y,z)=x+y+z$. We have also seen that the level surface P_λ is the interior of the convex hull of the triangle $X_\lambda Y_\lambda Z_\lambda$.

Thus we have a foliation $\mathcal{F}_p$ on Ω_3 whose leaves are the surfaces P_λ ($\lambda>0$). Of course, $\mathcal{F}_p$ is trivial since isomorphic to the product $P_1\times\mathbb{R}_+^*$.

5.3. The area foliation $\mathcal{F}_a$

The function $\mathcal{A}:(\mathbb{R}_+^*)^3\longrightarrow\mathbb{R}_+^*$ which associates to a triangle $\langle xyz\rangle$ its area is given by Héron formula :

$$\mathcal{A}(x,y,z)=\frac{1}{4}\sqrt{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)}. \tag{5.4}$$

The foliation $\mathcal{F}_a$ by which we will be interested is the foliation whose leaves are the level surfaces of this function.

• The surface at level s of the function $\mathcal{A}$ on the open set Ω_3 is exactly the surface at level s^2 of the function $\Phi=\mathcal{A}^2$. The benefit of working with Φ instead of $\mathcal{A}$ is that there is no more square root, which simplifies the calculations, among others that of the differential which plays a fundamental role. We consider then the function :

$$\Phi(x,y,z)=\frac{1}{16}(x+y+z)(-x+y+z)(x-y+z)(x+y-z). \tag{5.5}$$

• The differential of Φ has the form :

$$d\Phi(x,y,z)=\frac{1}{16}\{A(x,y,z)dx+B(x,y,z)dy+C(x,y,z)dz\}$$

where the functions A, B and C are given as follows :

$$
\begin{aligned}
A &= (-x+y+z)(x-y+z)(x+y-z) - (x+y+z)(x-y+z)(x+y-z) \\
&\quad +(x+y+z)(-x+y+z)(x+y-z) + (x+y+z)(-x+y+z)(x-y+z) \\
B &= (-x+y+z)(x-y+z)(x+y-z) + (x+y+z)(x-y+z)(x+y-z) \\
&\quad -(x+y+z)(-x+y+z)(x+y-z) + (x+y+z)(-x+y+z)(x-y+z) \\
C &= (-x+y+z)(x-y+z)(x+y-z) + (x+y+z)(x-y+z)(x+y-z) \\
&\quad +(x+y+z)(-x+y+z)(x+y-z) - (x+y+z)(-x+y+z)(x-y+z).
\end{aligned}
\tag{5.6}
$$

An easy but long computation shows that these three functions A, B et C are zero simultaneously only if $x = y = z = 0$, which can not happen since $(0, 0, 0)$ is not in Ω_3.

• If we fix the perimeter 2λ, the area function a is maximal, and so is the function Φ, when $x = y = z = \frac{2}{3}\lambda$; at this point Φ is equal to $\frac{\lambda^4}{27}$. These are the values taken by the function Φ on the open half line Δ whose equation is $x = y = z$.

Now let Ω_3^* be the open set $\Omega_3 \setminus \Delta$. At $u = (x, y, z) \in \Omega_3^*$, the differential $d_u\Phi$ has rank 1 ; then the set level of Φ passing through this point is a regular surface A, in fact an algebraic surface of degree 4. Its equation is :

$$(x+y+z)(-x+y+z)(x-y+z)(x+y-z) = 16\Phi(u).$$

Let G be the subgroup of $\mathrm{Isom}(\mathbb{R}^3)$ (the full group of isometries of the Euclidean space $\mathbb{R}^3$) generated by the rotation whose axis is Δ and angle $\frac{2\pi}{3}$ and the reflection σ with respect to the plane of equation $x = y$. (The restrictions of these elements to the plane of equation $x + y + z = 2\lambda$ is the group of isometries of the equilateral triangle $X_\lambda Y_\lambda Z_\lambda$.) It leaves the space Ω_3 invariant and also its boundary $\partial\Omega_3$, the half line Δ and the open sets Ω_3 and Ω_3^*. Then it acts on Ω_3 and fixes each leaf of $\mathcal{F}_a$; the same applies to the foliation $\mathcal{F}_p$.

