Near-critical swirling flow in a contracting duct: The case of plug axial flow with solid body rotation

Benjamin Leclaire,[a] Denis Sipp, and Laurent Jacquin
ONERA/DAFE, 8 rue des vertugadins, 92190 Meudon, France

(Received 19 June 2007; accepted 27 July 2007; published online 19 September 2007)

Rusak and Meder [AIAA J. **42**, 2284 (2004)] recently studied the behavior of a near-critical swirling flow in a weakly contracting duct. We investigate the particular inflow condition consisting of plug axial flow with solid body rotation, and introduce a new perturbation expansion specifically suited to that case. We show that the wall recirculation occurring in the exit plane as a result of the nonlinear excitation of the critical wave by the weak contraction is more accurately predicted. We also compute the near-critical flow for strong contractions and show that wall recirculations trapped inside the duct are obtained instead. *© 2007 American Institute of Physics*.
[DOI: 10.1063/1.2773767]

Swirling flows are characterized by a critical state that distinguishes between supercritical and subcritical states, similar to supersonic and subsonic states of compressible flows or to torrential and fluvial states of free surface flows. In supercritical flows, infinitesimal axisymmetric inertial waves only propagate downstream, whereas in subcritical flows they may propagate both upstream and downstream (see, for instance, Gallaire and Chomaz[1] for a synthesis). Subcritical flows are, moreover, characterized by their ability to sustain standing waves. In the critical regime, swirling flows have also been shown to respond nonlinearly to external perturbations. In a steady framework, Rusak and Meder,[2] for instance, recently explored the behavior of a near-critical swirling flow in a weakly contracting duct of finite length, via a weakly nonlinear analysis. They showed that the excitation of the critical wave by the duct contraction resulted in a flow with a wall deceleration in the exit plane. Upon applying their weakly nonlinear formalism in a range of parameters where the obtained perturbation becomes of order unity, they found flows with a recirculation at the exit plane wall.

We focus in this article on the particular inflow condition of plug axial flow with solid body rotation, which has received much attention in general theoretical analyses on swirling flows.[1,3–6] Our purpose is twofold. Considering first the case of a weak duct contraction, we introduce a new perturbation expansion exploiting the formal simplicity associated with this inflow condition, adapted from the unsteady analysis of Grimshaw and Yi.[3] This formalism allows to obtain flows with an incipient recirculation within the limit of validity of the expansion, and provides a more precise characterization thereof. We then investigate numerically the flow in a duct with a strong contraction, for which to this day only the velocity profiles in the exit plane have been characterized,[4] and discuss physically the differences that are observed with the case of the weak contraction.

We use cylindrical coordinates (r,θ,z), where r is the radius, θ the circumferential angle, and z the axial distance, and the velocity components (u,v,w) correspond, respectively, to the radial, azimuthal, and axial velocities. The steady axisymmetric incompressible motion of an inviscid fluid is more conveniently described by the axisymmetric stream function ψ, which is linked to the velocity via $u=-(1/r)\,\partial\psi/\partial z$ and $w=(1/r)\,\partial\psi/\partial r$. It is governed by a single equation, called Bragg-Hawthorne or Squire-Long equation (see, for instance, Batchelor[4] for its derivation):

$$\left(\psi_{zz}+\psi_{rr}-\frac{\psi_r}{r}\right)=r^2H'(\psi)-KK'(\psi). \tag{1}$$

Here, $H=p/\rho+(u^2+v^2+w^2)/2$ (where p and ρ stand for the pressure and density of the flow) and $K=rv$, respectively, denote the total head and the circulation, which only depend on ψ from the assumption of steady inviscid fluid.

We study the flow in a circular duct of finite length L and of varying radius $R(z)$. Note that in this study, the lengths are made dimensionless by use of the inlet radius $R(0)$, and the velocities by use of the inlet axial velocity. An adapted set of boundary conditions to be prescribed at the duct ends, axis, and wall reads[7–9]

$$\begin{gathered}\psi(r,0)=\psi_0(r)=r^2/2,\quad 0\le r\le 1,\\ K(r,0)=K_0(r)=\omega r^2/2,\quad 0\le r\le 1,\\ \psi_{zz}(r,0)=0,\quad 0\le r\le 1,\\ \psi_z(r,L)=0,\quad 0\le r\le R(L),\\ \psi(0,z)=0,\quad \psi(R(z),z)=1/2,\quad 0\le z\le L.\end{gathered} \tag{2}$$

