Three-dimensional centrifugal-type instabilities of two-dimensional flows in rotating systems

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This paper deals with the stability of incompressible inviscid planar basic flows in a rotating frame. We give a sufficient condition for such flows to undergo three-dimensional shortwave centrifugal-type instabilities. This criterion reduces to the Bradshaw–Richardson (1969) or Pedley (1969) criterion in the specific case of parallel shear flows subject to rotation, to Rayleigh's centrifugal criterion (1916) in the case of axisymmetric vortices in inertial frames, to the Kloosterziel and van Heijst (1991) criterion in the case of axisymmetric vortices subject to rotation and to Bayly's criterion (1988) in the case of general two-dimensional flows in inertial frames. The criterion states that a steady 2D basic flow subject to rotation Ω is unstable if there exists a streamline for which at each point $2(V/\mathcal{R}+\Omega)(W+2\Omega)<0$ where W is the vorticity of the streamline, $\mathcal{R}$ is the local algebraic radius of curvature of the streamline and V is the local norm of the velocity. If this condition is satisfied then the flow is unstable according to the geometrical optics method introduced by Lifschitz and Hameiri (1991), which consists in following wave packets along the flow trajectories using a Wentzel–Kramers–Brillouin formalism. When the streamlines are closed, it is further shown that a localized unstable normal mode can be constructed in the vicinity of a streamline. As an application, this new criterion is used to study the centrifugal-type instabilities in the Stuart vortices, which is a family of exact solutions describing a row of periodic co-rotating eddies. For each solution of that family and for each rotation parameter $f=2\Omega$, we give the unstable streamline interval, according to the criterion of instability. This criterion gives only a sufficient condition of centrifugal instability. The equations of the geometrical optics method are therefore numerically solved to obtain the true centrifugally unstable streamline intervals. It turns out that our criterion gives excellent results for highly concentrated vortices, i.e., the two approaches yield the same unstable streamline intervals. In less concentrated vortices, some streamlines undergo centrifugal instability although our criterion is not fulfilled. From these numerical results, another criterion of centrifugal instability for a flow with closed streamlines is conjectured which reduces to the change of sign of the absolute vorticity $W+2\Omega$ somewhere in the flow. © *2000 American Institute of Physics.* [S1070-6631(00)00107-0]

I. INTRODUCTION

In this paper, we give a simple criterion of centrifugal instability for inviscid, incompressible, two-dimensional (2D) planar flows in rotating systems. This criterion is a sufficient condition for instability or equivalently a necessary condition for stability. It is in accordance with the ones that already exist in some restrictive situations. For parallel shear flows in rotating systems, it amounts to the Pedley[1] or Bradshaw–Richardson[2] criterion. For axisymmetric flows in inertial frames, it yields the Rayleigh centrifugal criterion.[3] With axisymmetric flows in rotating frames, we obtain the criterion given by Kloosterziel and van Heijst[4] and Mutabazi *et al.*[5] And, with general 2D planar flows in inertial frames, we obtain Bayly's criterion.[6]

The proof of linear instability will be given by means of two different approaches. A first proof is based on the geometrical optics method introduced by Lifschitz and Hameiri.[7] This method, roughly speaking, reduces to following a wave packet along the flow trajectories, using a (Wentzel–Kramers–Brillouin) approximation. This wave packet is a localized shortwave perturbation characterized by a wave vector $\mathbf{k}$ and an amplitude vector $\mathbf{a}$. The flow is unstable if there exists a streamline on which the amplitude $\mathbf{a}(t)$ of a particular wave packet grows unboundedly as $t\to\infty$. A second proof of instability will be given in terms of unstable normal modes. If the flow lies in the (x,y) plane and the rotation axis is parallel to the z axis, these normal modes are sought under the form $[\mathbf{u}',p'](x,y,z,t)=\exp(ikz+st)[\tilde{\mathbf{u}},\tilde{p}]\times(x,y)$ where k is the vertical wave number and s the complex amplification rate. The flow is unstable if a normal mode exhibits a complex amplification rate s with a positive real part. It has been shown in the case of a Taylor–Green flow—which is an infinite 2D array of counter-rotating vortices[8]—subject to rotation, that these two approaches are consistent.[9,10] In particular, elliptic and centrifugal-type normal modes have been identified whose characteristics (spatial structure and eigenvalues) are in accordance with the results given by the geometrical optics method.

This paper is an extension to rotating frames of a paper due to Bayly.[6] It is also a continuation of the works of Cambon *et al.*[11] and Leblanc and Cambon[12] in the search of a

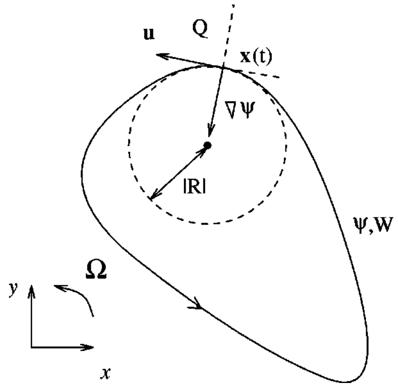

FIG. 1. Flow in the (x,y) plane. One streamline ψ is depicted. The flow is here assumed to be counterclockwise, so that $\nabla\psi$ and $\mathbf{u}$ have the disposition indicated. We have also shown the center of curvature, the radius of curvature $|\mathcal{R}|$, the quadrant $\mathbf{Q}$.

generalized criterion for the stability of two-dimensional flows in planes perpendicular to the rotation axis. Leblanc and Cambon[12] made a first attempt by suggesting that this criterion could be the local negativity of the second invariant of the inertial tensor somewhere in the flow. This effectively accounts for the special cases of Bradshaw–Richardson (or Pedley), Rayleigh and Kloosterziel and van Heijst, but it fails with Bayly's result.

Our criterion, which accounts for all of the above-mentioned criteria, is given in Sec. II. Then, as an application (Sec. III), we consider the case of Stuart vortices[13] which is a family of nonaxisymmetric eddies whose vorticity concentration can arbitrarily be varied. Using the geometrical optics method, it is shown that the criterion gives excellent results in the case of highly concentrated vortices. In other cases, some regions of the flow are shown to be centrifugally unstable even though our sufficient criterion is not fulfilled. This leads us to formulate (Sec. III C) a conjecture about a more general criterion of centrifugal instability, based only on the change of sign of the absolute vorticity of the basic flow. Note that Leblanc and Cambon[14] already gave some shortwave stability results on the case of Stuart vortices in a rotating frame, but their analysis was limited to elliptic and hyperbolic instabilities developing on stagnation points.