5.4. Let $\Psi : \Omega_3^* \longmapsto \left(\mathbb{R}_+^*\right)^2$ be the function :

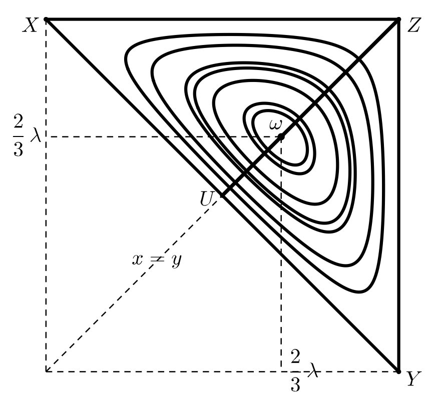
$$\Psi(x, y, z) = (p(x, y, z), \Phi(x, y, z)).$$

Up to a multiplicative factor, the matrix of its differential at $u = (x, y, z)$ is :

$$d_u\Psi = \begin{pmatrix} 1 & 1 & 1 \\ A(u) & B(u) & C(u) \end{pmatrix}$$

where A, B and C are the functions given by (5.6). It can be shown that these functions are equal only if $x = y = z$; then, for $u \in \Omega_3^*$, $d_u\Psi$ has rank 2. Thus, the level sets of Ψ are regular curves, leaves of a foliation $\mathcal{F}_*$ on Ω_3^*.

5.5. On Ω_3 we have a singular foliation $\mathcal{F} = \mathcal{F}_p \cap \mathcal{F}_a$. Its leaves of dimension 0 are the points of the open half line $\{\left(\frac{2}{3}\lambda, \frac{2}{3}\lambda, \frac{2}{3}\lambda\right) : \lambda \in \mathbb{R}_+\}$. The other leaves are of dimension 1 ; each one has equation $\Psi(u) =$ constant in the open set Ω_3^*. These curves define (by restriction) a foliation on each plaque P_λ (leaf of $\mathcal{F}_p$). To see what it is, this plaque is projected orthogonally on the plane $z = 0$; we obtain the foliation drawn on the picture below. We will explain what all this means.



The interior of the triangle XYZ is the projection (which we denote by Θ_λ) on the plane $z=0$ of the set P_λ of triangles $\langle x_\lambda y_\lambda z_\lambda\rangle$ with perimeter 2λ. Note that the boundary of P_λ is an equilateral triangle while XYZ is an isosceles and rectangle triangle. The foliation $\mathcal{F}$ on P_λ is isomorphic to the foliation on the picture via the diffeomorphism $f : P_\lambda \longrightarrow \Theta_\lambda$ defined by $f(x,y,z)=(x,y,0)$ with inverse $f^{-1}(x,y,0)=(x,y,2\lambda-x-y)$.

• The point ω with coordinates $\left(\frac{2}{3}\lambda, \frac{2}{3}\lambda\right)$ corresponds to the equilateral triangle $\langle xxx\rangle$ with maximal area. As we easily imagine, an equilateral triangle may never be deformed to an other one having the same perimeter and the same area.

• The curves at the interior of Θ_λ are leaves of a foliation of $\Theta_\lambda \setminus \{\omega\}$, each leaf corresponds to the set of triangles having the same area. It has $\lambda(2\lambda-x)(2\lambda-y)(x+y)=8c$ as equation where c is a constant varying in the interval $\left]0, \frac{8\lambda^4}{27}\right[$.

• The piece UZ of diagonal corresponds to isosceles triangles (for which $x=y$). In each leaf, there is exactly the projections of two isosceles triangles $\langle xxz\rangle$ and $\langle x'x'z'\rangle$.