The inflow condition of plug axial flow with solid body rotation is imposed by $\psi_0(r)=r^2/2$ and $K_0(r)=\omega r^2/2$; thus, the swirl number ω is equal to twice the ratio between the maximum azimuthal velocity and the axial velocity at the inlet. It is also worthwhile noting that conditions $K(r,0)=K_0(r)=\omega r^2/2$ and $\psi_{zz}(r,0)=0$ ensure that functions $H(\psi)$ and $K(\psi)$ are fixed by their respective values at the inlet (see, e.g., Buntine and Saffman[8]). As a consequence, the problem built by Eq. (1) together with boundary conditions (2) is only valid under the condition that $w>0$ everywhere. Flows with a recirculation or with entrance of fluid at the outlet, for

[a]Electronic mail: benjamin.leclaire@onera.fr

which there exists a region where $w<0$, are excluded from this analysis. Upon calculating $K(\psi)$ and $H(\psi)$ from their values at the inlet, Eq. (1) simplifies into[4]

$$\left(\psi_{zz}+\psi_{rr}-\frac{\psi_r}{r}\right)=\omega^2\left(\frac{r^2}{2}-\psi\right), \tag{3}$$

which is indeed *linear*, although no hypothesis of linearization has been done. This feature is really specific to plug axial flow with solid body rotation. However, the problem remains nonlinear via the boundary condition at the wall.

Considering first a weak duct contraction (see also Rusak and Meder[2]), we choose $R(z)=1-\varepsilon h(z)$ with $0<\varepsilon\ll 1$. Here, $h(z)$ is a positive increasing function such that $h(0)=0$ and $h(L)=1$; thus, $0<\varepsilon=1-R(L)\ll 1$ is the order of magnitude of the departure of the exit radius from unity. We also consider values of the swirl with departures from criticality of the same order of magnitude as the perturbation on the duct radius; that is, $\omega=\omega_c(1+\varepsilon\Delta)$, with Δ a constant control parameter of order unity. Supercritical (subcritical) flows therefore correspond to $\Delta<0$ ($\Delta>0$).

Since Eq. (3) is linear, the expansion for ψ can be sought with a leading-order perturbation of order unity (see, e.g., Grimshaw and Yi[3]). We therefore set

$$\psi(r,z)=\psi_0(r)+\phi^0(r,z)+\varepsilon\phi^1(r,z)+O(\varepsilon^2), \tag{4}$$

where $\psi_0(r)$ is the parallel flow [with a corresponding circulation equal to $K_0(r)$] that would be obtained for any ω if the duct were of constant cross section. We term this expansion *strongly nonlinear* since $\phi^0(r,z)$ is of the same order as the base flow $\psi_0(r)$. When injecting the decompositions for $R(z)$, ω and $\psi(r,z)$ in problem (2) and (3), the equation of motion becomes

$$\begin{aligned}\phi^0_{zz}+r\left(\frac{\phi^0_r}{r}\right)_r+\omega_c^2\phi^0+\varepsilon\bigg[\phi^1_{zz}+r\left(\frac{\phi^1_r}{r}\right)_r+\omega_c^2\phi^1\\ +2\Delta\omega_c^2\phi^0\bigg]+O(\varepsilon^2)=0.\end{aligned} \tag{5}$$

Among the boundary conditions, only that imposed at the wall leads to the following nontrivial expression, obtained via a Taylor expansion:

$$\begin{aligned}&\phi^0(1,z)+\varepsilon[\phi^1(1,z)-h(z)\phi^0_r(1,z)]+O(\varepsilon^2)\\ &\quad=\varepsilon h(z)+O(\varepsilon^2),\quad 0\le z\le L.\end{aligned} \tag{6}$$

Retaining only the terms of order unity, the problem verified by the critical wave $\phi^0(r,z)$ is then obtained:

$$\begin{aligned}&\phi^0_{zz}+r\left(\frac{\phi^0_r}{r}\right)_r+\omega_c^2\phi^0=0,\\ &\phi^0(r,0)=0,\quad \phi^0_z(r,L)=0,\quad 0\le r\le 1,\\ &\phi^0(0,z)=0,\quad \phi^0(1,z)=0,\quad 0\le z\le L.\end{aligned} \tag{7}$$