II. A SUFFICIENT CRITERION OF CENTRIFUGAL INSTABILITY

A. Presentation

We consider an inviscid incompressible flow in a frame rotating at a constant angular velocity $\mathbf{\Omega}=\Omega\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the unit vector in the positive z direction. We use the Euler equations referred to the rotating frame, so that a Coriolis term $2\mathbf{\Omega}\times\mathbf{u}$ has to be considered. The relative motion of the basic flow $\mathbf{u}$ is supposed to be two dimensional and to lie in a plane perpendicular to $\hat{\mathbf{z}}$, so that there exists a stream function $\psi(x,y)$ such as $\mathbf{u}(x,y)=\nabla\times[\psi(x,y)\hat{\mathbf{z}}]$ (see Fig. 1). The basic flow is a steady solution of the modified Euler equations:

$$\mathbf{u}\cdot\nabla\mathbf{u}+2\mathbf{\Omega}\times\mathbf{u}=-\nabla p, \quad \nabla\cdot\mathbf{u}=0. \tag{1}$$

The relative vorticity, $\mathbf{W}=\nabla\times\mathbf{u}=W\hat{\mathbf{z}}$ where $W=-\nabla^2\psi$, is therefore constant on each streamline ψ and is a function of ψ, $W=W(\psi)$. In the following, we will consider the local algebraic radius of curvature $\mathcal{R}$ at a given point of a streamline. This quantity is defined at all points of a particle trajectory $[\mathbf{x}(t)=x(t)\hat{\mathbf{x}}+y(t)\hat{\mathbf{y}}$ where $\mathbf{x}'=d\mathbf{x}/dt=\mathbf{u}]$ as

$$\mathcal{R}(x,y)=\frac{(x'^2+y'^2)^{3/2}}{y''x'-x''y'}=\frac{V^3}{(\nabla\psi)\cdot[\mathbf{u}\cdot\nabla\mathbf{u}]}, \tag{2}$$

where $V(x,y)=|\mathbf{u}(x,y)|$ is the norm of the velocity field. In this formula, $\nabla\psi$ leads to the x' and y' terms while $\mathbf{u}\cdot\nabla\mathbf{u}$ yields the x'' and y'' terms—in a stationary flow field $d^2\mathbf{x}/dt^2=d\mathbf{u}/dt=\mathbf{u}\cdot\nabla\mathbf{u}$. For the reader's sake, the term $(\nabla\psi)\cdot[\mathbf{u}\cdot\nabla\mathbf{u}]$ can also be written as (in tensorial notation with indices and Einstein convention): $(\partial\psi/\partial x_i)u_j(\partial u_i/\partial x_j)$. Here, $\mathcal{R}>0$ if the flow is locally counterclockwise and $\mathcal{R}<0$ if the flow is locally clockwise.

Let us introduce the following quantities:

$$\delta(\mathbf{x})=2\left(\frac{V}{\mathcal{R}}+\Omega\right)(W+2\Omega), \tag{3}$$

$$\Delta(\psi)=\max_{\psi}\delta(\mathbf{x}), \tag{4}$$

where $\max_{\psi}$ denotes the maximum over the streamline ψ. The flow is unstable if there exists ψ_0 such as

$$\Delta(\psi_0)<0. \tag{5}$$

It will be shown below that instability is achieved both with respect to the geometrical optics method (see Sec. II C) and with respect to a classical normal mode analysis (see Sec. II D). The perturbations $(\mathbf{u}',p')$ are three-dimensional (3D) in both cases and are governed by the linearized modified Euler equations:

$$\frac{\partial\mathbf{u}'}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}'+\mathbf{u}'\cdot\nabla\mathbf{u}+2\mathbf{\Omega}\times\mathbf{u}'=-\nabla p', \quad \nabla\cdot\mathbf{u}'=0. \tag{6}$$

The criterion $\Delta(\psi)<0$ compares the sign of the absolute angular velocity of the particle $V/\mathcal{R}+\Omega$ to the sign of its absolute vorticity $W+2\Omega$. If these two quantities have opposite signs along a whole streamline ψ_0, then the flow is unstable. The quantity $\delta(\mathbf{x})$ is a polynom of the second order in Ω. The two roots of this polynom, $-V/\mathcal{R}$ and $-W/2$, have a special importance since if Ω is in the interval bounded by the two roots, then $\delta(\mathbf{x})<0$. The two roots are linked since:

$$\frac{V}{\mathcal{R}}=\frac{W}{2}+\mathcal{S}, \tag{7}$$

where $\mathcal{S}=\mathbf{t}\cdot[1/2(\mathcal{L}+\mathcal{L}^T)\mathbf{n}]$. Here $\mathcal{L}=\nabla\mathbf{u}$ designates the velocity gradient tensor. The term $\mathcal{S}$ is the nondiagonal term of the symmetric part of the velocity gradient tensor in the Serret–Frenet vector basis $(\mathbf{t}=\mathbf{u}/V,\mathbf{n}=\nabla\psi/V)$. It can therefore be referred to as the intrinsic shear—note that in the case of parallel shear flows $U(y)$, this definition yields twice the usual shear rate: $\mathcal{S}=2\partial U/\partial y$. Equation (7) shows that the relative angular velocity of a fluid particle $V/\mathcal{R}$ is the sum of two terms. The first one, $W/2$, is due to the vorticity

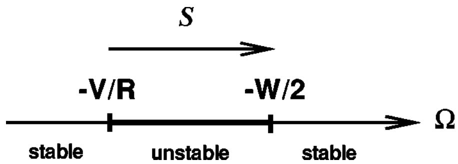

FIG. 2. A streamline is centrifugally unstable if the system rotation Ω lies in the interval bounded by $-V/\mathcal{R}$ and $-W/2$ along the whole streamline. The width of this interval is $|\mathcal{S}|$. $-W/2$ is a fixed bound of the interval since the vorticity is constant along the streamline whereas $-V/\mathcal{R}$ can change along the streamline. For centrifugal instability, $\mathcal{S}$ is required to be either strictly positive (as it is in the figure) or strictly negative along the whole streamline, so that Ω can lie in the specified unstable interval all along the streamline.

of the basic flow and the second one, $\mathcal{S}$, represents the angular velocity due to the shear. The criterion $\Delta(\psi)<0$ can be satisfied only if the angular velocity Ω lies at each point of the streamline ψ between $-V/\mathcal{R}$ and $-W/2$. This requires that the intrinsic shear $\mathcal{S}=V/\mathcal{R}-W/2$ be either strictly positive or strictly negative along a whole streamline ψ (see Fig. 2). The physical mechanism of instability can therefore be traced back to the stretching of the vorticity of the perturbation by the intrinsic shear $\mathcal{S}$.

B. Special cases

In the particular case where the streamlines are circular, the criterion $\Delta(\psi)<0$ is identical to the one given by Kloosterziel and van Heijst[4] and Mutabazi *et al.*:[5]

$$2\left(\frac{U(r)}{r}+\Omega\right)(W(r)+2\Omega)<0 \quad \text{for some radius } r_0, \tag{8}$$

where $U(r)$ is the orthoradial velocity of the basic flow. If in addition the background rotation is zero, then we obtain Rayleigh's[3] criterion:

$$\frac{U(r)}{r}W(r)<0 \quad \text{for some radius } r_0. \tag{9}$$

If the radius of curvature $\mathcal{R}$ tends to infinity, we are led to the Pedley[1] or Bradshaw–Richardson[2] criterion, which gives sufficient conditions for a parallel shear flow $U(y)$ to be destabilized:

$$2\Omega\left(2\Omega-\frac{dU}{dy}\right)<0 \quad \text{for some } y_0. \tag{10}$$

In an inertial frame, we also get Bayly's[6] result which applies to general planar basic flows with convex streamlines: ''sufficient conditions for centrifugal instability are that streamlines be convex closed curves in some region of the flow, with the magnitude of the circulation decreasing outward.'' In our formalism, this can be restated in the following manner:

$$\frac{V}{\mathcal{R}}W<0 \quad \text{on a whole streamline } \psi_0. \tag{11}$$