5.6. Geoffrey Letellier constructed two lines of isosceles triangles : $\langle x_\lambda y_\lambda z_\lambda\rangle$ and $\langle x'_\lambda y'_\lambda z'_\lambda\rangle$ where $\lambda \in \mathbb{R}_+^*$ with $x_\lambda = y_\lambda = \frac{11}{14}\lambda$, $z_\lambda = \frac{3}{7}\lambda$ and $x'_\lambda = y'_\lambda = \frac{4}{7}\lambda$, $z'_\lambda = \frac{6}{7}\lambda$. They are such that, for any $\lambda \in \mathbb{R}_+^*$:

$$\begin{cases} \langle x_\lambda x_\lambda z_\lambda\rangle \text{ and } \langle x'_\lambda x'_\lambda z'_\lambda\rangle \text{ have the same perimeter } 2\lambda. \\ \langle x_\lambda x_\lambda z_\lambda\rangle \text{ and } \langle x'_\lambda x'_\lambda z'_\lambda\rangle \text{ have the same area } \frac{3\lambda^2}{7\sqrt{7}}. \\ \langle x_\lambda x_\lambda z_\lambda\rangle \text{ and } \langle x'_\lambda x'_\lambda z'_\lambda\rangle \text{ are not isometric.} \end{cases}$$

For instance, the two isosceles triangles $x=y=11$, $z=6$ and $x'=y'=8$, $z'=12$ have the same perimeter equal to 28 and the same area equal to $12\sqrt{7}$.

• Finally one can see on the picture that all the situation is invariant by the reflection σ (symmetry with respect to the diagonal $x=y$) while that on the triangle P_λ is invariant by the full group G. ♢

6. Some results related to the perimeter and the area

The following well known classical results are among the most beautiful theorems that we can cite in Euclidean elementary geometry of the plane.

6.1. Theorem (Isoperimetric inequality). *Among all the convex polygons with prescribed perimeter, the regular polygon is the one whose area is maximum.*

For a sketch of proof, see for instance [Han]. In the same order of ideas, we also have the following theorem. Its proof is not difficult but it is a bit long and not immediate. (And the reader can even attempt to reproduce it himself!)

6.2. Theorem. *Among all convex polygons whose sides have given lengths, the inscribable polygon is the one whose area is maximum.*

Using the analytic expression of the function "area" $\mathcal{A} : \omega \in \Omega_n \longrightarrow \text{area}(\omega) \in \mathbb{R}$, we prove the following result related to the two theorems above. (It was also partially established, by a different method, in [Khi].)

6.3. Theorem. *We have the following results.*

(1) *For any real number $L>0$, the differentiable manifold $p^{-1}(\{L\})$ consisting of all polygons with perimeter L, is diffeomorphic to a convex open set $\Omega_{n,L}$ of $\mathbb{R}^{2n-4}$ and the restriction $\mathcal{A}_L : \Omega_{n,L} \to \mathbb{R}$ of $\mathcal{A}$ to $\Omega_{n,L}$ admits a critical point at the unique regular polygon ω_L of perimeter L.*

(2) *The convex polygons whose sides have given lengths form a differentiable manifold diffeomorphic to an open convex set of $\mathbb{R}^{n-3}$ and the restriction of the function $\mathcal{A}$ to this open set admits a critical point at its unique inscribable polygon.*

Proof. Recall that, for any $\omega = (t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_n$, we have :

$$p(\omega) = \text{perimeter}(\omega) = t_1 + x_1 + \cdots + x_{n-2} + t_{n-1}$$

$$\mathcal{A}(\omega) = \text{area}(\omega) = \frac{1}{4}\sqrt{f(\omega_1)} + \cdots + \frac{1}{4}\sqrt{f(\omega_{n-2})}$$

with :

$$\begin{cases} \omega_k = (t_k, x_k, t_{k+1}) \in \mathcal{V} \text{ for } k \in \{1, \cdots, n-2\} \\ f(x,y,z) = (x+y+z)(-x+y+z)(x-y+z)(x+y-z) \text{ for } (x,y,z) \in \mathcal{V}. \end{cases}$$

Setting $h(v) = \frac{1}{4}\sqrt{f(v)}$ for $v \in \mathcal{V}$, we obtain :

$$\mathcal{A}(\omega) = h(\omega_1) + \cdots + h(\omega_{n-2})$$

for any $\omega = (t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_n$.