Since all boundary conditions are homogeneous, this problem is an eigenvalue problem for ω_c, whose corresponding eigenvector is the critical wave $\phi^0(r,z)$. From the studies of Wang and Rusak,[5,9] it can be shown that $\phi^0(r,z)$ has to be sought under the form

$$\phi^0(r,z)=A\sin\left(\frac{\pi z}{2L}\right)rJ_1(j_{1,1}r), \tag{8}$$

where A is a constant amplitude undetermined at leading order, $j_{1,1}$ denotes the first nontrivial root of the Bessel function of order one of the first kind J_1, and the so-called *critical swirl in a pipe* ω_c is given by $\omega_c^2=\omega_B^2+\pi^2/(4L^2)$. Here, ω_B stands for the critical swirl for a *parallel* flow in a duct of infinite length determined by Benjamin;[10] therefore, ω_c includes a contribution of the finite-length effects. For our specific inlet flow, it is known that $\omega_B=j_{1,1}\approx 3.8317$.

At order ε, the problem for $\phi^1(r,z)$ is obtained, with forcing terms from the order unity appearing in both the motion equation and the boundary condition at the wall:

$$\begin{aligned}&\phi^1_{zz}+r\left(\frac{\phi^1_r}{r}\right)_r+\omega_c^2\phi^1=-2\Delta\omega_c^2\phi^0,\\ &\phi^1(r,0)=0,\quad \phi^1_z(r,L)=0,\quad 0\le r\le 1,\\ &\phi^1(0,z)=0,\quad \phi^1(1,z)=h(z)[1+\phi^0_r(1,z)],\quad 0\le z\le L.\end{aligned} \tag{9}$$

In particular, the wall boundary condition expresses the conservation of mass, which was not enforced at leading order as seen in problem (7). Since the left-hand side of the equation of motion in problem (9) is the same as in problem (7), Fredholm's theorem applies: problem (9) admits a solution only under the condition that a compatibility condition be fulfilled. This condition is obtained by multiplying the motion equation in (9) by $\phi^0(r,z)/r$ and integrating on the whole domain. After some integration by parts, and upon using Eqs. (7) and the boundary conditions of problem (9), one obtains a balance between the forcing terms in the bulk flow and at the wall:

$$\int_0^L\frac{\phi^0_r}{r}(1,z)\phi^1(1,z)dz=2\Delta\omega_c^2\int_0^L\int_0^1\frac{\phi^0(r,z)^2}{r}dr\,dz. \tag{10}$$

Upon replacing $\phi^0(r,z)$ and $\phi^1(1,z)$, one finally gets the amplitude equation $bI_2+b^2AI_3-2A\Delta\omega_c^2I_1=0$, which yields the expression for A as a function of Δ:

$$A=\frac{I_2b}{2\Delta\omega_c^2I_1-b^2I_3}. \tag{11}$$

Here, the following notations have been introduced for clarity:

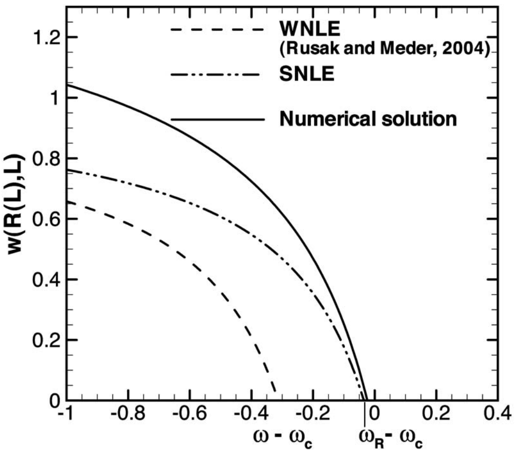

FIG. 1. Axial velocity $w[R(L),L]$ at the exit plane wall of a weakly contracting duct with $L=1$, $R(L)=0.9$ ($\varepsilon=0.1$). Comparison of the present strongly nonlinear expansion (SNLE) with the weakly nonlinear expansion (WNLE) of Rusak and Meder (Ref. 2) and with the numerical solution.