C. Proof of instability by means of the geometrical optics method

1. Presentation of the geometrical optics method

Following Lifschitz and Hameiri,[7] the basic flow field $[\mathbf{u}(\mathbf{x}),p(\mathbf{x})]$ is perturbed by the following WKB-form velocity/pressure field:

$$\begin{bmatrix}\mathbf{u}'\\ p'\end{bmatrix}(\mathbf{x},t)=\exp\left[i\,\frac{\phi(\mathbf{x},t)}{\epsilon}\right]\left\{\begin{bmatrix}\mathbf{a}\\ \pi\end{bmatrix}(\mathbf{x},t)+\epsilon\begin{bmatrix}\mathbf{a}_\epsilon\\ \pi_\epsilon\end{bmatrix}(\mathbf{x},t)\right\}+\epsilon\begin{bmatrix}\mathbf{u}_r\\ \pi_r\end{bmatrix}(\mathbf{x},t), \tag{12}$$

where ϵ is a small parameter and ϕ is a phase field which is real. In the following, $\mathbf{k}=\nabla\phi$ is the wave vector of the perturbation field. This perturbation is substituted into the incompressible linearized Euler equations (6). The resulting equations contain terms of various orders in ϵ. Equating the lowest-order terms yields $\pi=0$, the incompressibility condition $\mathbf{k}\cdot\mathbf{a}=0$ and the eikonal equation:

$$(\partial_t+\mathbf{u}\cdot\nabla)\phi=0, \tag{13}$$

i.e., the phase field ϕ is passively advected. The next-lowest-order terms yield the evolution equation for the velocity envelope function $\mathbf{a}$:

$$(\partial_t+\mathbf{u}\cdot\nabla)\mathbf{a}+\mathcal{L}(\mathbf{x})\mathbf{a}+2\boldsymbol{\Omega}\times\mathbf{a}=-i\mathbf{k}\pi_\epsilon. \tag{14}$$

Projecting this equation in the plane perpendicular to $\mathbf{k}$, i.e., multiplying this equation by the operator $\mathcal{I}-(\mathbf{k}\otimes\mathbf{k})/|\mathbf{k}|^2$ where $\mathcal{I}$ is the identity tensor and $\otimes$ the tensor product, yields

$$(\partial_t+\mathbf{u}\cdot\nabla)\mathbf{a}=\left(\frac{2\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}-\mathcal{I}\right)\mathcal{L}(\mathbf{x})\mathbf{a}+\left(\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}-\mathcal{I}\right)(2\boldsymbol{\Omega}\times\mathbf{a}). \tag{15}$$

Equations (13) and (15) form a system evolving locally along particle trajectories. We can therefore write them as

$$\frac{d\mathbf{x}}{dt}=\mathbf{u}(\mathbf{x}), \tag{16}$$

$$\frac{d\mathbf{k}}{dt}=-\mathcal{L}^T(\mathbf{x})\mathbf{k}, \tag{17}$$

$$\frac{d\mathbf{a}}{dt}=\left(\frac{2\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}-\mathcal{I}\right)\mathcal{L}(\mathbf{x})\mathbf{a}+\left(\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}-\mathcal{I}\right)(2\boldsymbol{\Omega}\times\mathbf{a}). \tag{18}$$

Here $\mathbf{x}(t)$ is the position at time t of a fluid particle and the superscript T is the transpose. For the reader's sake, we give these three equations, in tensorial notations with indices:

$$\frac{dx_i}{dt}=u_i, \tag{19}$$

$$\frac{dk_i}{dt}=-\mathcal{L}_{mi}k_m, \tag{20}$$

$$\frac{da_i}{dt}=\left(\frac{2k_ik_m}{|\mathbf{k}|^2}-\delta_{im}\right)\mathcal{L}_{mn}a_n+2\left(\frac{k_ik_m}{|\mathbf{k}|^2}-\delta_{im}\right)\epsilon_{mnp}\Omega_na_p, \tag{21}$$

where $\mathcal{L}_{mn}=\partial u_m/\partial x_n$ is evaluated at a point $\mathbf{x}$ of the trajectory. Lifschitz and Hameiri[7] proved that a sufficient criterion for instability is that the above systems have at least one solution for which the amplitude $\mathbf{a}(t)$ increases unboundedly as $t\rightarrow\infty$.

This WKB method can be considered as an extension of techniques that exist for homogeneous flows—for which $\mathcal{L}$ is space uniform—to general flows—for which $\mathcal{L}(\mathbf{x})$ is space dependant. As a matter of fact, Eqs. (17) and (18) have already been extensively used with homogeneous flows in the context of stability studies[15–19] and in the context of homogeneous rapid distortion theory[20–23] (RDT). Disturbances are then easily found under time-dependant Fourier modes $\mathbf{a}(t)\exp[i\mathbf{k}(t)\cdot\mathbf{x}]$ with $\mathbf{a}$ and $\mathbf{k}$ governed by (17) and (18). Then the shortwave asymptotics allows to consider application to general flows for which $\mathcal{L}(\mathbf{x})$ is space dependant, using a WKB ''mathematical zoom,'' provided the length scale of the disturbance be smaller than any length scale of the background flow. The parameter ϵ in (12) provides this scale separation, so that homogeneous stability and RDT equations are recovered at the leading order for a wave packet which is convected and distorted following individual trajectories.

2. Proof of instability

In this section, we prove that if there exists a streamline ψ_0 such as $\Delta(\psi_0)<0$, then there exists a wave packet for which the amplitude $\mathbf{a}(t)$ increases unboundedly as $t\rightarrow\infty$.

The condition $\Delta(\psi_0)<0$ can be satisfied in the two following cases:

$$\Omega+W(\psi_0)/2<0 \quad \text{and} \quad \Omega+\min_{\psi_0} V/\mathcal{R}>0 \tag{22}$$

or

$$\Omega+W(\psi_0)/2>0 \quad \text{and} \quad \Omega+\max_{\psi_0} V/\mathcal{R}<0, \tag{23}$$

where $\min_{\psi_0}$ and $\max_{\psi_0}$ designate the minimum and maximum over the streamline ψ_0. Only the first case (22) will be treated here, the demonstration of the other case being similar. The proofs follow the treatment given by Bayly.[6] Here, we extend his formalism to the rotating case.

Let us consider spanwise perturbations where the wave vector is perpendicular to the flow field ($\mathbf{k}\|\hat{\mathbf{z}}$). Equation (18) therefore reduces to

$$\frac{d\mathbf{a}}{dt}=-\mathcal{L}(\mathbf{x})\mathbf{a}-2\boldsymbol{\Omega}\times\mathbf{a}. \tag{24}$$

We now briefly discuss the pressureless nature of such perturbations. It is important to quote that Eq. (18) includes in general a contribution from pressure through the terms $\mathbf{k}\otimes\mathbf{k}/|\mathbf{k}|^2$. But this contribution disappears for some orientations of the wave vector, especially when $\mathbf{k}$ is perpendicular to the plane of the background 2D flow (and thus aligned with the angular velocity vector $\boldsymbol{\Omega}$). Hence, Eq. (24) can also be obtained by throwing out the pressure term in Eqs. (6) and therefore in Eq. (14). The primary function of pressure in shortwave perturbations is to maintain at all times the incompressibility of the perturbation field ($\mathbf{k}\perp\mathbf{a}$). Now, for spanwise perturbations (and only for these perturbations), if $\mathbf{k}\perp\mathbf{a}$ at $t=0$, then $\mathbf{k}\perp\mathbf{a}$ for all times, without having to invoke the pressure: considering Eq. (14), if $\mathbf{a}$ lies in the horizontal plane, then the distortions induced by the velocity gradient tensor, $-\mathcal{L}\mathbf{a}$, and those induced by the Coriolis force, $-2\boldsymbol{\Omega}\times\mathbf{a}$, are in the horizontal plane. The pressure field is therefore not necessary for such spanwise perturbations to remain incompressible, so that they can be referred to as pressureless perturbations. The role of the spanwise (and therefore pressureless) perturbations was analyzed by Leblanc and Cambon[12,14] for elliptic and hyperbolic instabilities developing on stagnation points and by Bayly[6] for centrifugal instabilities. In the context of shortwave asymptotics, it is important to note that the term pressureless is not a mathematical artifact as it is in the usual displaced-particle argument.[4,24] In this latter theory, one just throws away pressure without justification and one obtains, almost fortuitously, the correct criterion. On the other hand, with shortwave asymptotics, the fact that the unstable perturbations are pressureless is a consequence and stems from the fact that the wave vector of these perturbations is perpendicular to the basic flow field.