Point (1)

Let $L \in]0, +\infty[$. For any $\omega = (t_1, x_1, t_2, ..., t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_n$, we have :

$$p(\omega) = L \Leftrightarrow t_1 = L - x_1 - \cdots - x_{n-2} - t_{n-1}$$

Then, by considering the affine map $T : \mathbb{R}^{2n-4} \to \mathbb{R}^{2n-3} = \mathbb{R} \times \mathbb{R}^{2n-4}$ given by :

$$u = (x_1, t_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \longmapsto T(u) = (t_1(u), u)$$

where $t_1(u) = L - x_1 - \cdots - x_{n-2} - t_{n-1}$, we see that $p^{-1}(\{L\})$ is naturally identified to the convex open set $\Omega_{n,L} = T^{-1}(\Omega_n)$ of $\mathbb{R}^{2n-4}$.

For any $u = (x_1, t_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_{n,L}$, we have :

$$\mathcal{A}_L(u) = h(t_1(u), x_1, t_2) + h(t_2, x_2, t_3) + \cdots + h(t_{n-2}, x_{n-2}, t_{n-1}).$$

Set $t_1 = t_1(u)$ and $\omega_k = (t_k, x_k, t_{k+1})$ for $k \in \{1, \cdots, n-2\}$. The partial derivatives for any $(x,y,z) \in \mathcal{V}$ are :

$$\begin{cases} \frac{\partial h}{\partial x}(x,y,z) = \frac{x\left(-x^2+y^2+z^2\right)}{2\sqrt{(x+y+z))(-x+y+z)(x-y+z)(x+y-z)}} \\ \frac{\partial h}{\partial y}(x,y,z) = \frac{y\left(x^2-y^2+z^2\right)}{2\sqrt{(x+y+z))(-x+y+z)(x-y+z)(x+y-z)}} \\ \frac{\partial h}{\partial z}(x,y,z) = \frac{z\left(x^2+y^2-z^2\right)}{2\sqrt{(x+y+z))(-x+y+z)(x-y+z)(x+y-z)}} \end{cases} \tag{6.1}$$

$$\begin{cases} \frac{\partial \mathcal{A}_L}{\partial x_1}(u) = \frac{\partial}{\partial x_1}\left[h\left(t_1(u), x_1, t_2\right)\right] = -\frac{\partial h}{\partial x}(\omega_1) + \frac{\partial h}{\partial y}(\omega_1) = \\ \frac{(t_1-x_1)(t_1+x_1-t_2)(t_1+x_1+t_2)}{2\sqrt{(t_1+x_1+t_2)(-t_1+x_1+t_2)(t_1-x_1+t_2)(t_1+x_1-t_2)}} \\ \frac{\partial \mathcal{A}_L}{\partial t_{n-1}}(u) = \frac{\partial}{\partial t_{n-1}}\left[h\left(t_1(u), x_1, t_2\right) + h\left(t_{n-2}, x_{n-2}, t_{n-1}\right)\right] = -\frac{\partial h}{\partial x}(\omega_1) + \frac{\partial h}{\partial z}(\omega_{n-2}) \end{cases}$$

and for $k \in \{2, \cdots, n-2\}$:

$$\begin{cases} \frac{\partial \mathcal{A}_L}{\partial t_k}(u) = \frac{\partial}{\partial t_k}\left[h\left(t_{k-1}, x_{k-1}, t_k\right) + h\left(t_k, x_k, t_{k+1}\right)\right] = \frac{\partial h}{\partial z}(\omega_{k-1}) + \frac{\partial h}{\partial x}(\omega_k) \\ \frac{\partial \mathcal{A}_L}{\partial x_k}(u) = \frac{\partial}{\partial x_k}\left[h\left(t_1(u), x_1, t_2\right) + h\left(t_k, x_k, t_{k+1}\right)\right] = -\frac{\partial h}{\partial x}(\omega_1) + \frac{\partial h}{\partial y}(\omega_k) \end{cases} \tag{6.2}$$

($\star$) If the sides of the polygon are not all equal, there are two consecutive ones with a common vertex M and different lengths.

By changing the numbering of the vertices, one can assume $M = M_1$. In these conditions, the lengths $t_1 = M_n M_1$ and $x_1 = M_1 M_2$ are different and this implies $\dfrac{\partial \mathcal{A}_L}{\partial x_1}(u) \neq 0$ and that u is not a critical point of $\mathcal{A}_L$.

($\star$) If the lengths of all the sides are equal, then $t_1 = x_1 = x_2 = ... = x_{n-2} = t_{n-1}$ and a necessary condition for this polygon to be a critical point of $\mathcal{A}_L$, is $\frac{\partial \mathcal{A}_L}{\partial t_k}(u) = 0$ for any $k \in \{2, \cdots, n-2\}$, or :

$$\frac{\partial h}{\partial z}(t_{k-1}, x_k, t_k) = -\frac{\partial h}{\partial x}(t_k, x_k, t_{k+1}) \text{ pour tout } k \in \{2, \cdots, n-2\}.$$

This implies :

$$\left(\frac{\partial h}{\partial z}(t_{k-1}, x_k, t_k)\right)^2 - \left(\frac{\partial h}{\partial x}(t_k, x_k, t_{k+1})\right)^2 = 0 \text{ for any } k \in \{2, \cdots, n-2\}.$$

The development of this relationship leads to next equality :

$$\frac{(t_{k+1}t_{k-1}+x_k^2-t_k^2)(t_{k+1}t_{k-1}-x_k^2+t_k^2)(t_{k+1}+t_{k-1})(t_{k+1}-t_{k-1})x_k^2t_k^2}{\Sigma}=0$$

where :

$$\begin{aligned}\Sigma = \quad & (t_{k+1}+x_k+t_k)(t_{k+1}+x_k-t_k)(t_{k+1}-x_k+t_k)(t_{k+1}-x_k-t_k)\\ & (t_{k-1}+x_k+t_k)(t_{k-1}+x_k-t_k)(t_{k-1}-x_k+t_k)(t_{k-1}-x_k-t_k).\end{aligned}$$

Thus $(t_{k+1}-t_{k-1})(t_{k+1}t_{k-1}-x_k^2+t_k^2)(t_{k+1}t_{k-1}+x_k^2-t_k^2)=0$. The proof ends as that of the Theorem 4.2. We thus obtain the inscriptibility of all the quadrilaterals $(M_n, M_{k-1}, M_k, M_{k+1})$ for $k \in \{2, \cdots, n-2\}$ and then the inscriptibility of the polygon $(M_1, \cdots, M_n)$. But since the latter has all its sides of the same length, it is necessarily regular.

Finally the singular points of the map $\mathcal{A}_L$ are the regular polygons of $\Omega_{n,L}$, that is, the unique regular polygon ω_L of perimeter L.

Point (2)

Let $v=(\overline{t}_1, \overline{x}_1, \cdots, \overline{x}_{n-2}, \overline{t}_{n-1}) \in \mathcal{V}_n$ and let F_v be the set of convex polygons whose sides are $\overline{t}_1$, $\overline{x}_1$,$\cdots$,$\overline{x}_{n-2}$, $\overline{t}_{n-1}$. We have seen (by the Proposition 3.2) that this set is a submanifold of Ω_n, diffeomorphic to a convex open set of $\mathbb{R}^{n-3}$.