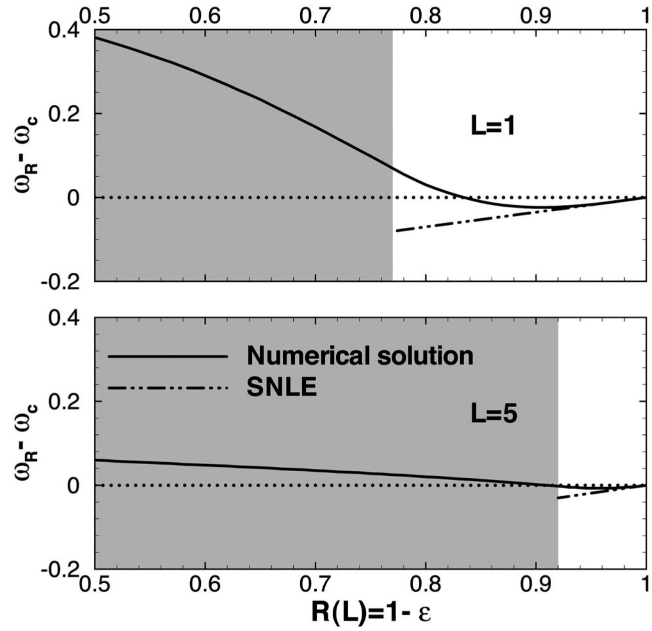

FIG. 2. Value of $\omega_R-\omega_c$ (ω_R being the swirl at which a wall recirculation is obtained) as a function of the exit duct radius $R(L)$, for $L=1$ and $L=5$. Comparison between the SNLE [valid in the vicinity of $R(L)=1$] and the numerical solution. In the shaded region, the recirculation is obtained for $z<L$, inside the duct.

$$b=j_{1,1}J_0(j_{1,1}), \quad I_1=\int_0^L\int_0^1 \sin^2\left(\frac{\pi z}{2L}\right) r J_1^2(j_{1,1}r)\,dr\,dz,$$

$$I_2=\int_0^L h(z)\sin\left(\frac{\pi z}{2L}\right)dz, \quad I_3=\int_0^L h(z)\sin^2\left(\frac{\pi z}{2L}\right)dz,$$

with $I_1\approx 4.0554\times 10^{-2}$, $b\approx -1.5433$, and J_0 denoting the Bessel function of the first kind of order zero. Furthermore, since $h(z)>0$, one obtains $I_2>0$ and $I_3>0$.

We now seek the value ω_R of the inlet swirl number at which a recirculation is obtained, which also coincides with the limit of validity of the model. To that aim, we consider a supercritical flow approaching criticality (i.e., with $\Delta<0$). At order unity, the flow axial velocity is given by $w(r,z)=1+Aj_{1,1}\sin[\pi z/(2L)]J_0(j_{1,1}r)$. From (11) it is found that $A>0$. Besides, $\sin[\pi z/(2L)]$ is increasing on the interval $[0,L]$, and J_0 is decreasing on $[0,j_{1,1}]$ with $J_0(j_{1,1})<0$. Therefore, the minimum of $w(r,z)$ is obtained in the exit plane, at the duct wall. From a Taylor expansion around $r=1$, the condition $w=0$ yields the limit value for A as $A_{\max}=-1/b\approx 0.6480$. The present expansion thus shows that the recirculation is obtained at the exit plane wall, and allows to derive the following expression of ω_R as a function of the exit radius $R(L)$:

$$\omega_R=\omega_c-[1-R(L)]\frac{b^2(I_2-I_3)}{2\omega_c I_1}. \tag{12}$$

It is thus confirmed that in the case of plug axial inflow with solid body rotation, the strongly nonlinear response of the near-critical flow due to the weak contraction may lead to a wall recirculation in the exit plane. It should also be emphasized here that the obtained resonant flow is intrisically nonparallel, since even for values of $R(L)$ very close to 1, which guarantee that $|R'(z)|\ll 1$ for $0<z<L$, large axial gradients are observed.