Equation (24) yields the following equation for the horizontal component of $\mathbf{a}$:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{a}\cdot\mathbf{u}\\ \mathbf{a}\cdot\boldsymbol{\nabla}\psi\end{pmatrix}=\begin{pmatrix}0 & W+2\Omega\\ -2(V/\mathcal{R}+\Omega) & 2V'/V\end{pmatrix}\begin{pmatrix}\mathbf{a}\cdot\mathbf{u}\\ \mathbf{a}\cdot\boldsymbol{\nabla}\psi\end{pmatrix}, \tag{25}$$

where $V'=dV/dt=\mathbf{u}\cdot\boldsymbol{\nabla}V$. Equation (25) is obtained by using the relation $(\boldsymbol{\nabla}\psi)\cdot(\mathcal{L}\mathbf{u})=V^3/\mathcal{R}$ and the fact that the antisymmetric part of the velocity gradient tensor $(\mathcal{L}-\mathcal{L}^T)/2$ equals the operator $(\mathbf{W}/2)\times(\,)$ where $\times$ denotes the cross product.

At each point $\mathbf{x}$ of the streamline ψ_0, the vectors $\mathbf{u}$ and $\boldsymbol{\nabla}\psi$ divide the plane into four quadrants. The quadrant $\mathbf{Q}(\mathbf{x})$ consists of all vectors $\mathbf{v}$ satisfying $\mathbf{v}\cdot\mathbf{u}>0$ and $\mathbf{v}\cdot\boldsymbol{\nabla}\psi<0$ (see Fig. 1). We first prove that if $\mathbf{a}(t=0)$ lies in the quadrant $\mathbf{Q}(t=0)$, then $\mathbf{a}(t)$ remains in the quadrant $\mathbf{Q}(t)$ for all $t>0$. In other words, we show that if $\mathbf{a}\cdot\mathbf{u}>0$ and $\mathbf{a}\cdot\boldsymbol{\nabla}\psi<0$ hold at $t=0$ then they continue to hold for all $t>0$. To prove this, suppose that this were to be violated. So there must be some time $t>0$ when either

$$\mathbf{a}\cdot\boldsymbol{\nabla}\psi=0 \quad \text{with} \quad \mathbf{a}\cdot\mathbf{u}>0 \quad \text{and} \quad (d/dt)(\mathbf{a}\cdot\boldsymbol{\nabla}\psi)\geqslant 0 \tag{26}$$

or

$$\mathbf{a}\cdot\mathbf{u}=0 \quad \text{with} \quad \mathbf{a}\cdot\boldsymbol{\nabla}\psi<0 \quad \text{and} \quad (d/dt)(\mathbf{a}\cdot\mathbf{u})\leqslant 0 \tag{27}$$

must hold. We show that neither of these two assumptions can occur.

First, suppose that $\mathbf{a}\cdot\boldsymbol{\nabla}\psi=0$ occurred with $\mathbf{a}\cdot\mathbf{u}>0$ for some time t. Therefore $\mathbf{a}=c\mathbf{u}$ for some positive number c. Thus, from Eq. (25), we find

$$\frac{d}{dt}(\mathbf{a}\cdot\boldsymbol{\nabla}\psi)=-2cV^2(V/\mathcal{R}+\Omega). \tag{28}$$

Now, assumption (22) yields $V/\mathcal{R}+\Omega>0$ along ψ_0, so that $(d/dt)(\mathbf{a}\cdot\boldsymbol{\nabla}\psi)<0$ which contradicts (26).

Secondly, suppose that $\mathbf{a}\cdot\mathbf{u}=0$ with $\mathbf{a}\cdot\nabla\psi<0$ for some time t. Now, Eq. (25) yields

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{u})=(W+2\Omega)(\mathbf{a}\cdot\nabla\psi) \tag{29}$$

so that $(d/dt)(\mathbf{a}\cdot\mathbf{u})>0$, since $W+2\Omega<0$ [assumption (22)]. This contradicts (27).

Now, using the fact that the amplitude $\mathbf{a}(t)$ remains at all times in the quadrant $\mathbf{Q}(t)$, it can be shown that it grows exponentially. Consider the angle $\theta(t)$ between $\mathbf{u}(\mathbf{x}(t))$ and $\mathbf{a}(t)$. Since (24) is linear in $\mathbf{a}$, θ satisfies a first-order ordinary differential equation depending only on θ itself and $\mathbf{x}(t)$. Equation (28) shows that $d\theta/dt(\theta=0)$ $[=2(V/\mathcal{R}+\Omega)]$ is separated by a finite amount from zero. If the streamline ψ_0 is closed, it can be deduced that there is a finite angle $\theta_c>0$ such as $d\theta/dt(\theta=\theta_c)$ is greater than or equal to zero on the whole streamline ψ_0. This fact remains true in the case of parallel shear flows where $\mathcal{R}\rightarrow\infty$ and $V=cst$. So if $\theta(t=0)\geqslant\theta_c$, then $\theta(t)\geqslant\theta_c$. This implies, using (29), that

$$\mathbf{a}(t)\cdot\mathbf{u}(\mathbf{x}(t))\geqslant\exp[-t(W+2\Omega)\tan\theta_c] \times[\mathbf{a}(t=0)\cdot\mathbf{u}(\mathbf{x}(t=0))]. \tag{30}$$

In the case of a closed streamline $\psi_0, \mathbf{u}(\mathbf{x}(t))$ is periodic with a period $T(\psi_0)$, so that $\mathbf{a}(nT(\psi_0))$ grows exponentially and monotically as $t\rightarrow\infty$. The same conclusion, i.e., the exponential growth of $\mathbf{a}(t)$ as $t\rightarrow\infty$, is reached in the case of parallel shear flows. The flow is therefore unstable with respect to the geometrical optics method.

The fact that $\mathbf{a}(t)$ always remains in the quadrant $\mathbf{Q}(t)$ for all times means that the perturbation exchanges fluid from the region $\psi<\psi_0$ with fluid from the region $\psi>\psi_0$. We thus retrieve one of the basic features of centrifugal instability.