On the other hand, the area function $F_v \xrightarrow{\mathcal{A}} \mathbb{R}$ is given, for $t=(t_2, t_3, \cdots, t_{n-2}) \in F_v$, by :

$$\mathcal{A}(t)=h(\overline{t}_1, \overline{x}_1, t_2)+h(t_2, \overline{x}_2, t_3)+\cdots+h(t_{n-2}, \overline{x}_{n-2}, \overline{t}_{n-1}). \tag{6.3}$$

Setting :

$$\begin{cases}\omega_1=(\overline{t}_1, \overline{x}_1, t_2)\\ \\ \omega_k=(t_k, \overline{x}_k, t_{k+1}) \text{ for } k \in \{2, \cdots, n-3\}\\ \\ \omega_{n-2}=(t_{n-2}, \overline{x}_{n-2}, \overline{t}_{n-1}),\end{cases}$$

one can express the partial derivatives of $\mathcal{A}$ as follows :

$$\frac{\partial \mathcal{A}}{\partial t_k}(t)=\frac{\partial}{\partial t_k}\left[h\left(t_{k-1}, \overline{x}_{k-1}, t_k\right)+h\left(t_k, \overline{x}_k, t_{k+1}\right)\right]=\frac{\partial h}{\partial z}(\omega_{k-1})+\frac{\partial h}{\partial x}(\omega_k).$$

A critical point $t \in F_v$ of the area function must satisfy :

$$\frac{\partial h}{\partial z}(\omega_{k-1})+\frac{\partial h}{\partial x}(\omega_k)=0 \text{ and then } \left(\frac{\partial h}{\partial z}(\omega_{k-1})\right)^2-\left(\frac{\partial h}{\partial x}(\omega_k)\right)^2=0.$$

Setting $\omega_{k-1}=(u,v,w)$ and $\omega_k=(w,s,r)$, we obtain $\frac{\alpha}{\beta}=0$ where :

$$\begin{aligned}\alpha = \quad & w^2\left(r^2uv+rsu^2+rsv^2-rsw^2+s^2uv-uvw^2\right)\\ & \left(-r^2uv+rsu^2+rsv^2-rsw^2-s^2uv+uvw^2\right).\end{aligned}$$

and $\beta=(r-s-w)(r+s-w)(r-s+w)(r+s+w)(-u-v+w)(u-v+w)(-u+v+w)(u+v+w)$ or :

$$\begin{gathered}\left[(rsu^2+rsv^2-rsw^2)+(s^2uv-uvw^2+r^2uv)\right]\\ \left[(rsu^2+rsv^2-rsw^2)-(s^2uv-uvw^2+r^2uv)\right]=0.\end{gathered}$$

This implies :

$$\left(rsu^2+rsv^2-rsw^2\right)^2-\left(s^2uv-uvw^2+r^2uv)\right)^2=0$$

or :

$$rs(u^2+v^2-w^2)=\pm uv(r^2+s^2-w^2).$$

Then we deduce :

$$\cos\widehat{M_{k-1}}=\pm\cos\widehat{M_{k+1}}$$

which implies that the quadrilateral $(M_n, M_{k-1}, M_k, M_{k+1})$ is inscribable for any index $k \in \{2, \cdots, n-2\}$. Hence the polygon t is inscribable.

Conversely, if $t \in F_v$ is inscribable then, one can prove easily that t is a critical point of the area function.

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Université Polytechnique des Hauts-de-France
LAMAV, FR CNRS 2956,
ISTV2, Le Mont Houy
59313 Valenciennes Cedex 9
FRANCE

aziz.elkacimi@uphf.fr
abdellatif.zeggar@uphf.fr