The general criterion presented by Leblanc and Cambon[12] is also based on the components of $\mathbf{a}$ in the frame of reference attached to the trajectories (the Serret–Frenet vector basis), with a system of equations close to (25). The difference lies in the fact that they chose a normed frame, so that they analyzed the quantities $(d/dt)(\mathbf{a}\cdot\mathbf{t})$ and $(d/dt)(\mathbf{a}\cdot\mathbf{n})$ rather than $(d/dt)(\mathbf{a}\cdot\mathbf{u})$ and $(d/dt)(\mathbf{a}\cdot\nabla\psi)$. As a result, their matrix is not triangular but full. Now, it is precisely the special reduced form of the matrix involved in (25) that enables one to prove a criterion of instability. The matrix in (25) actually combines in a more suitable way advection, distortion and rotation effects than the matrix does in the paper of Leblanc and Cambon. This explains the failure of the Leblanc and Cambon criterion to account for the centrifugal instability. Note that Scorer and Wilson[25] were the first to achieve a stability analysis by considering the vorticity equation in the Serret–Frenet vector basis.

D. Proof of instability by means of a normal mode analysis

We consider closed streamlines ψ with $\Delta(\psi)<0$. We show that a localized normal mode can be constructed in the neighborhood of a streamline. We follow Bayly's formalism[6] and extend it to the rotating case which is considered here. Normal modes are sought in the usual way by considering a vertical wavelength k and a complex amplification rate s:

$$\begin{bmatrix}\mathbf{u}'\\ p'\end{bmatrix}(x,y,z,t)=\exp(ikz+st)\begin{bmatrix}\tilde{\mathbf{u}}\\ \tilde{p}\end{bmatrix}(x,y). \tag{31}$$

The main idea is to use a particular vector field basis $\mathbf{f}_i$ for the representation of the eigenmode $\tilde{\mathbf{u}}(x,y)$:

$$\tilde{\mathbf{u}}(x,y)=\tilde{u}(x,y)\mathbf{f}_1(x,y)+\tilde{v}(x,y)\mathbf{f}_2(x,y) +\tilde{w}(x,y)\mathbf{f}_3(x,y), \tag{32}$$

$\mathbf{f}_i$ is constructed using the eigenvalues m_i and the eigenvectors $\mathbf{e}_i$ of the fundamental Floquet matrix associated to the differential equation (24). This matrix is $\mathcal{A}(T(\psi))$ where $T(\psi)$ refers to the time period on the streamline ψ and $\mathcal{A}(t)$ is obtained from

$$\frac{d\mathcal{A}}{dt}=-\mathcal{L}(\mathbf{x})\mathcal{A}-2\mathbf{\Omega}\times\mathcal{A}, \quad \mathcal{A}(0)=\mathcal{I}. \tag{33}$$

Since the basic flow lies in the (x,y) plane, the eigenvalues/eigenvectors $(m_i,\mathbf{e}_i)$ can be taken as follows: $(m_3=1,\mathbf{e}_3=\hat{\mathbf{z}})$ and $(\mathbf{e}_1,\mathbf{e}_2)$ lie in the (x,y) plane. From the incompressibility of the basic flow, the two corresponding complex eigenvalues (m_1,m_2) where $|m_1|\geqslant|m_2|$ must multiply to 1 : $m_1m_2=1$. As $\Delta(\psi)<0$, there exists in Eq. (24) a perturbation whose amplitude $\mathbf{a}(t)$ exponentially increases as $t\rightarrow\infty$ (see Sec. II C 2). This can be restated here by saying that m_1 and m_2 are real and reciprocals, with $|m_1|>1$ and $|m_2|<1$. The Floquet exponents, which are the logarithms of $|m_i|$ divided by the period $T(\psi)$ are therefore as follows: $s_1=\sigma$, $s_2=-\sigma$ and $s_3=0$ where $\sigma>0$. The vector fields $\mathbf{f}_i$ are then defined in the following way: $\mathbf{f}_i(\mathbf{x})=\exp[-s_it]\mathcal{A}(t)\mathbf{e}_i$ where $\mathbf{x}(t)$ is a particle trajectory. From Sec. II C 2, it follows that $\mathbf{f}_1$ lies in the quadrant $\mathbf{Q}(\mathbf{x})$. Considering σ as a function of ψ, we shall further assume that $\sigma(\psi)$ takes a quadratic maximum on a given streamline ψ_0, i.e., $\sigma'(\psi_0)=0$ and $-\sigma''(\psi_0)>0$. This streamline, where $\sigma(\psi_0)>0$, turns out to be the one in whose neighborhood we can construct a localized instability.

In the limit $k\rightarrow\infty$, the eigenmodes are sought with the following asymptotic behavior in k:

$$\tilde{u}=\mathcal{U}, \quad \tilde{v}=k^{-1}\mathcal{V}, \quad \tilde{w}=k^{-1/2}\mathcal{W}, \quad \tilde{p}=k^{-3/2}\mathcal{P} \tag{34}$$

with $s=\sigma(\psi_0)-\mu/k$ and where $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ have been introduced in (32) and μ is a constant to be determined. Hence, as $k\rightarrow\infty$, the amplification rate s of the constructed eigenmode (31) converges towards the predicted maximum value of the geometrical optics method $\sigma(\psi_0)$. The solution (34) is localized within a region of width $O(k^{-1/2})$ around the streamline ψ_0. The following rescaled stream function coordinate is therefore considered: $\eta=k^{1/2}(\psi-\psi_0)$, so that $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, $\mathcal{P}$ are now functions of η. Introducing these expansions in the linearized Euler equations, we are led to the quantum harmonic oscillator:[12,26]

$$\frac{d^2\mathcal{U}}{d\eta^2}+\left(\frac{\mu}{C(\psi_0)}-\lambda^2\eta^2\right)\mathcal{U}=0 \quad \text{with } \mathcal{U}(\pm\infty)=0, \tag{35}$$

where

$$C(\psi_0)=\frac{1}{T(\psi_0)}\int_0^{T(\psi_0)}(\mathbf{f}_1^\dagger\cdot\nabla\psi)\times\left[\sigma(\psi_0)+\frac{d}{dt}\right](\mathbf{f}_1\cdot\nabla\psi)\,dt, \tag{36}$$

$$\lambda^2=-\frac{\sigma''(\psi_0)}{2C(\psi_0)}. \tag{37}$$

In these equations, $\mathbf{f}_i^\dagger(x,y)$ is the adjoint vector field corresponding to $\mathbf{f}_i(x,y)$:$\mathbf{f}_i^\dagger(x,y)\cdot\mathbf{f}_j(x,y)=\delta_{ij}$.

Now, the condition $\Delta(\psi_0)<0$ implies that $C(\psi_0)>0$. This can be shown by combining the proof given in Bayly's paper[6] and the modifications introduced in Sec. II C 2 to cope with the rotating frame. At last, $-\sigma''(\psi_0)$ is positive by assumption, so that $\lambda^2>0$. Equation (35) has therefore the following solution:

$$\mathcal{U}(\eta)=\exp\left(-\frac{\lambda\eta^2}{2}\right) \tag{38}$$

with $\mu=\lambda C(\psi_0)$. This eigenmode is exponentially concentrated in the neighborhood of the streamline ψ_0 on a characteristic length scale $1/\sqrt{\lambda}$. It's amplification rate is $s=\sigma(\psi_0)-\mu/k$ with $\sigma(\psi_0)>0$. The flow is therefore unstable with respect to a classical normal mode analysis.

From a physical point of view, the constructed unstable normal mode again reflects the basic physics of the centrifugal instability. The localized modes take the form of highly elongated eddies. The fluid motion in these instabilities is predominantly in the horizontal plane along the unstable $\mathbf{f}_1$ direction, which belongs to the quadrant $\mathbf{Q}(\mathbf{x})$. This again shows that an exchange of fluid occurs between the region $\psi<\psi_0$ and the region $\psi>\psi_0$. This exchange is modulated in the z direction by the term $\exp(ikz)$. In order to preserve exact incompressibility, a pressure field and a z velocity component are also present in the perturbation, but at higher order in k^{-1}. This again reflects the pressureless nature of the centrifugal instability.

III. CENTRIFUGAL-TYPE INSTABILITIES IN STUART VORTICES

In this section, we apply the criterion $\Delta(\psi)<0$ to the Stuart vortices, which is a particular nonaxisymmetric two-dimensional steady flow. We compare the unstable streamline interval given by this criterion to the one given by the numerical evaluation of the Floquet exponent $\sigma(\psi)$ associated to Eq. (33). This comparison shows that the criterion works well for highly concentrated vortices. We end this section by a discussion on the case of centrifugal unstable streamlines which do not satisfy the sufficient criterion proposed above.

A. The Stuart vortices

The Stuart vortices constitute a family of exact solutions of the Euler equations often used to model 2D mixing layers. The nondimensional stream function is given by

$$\psi=\log(\cosh y-\rho\cos x), \tag{39}$$

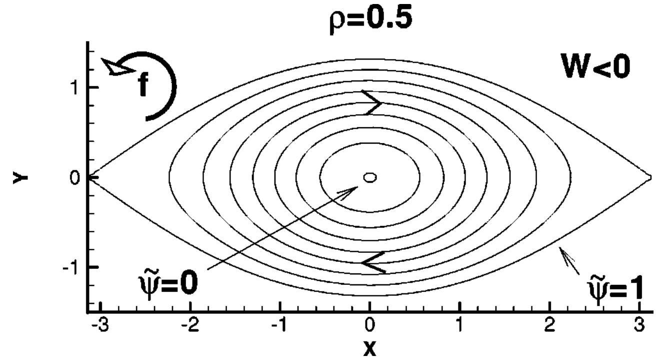

FIG. 3. Streamlines of the Stuart vortices for $\rho=0.5$. The flow is periodic in the x direction with period 2π.

where $0\leq\rho\leq1$ is a constant. If $\rho=0$, the hyperbolic-tangent mixing layer is recovered, whereas for $\rho=1$, we obtain a single row of co-rotating point vortices, with circulation -4π, periodically spaced along the x axis with period 2π. If $0<\rho<1$, the shear-layer exhibits periodic two-dimensional co-rotating eddies with smooth vorticity distribution, each of which has the circulation -4π. For a given parameter $0<\rho<1$, the stream function is minimum in the center of the vortices $\psi=\psi_{\min}=\log(1-\rho)$, increases monotically outward and reaches the value $\psi_{\max}=\log(1+\rho)$ on the streamline that separates the eddies. A given streamline will be referred in the following by $\tilde{\psi}=(\psi-\psi_{\min})/(\psi_{\max}-\psi_{\min})$. Thus, $0\leq\tilde{\psi}\leq1$ parametrizes the streamlines. $\tilde{\psi}=0$ represents the center of the vortices and $\tilde{\psi}=1$ the streamline separating the eddies. In Fig. 3, we have sketched the isovalues of the stream function in the particular case $\rho=0.5$. We consider this flow in a frame rotating at angular velocity Ω. In the following, we use $f=2\Omega$ to specify the level of background rotation.

B. Application and evaluation of the criterion $\Delta(\tilde{\psi})<0$

In this section, we apply the criterion $\Delta(\tilde{\psi})<0$ to obtain centrifugally unstable streamline intervals in the Stuart vortices. We compare these intervals to those obtained with the condition $\sigma(\tilde{\psi})>0$. The Floquet exponent $\sigma(\tilde{\psi})$ associated to the matrix $\mathcal{A}(T(\tilde{\psi}))$ is numerically obtained by integrating Eq. (33) using a fourth order Runge–Kutta scheme.

Figure 4 is relative to the case $\rho=0.9$, $f=2$. The solid line with filled circles represents the Floquet exponent $\sigma(\tilde{\psi})$ and the solid line with empty triangles refers to $\Delta(\tilde{\psi})$. There are several unstable zones where $\sigma(\tilde{\psi})>0$. We focus on the one which exactly starts at the streamline $\tilde{\psi}$ where $\Delta(\tilde{\psi})$ becomes negative. This is the centrifugally unstable region. The extent of the unstable streamline interval ($\sigma(\tilde{\psi})>0$) almost corresponds to the extent of the region where $\Delta(\tilde{\psi})<0$. In this case, the criterion $\Delta(\tilde{\psi})<0$ gives excellent results to predict the centrifugally unstable region. However, considering the end of the centrifugally unstable streamline interval, we note that $\Delta(\tilde{\psi})$ becomes positive before the end of the unstable zone is reached. Figure 5 shows the case $\rho=0.6$, $f=1$ and Fig. 6 the case $\rho=0.3$, $f=1$. We again

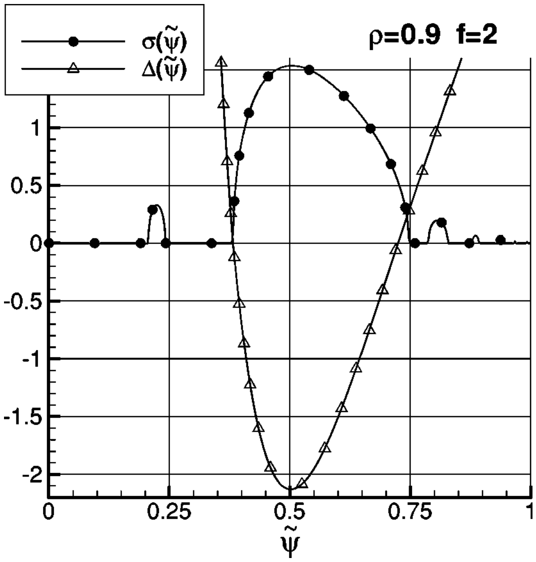

FIG. 4. Case $\rho=0.9$, $f=2$: Floquet exponent $\sigma(\tilde{\psi})$ and criterion $\Delta(\tilde{\psi})$.

consider the unstable intervals ($\sigma(\tilde{\psi})>0$) which exactly start at the streamline $\tilde{\psi}$ where $\Delta(\tilde{\psi})$ becomes negative. We notice that the smaller ρ, the worse the agreement between the upper interval bound obtained with the $\sigma(\tilde{\psi})>0$ condition and the one given by the $\Delta(\tilde{\psi})<0$ condition. In Fig. 6, $\Delta(\tilde{\psi})$ is even always strictly positive, except on one streamline $\tilde{\psi}_0$ where $\Delta(\tilde{\psi}_0)=0$. In conclusion, as ρ decreases, the criterion $\Delta(\tilde{\psi})<0$ is less and less efficient to predict the extent of the centrifugally unstable streamline interval.

At last, an evaluation of the criterion $\Delta(\tilde{\psi})<0$ with regards to the condition $\sigma(\tilde{\psi})>0$ for all values of ρ and f is given in Figs. 7 and 8. In Fig. 7, the white region shows the centrifugally unstable domain in the $(\tilde{\psi},\rho)$ plane, according to $\Delta(\tilde{\psi})<0$. $\Delta(\tilde{\psi})<0$ is fulfilled if there exists f such as either $-W/2<f/2<-\max_{\tilde{\psi}} V/\mathcal{R}$ or $-\min_{\tilde{\psi}} V/\mathcal{R}<f/2<$

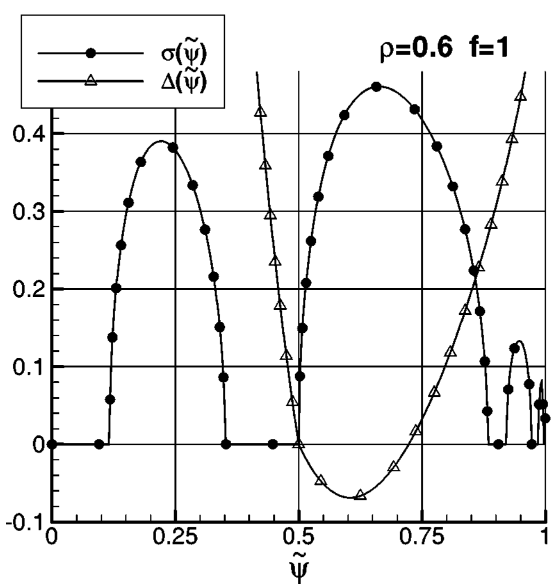

FIG. 5. Case $\rho=0.6$, $f=1$: Floquet exponent $\sigma(\tilde{\psi})$ and criterion $\Delta(\tilde{\psi})$.

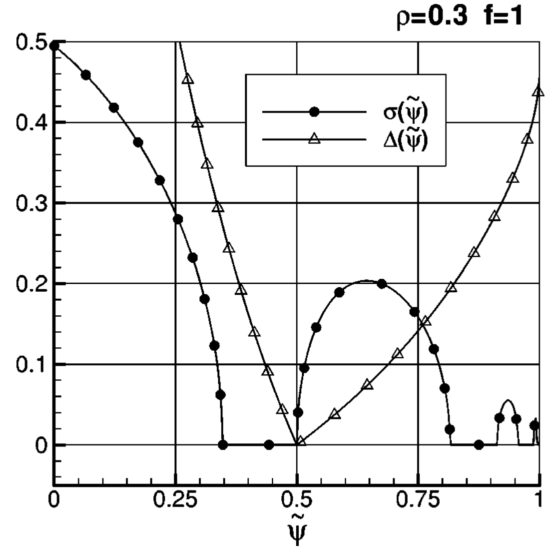

FIG. 6. Case $\rho=0.3$, $f=1$: Floquet exponent $\sigma(\tilde{\psi})$ and criterion $\Delta(\tilde{\psi})$.

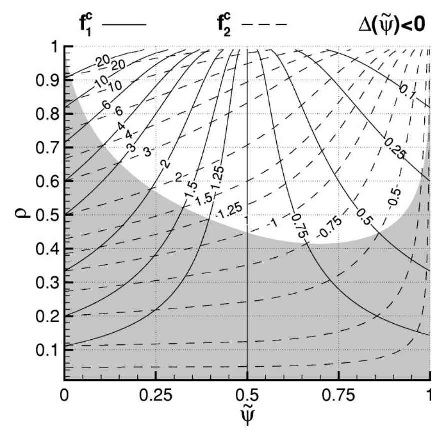

FIG. 7. Unstable $(\tilde{\psi},\rho)$ region (in white) according to the criterion $\Delta(\tilde{\psi})<0$. The solid and dashed lines represent the bounds of the unstable f intervals: $f_1^c<f<f_2^c$.

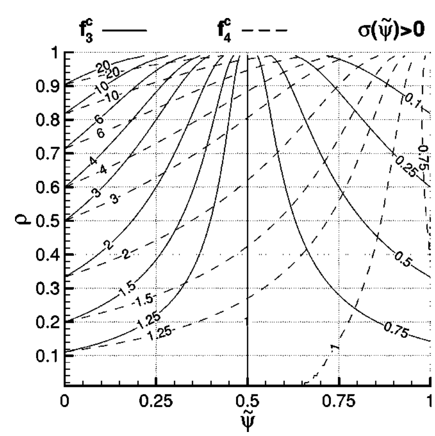

FIG. 8. Unstable $(\tilde{\psi},\rho)$ region (the whole plane) according to the criterion $\sigma(\tilde{\psi})>0$. The solid and dashed lines represent the bounds of the centrifugally unstable f intervals: $f_3^c<f<f_4^c$.

$-W/2$. Instability requires that either $W/2-\max_{\tilde{\psi}} V/\mathcal{R}>0$ or $-W/2+\min_{\tilde{\psi}} V/\mathcal{R}>0$. The last condition is never satisfied in Stuart vortices because, as can easily be shown, on all streamlines there exists points where the shear $\mathcal{S}=-W/2+V/\mathcal{R}$ is negative. On the other hand, the first condition ($\mathcal{S}$ always negative) is fulfilled in the white region of Fig. 7. The flow is centrifugally unstable if f lies in the interval $f_1^c<f<f_2^c$ where $f_1^c=-W(\tilde{\psi},\rho)$ and $f_2^c=-2\max_{\tilde{\psi}} V/\mathcal{R}$. The solid lines in Fig. 7 represent f_1^c and the dashed lines, f_2^c.

In Fig. 8, we have sketched the true centrifugally unstable f intervals in the $(\tilde{\psi},\rho)$ plane, using the condition $\sigma(\tilde{\psi})>0$. It turns out that the whole plane $(\tilde{\psi},\rho)$ is centrifugally unstable. The solid and dashed lines represent the two bounds f_3^c and f_4^c of the f interval for which instability occurs: $f_3^c<f<f_4^c$. It is shown that f_3^c always coincides with f_1^c, so that the solid lines in Figs. 7 and 8 are identical. Also $f_4^c>f_3^c$ in the whole $(\tilde{\psi},\rho)$ plane. Consequently, an unstable region always begins for values of $\tilde{\psi}$ and f such as $W(\tilde{\psi})+f=0$. The end of the unstable zones, f_4^c, coincides with f_2^c only for sufficiently peaked vortices (high ρ). In this case, the criterion $\Delta(\tilde{\psi})<0$ becomes necessary to capture centrifugal-type instabilities. But, the smaller ρ, the greater the difference $f_4^c-f_2^c$. For values of ρ and $\tilde{\psi}$ which are in the shaded region of Fig. 7, centrifugally unstable regions always exist although $\Delta(\tilde{\psi})$ is never negative.

C. A conjecture

The above results show that the criterion $\Delta(\tilde{\psi})<0$ is too restrictive in the cases where ρ is small. On the other hand, the results also show that the streamline ψ where the sign of $W(\psi)+f$ changes, always corresponds to the beginning of a centrifugally unstable streamline interval. The same phenomenon is observed in Taylor–Green flows.[10] This leads us to conjecture that a general steady two-dimensional flow with closed streamlines subject to rotation Ω undergoes centrifugal-type instabilities if there exists a streamline ψ_0 where the sign of the absolute vorticity changes, i.e., the sign of the function $\psi\rightarrow W(\psi)+2\Omega$ changes at ψ_0.

We have not been able to prove this conjecture in the general case but it can be done in axisymmetric flows. In the latter case, the spanwise perturbations are governed by (25) with $V'=0$ and $\mathcal{R}=r$. The matrix in this equation is therefore constant and the eigenvalues $\sigma(r)$ verify $\sigma(r)^2+\delta(r)=0$, so that a streamline r is stable if $\delta(r)>0$ and unstable if $\delta(r)<0$. Now, in axisymmetric flows, the absolute vorticity $\alpha(r)=W(r)+2\Omega$ and the absolute angular velocity $\beta(r)=U(r)/r+\Omega$ are linked through:

$$\alpha(r)=2\beta(r)+r\frac{d\beta}{dr}. \tag{40}$$

Supposing that the sign of the absolute vorticity $\alpha(r)$ changes at some radius r_0 and using the above relation and the Taylor series of $\alpha(r)$ and $\beta(r)$ at the radius r_0, it can be proved that the absolute angular velocity $\beta(r)$ keeps the same sign in the vicinity of r_0. The sign of the quantity $\delta(r)=2\alpha(r)\beta(r)$ therefore changes at r_0, so that r_0 separates a stable region from an unstable region. This argument also proves that in an axisymmetric configuration, spanwise shortwave perturbations can only undergo instabilities of the centrifugal type.

IV. CONCLUSION

In this paper, we have presented a new criterion which gives a sufficient condition for centrifugal-type instabilities to occur in a general inviscid steady two-dimensional flow. This criterion is a generalization of the Rayleigh, Kloosterziel and van Heijst, Mutabazi *et al.*, Pedley, Bradshaw-Richardson and Bayly's criteria. The proof of instability has been given using both the geometrical optics approach and a classical normal mode analysis. This criterion shows that a 2D basic flow may undergo centrifugal instability only if its intrinsic shear is either positive or negative along a whole streamline. In agreement with the basic mechanisms of centrifugal instability, we have considered spanwise perturbations, which are free from pressure effects. These perturbations are shown to induce inward/outward exchanges of fluid in a vortex, the fluid motion lying at first order in the horizontal plane.

In the second part of this article, we applied the results to the Stuart vortices. We gave for each member of that family and for each streamline $\tilde{\psi}$, the unstable $f=2\Omega$ interval according to the criterion $\Delta(\tilde{\psi})<0$: $f_1^c<f<f_2^c$. We found that only sufficiently peaked vorticity distributions (high ρ) satisfy the criterion. But, this criterion is only a sufficient condition: a streamline $\tilde{\psi}$ can undergo centrifugal-type instabilities even though $\Delta(\tilde{\psi})>0$. For this purpose, we have numerically calculated the real unstable f interval $f_3^c<f<f_4^c$ by computing the Floquet exponents $\sigma(\tilde{\psi})$. This analysis showed that for sufficiently peaked vorticity distributions we obtain the same unstable intervals in f as with the criterion $\Delta(\tilde{\psi})<0$. For these vorticity distributions, our criterion becomes therefore necessary to detect centrifugal-type instabilities. For vorticity distributions which are less concentrated, the angular rotation f_1^c is still the beginning of an unstable region. But its extent cannot be predicted by the criterion $\Delta(\tilde{\psi})<0$.

This led us to suggest another criterion of centrifugal instability, which is based on the change of sign of the absolute vorticity $W+2\Omega$. A general proof of this statement has not been given. But, we have proved that this conjecture is true in the circular case.

[1] T. J. Pedley, ''On the instability of viscous flow in a rapidly rotating pipe,'' J. Fluid Mech. **35**, 97 (1969).

[2] P. Bradshaw, ''The analogy between streamline curvature and buoyancy in turbulent shear flow,'' J. Fluid Mech. **36**, 177 (1969).

[3] J. W. S. Rayleigh, ''On the dynamics of revolving flows,'' Proc. R. Soc. London, Ser. A **93**, 148 (1916).

[4] R. C. Kloosterziel and G. J. F. van Heijst, ''An experimental study of unstable barotropic vortices in a rotating fluid,'' J. Fluid Mech. **223**, 1 (1991).

[5] I. Mutabazi, C. Normand, and J. E. Wesfreid, ''Gap size effects on centrifugally and rotationally driven instabilities,'' Phys. Fluids A **4**, 1199 (1992).

[6] B. J. Bayly, ''Three-dimensional centrifugal-type instabilities in inviscid two-dimensional flows,'' Phys. Fluids **31**, 56 (1988).

[7]A. Lifschitz and E. Hameiri, ‘‘Local stability conditions in fluid dynamics,’’ Phys. Fluids A **3**, 2644 (1991).
[8]G. I. Taylor and A. E. Green, ‘‘Mechanism of the production of small eddies from large ones,’’ Proc. R. Soc. London, Ser. A **158**, 499 (1937).
[9]D. Sipp and L. Jacquin, ‘‘Elliptic instability in two-dimensional flattened Taylor–Green vortices,’’ Phys. Fluids **10**, 839 (1998).
[10]D. Sipp, E. Lauga, and L. Jacquin, ‘‘Vortices in rotating systems: centrifugal, elliptic and hyperbolic type instabilities,’’ Phys. Fluids **11**, 3716 (1999).
[11]C. Cambon, J.-P. Benoit, L. Shao, and L. Jacquin, ‘‘Stability analysis and large-eddy simulation of rotating turbulence with organized eddies,’’ J. Fluid Mech. **278**, 175 (1994).
[12]S. Leblanc and C. Cambon, ‘‘On the three-dimensional instabilities of plane flows subjected to Coriolis force,’’ Phys. Fluids **9**, 1307 (1997).
[13]J. T. Stuart, ‘‘On finite amplitude oscillations in laminar mixing layers,’’ J. Fluid Mech. **29**, 417 (1967).
[14]S. Leblanc and C. Cambon, ‘‘Effects of the Coriolis force on the stability of Stuart vortices,’’ J. Fluid Mech. **356**, 353 (1998).
[15]R. R. Lagnado, N. Phan-Thien, and L. G. Leal, ‘‘The stability of two-dimensional linear flows,’’ Phys. Fluids **27**, 1094 (1984).
[16]B. J. Bayly, ‘‘Three-dimensional instability of elliptical flow,’’ Phys. Rev. Lett. **57**, 2160 (1986).
[17]A. D. D. Craik and W. O. Criminale, ‘‘Evolution of wavelike disturbances in shear flows: a class of exact solutions of the Navier-Stokes equations,’’ Proc. R. Soc. London, Ser. A **406**, 13 (1986).
[18]M. J. Landman and P. G. Saffman, ‘‘The three-dimensional instability of strained vortices in a viscous fluid,’’ Phys. Fluids **30**, 2339 (1987).
[19]F. Waleffe, ‘‘On the three-dimensional instability of strained vortices,’’ Phys. Fluids A **2**, 76 (1990).
[20]G. K. Batchelor and I. Proudman, ‘‘The effect of rapid distorsion on a fluid in turbulent motion,’’ Q. J. Mech. Appl. Math. **7**, 83 (1954).
[21]A. A. Townsend, *The Structure of Turbulent Shear Flow* (Cambridge University Press, Cambridge, 1976).
[22]C. Cambon, C. Tessèdre, and D. Jeandel, ‘‘Etude d'effets couplés de déformation et de rotation sur une turbulence homogène,’’ J. Méc. Théor. Appl. **4**, 629 (1985).
[23]C. Cambon and J. F. Scott, ‘‘Linear and nonlinear models of anisotropic turbulence,’’ Annu. Rev. Fluid Mech. **31**, 1 (1999).
[24]D. J. Tritton and P. A. Davies, ‘‘Instabilities in geophysical fluid dynamics,’’ in *Hydrodynamics Instabilities and the Transition to Turbulence*, edited by H. Swinney and J. Gollub (Springer-Verlag, Berlin, 1981), p. 229.
[25]R. S. Scorer and S. D. R. Wilson, ‘‘Secondary instability in steady gravity waves,’’ Q. J. R. Meteorol. Soc. **89**, 532 (1963).
[26]C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).