We now illustrate these results by considering a weakly contracting duct of length $L=1$, with a radius defined by $h(z)=0.5[1-\cos(\pi z/L)]$ and $\varepsilon=1-R(L)=0.1$. Such values lead to $I_2\approx 0.4244$ and $I_3=0.375$. Figure 1 compares the values of the wall axial velocity in the exit plane as a function of $\omega-\omega_c$ given by the present strongly nonlinear expansion (SNLE), together with the values obtained with the weakly nonlinear expansion (WNLE) of Rusak and Meder[2] and the numerical solution of Eq. (3) with boundary conditions (2). This solution was obtained with a Chebyshev collocation method using a curvilinear coordinate transformation mapping the duct geometry into a square computational domain. As expected, since the SNLE takes into account a perturbation of order unity which is necessary for dealing with vanishing axial velocities, it fits more accurately to the numerical solution than the WNLE, in particular in the vicinity of the limit value $\omega-\omega_c=\omega_R-\omega_c$. As $|\omega-\omega_c|$ increases, the SNLE progressively shifts from the numerical solution since the hypothesis of near-critical flow progressively becomes invalid.

We now investigate the flow behavior for decreasing values of $R(L)$; i.e., increasing values of ε. Figure 2 plots the values of $\omega_R-\omega_c$ obtained with the numerical simulation for two ducts of lengths $L=1$ and $L=5$, as a function of $R(L)$. Corresponding values predicted by the SNLE [Eq. (12)] are also plotted for comparison for $R(L)$ close to unity. As above, a very good agreement is found in this region. However, when the contraction is strong enough so that finite values of ε are reached [$R(L)\leq 0.77$ for $L=1$ and $R(L)\leq 0.92$ for $L=5$, see the shaded rectangles in Fig. 2], the numerical results show that a new behavior sets in, since then the recirculation appearing at ω_R is not observed in the exit plane, but *inside* the duct. As $R(L)$ further decreases from these values, the location of this trapped recirculation is observed to progressively shift upstream in the duct. Note that we did not

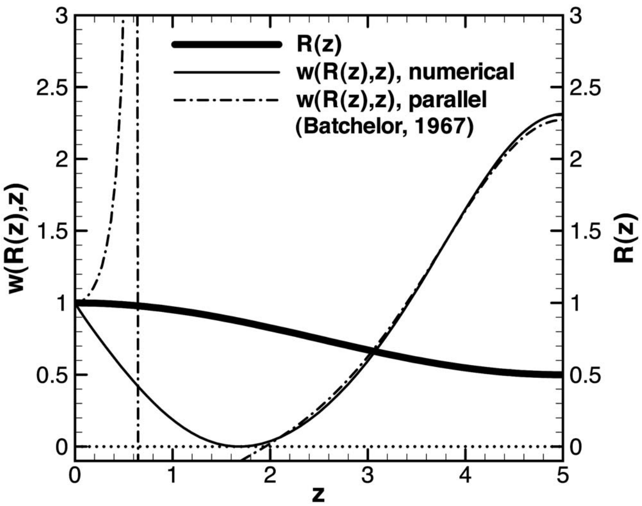

FIG. 3. Axial velocity $w[R(z),z]$ at the wall of a contracting duct with $L=5$, $R(L)=0.5$, in a situation of incipient recirculation ($\omega-\omega_c=\omega_R-\omega_c=0.0594$). Comparison of the numerical solution with the parallel approximation of Batchelor (Ref. 4).

plot the values of $\omega_R-\omega_c$ given by the SNLE in the shaded zones of Fig. 2. As a matter of fact, extending the results of the SNLE to this range of $R(L)$ still leads to flows with a wall recirculation in the exit plane [as shown above in the derivation of (12)], whereas this is no longer the case.

The reason for this difference between weak and strong contractions is that for a sufficiently strong contraction, the flow does not remain near-critical in the whole duct, as was implicitly the case when deriving the SNLE. Considering such a strong contraction, this may be justified by analyzing the evolution with z of the *local* swirl number $S(z)=2v[R(z),z]/\bar{W}(z)$, where $v[R(z),z]$ and $\bar{W}(z)=2/R^2(z)\int_0^{R(z)} rw(r,z)dr$ respectively denote the wall azimuthal velocity and the mean axial velocity built from the volume flow rate at the considered axial location z. Note that such a definition leads to $S(0)=\omega$. Using the conservation of mass and of circulation one gets $\bar{W}(z)=1/R^2(z)$ and $v[R(z),z]=\omega/[2R(z)]$, so that $S(z)=\omega R(z)$. Consequently, if $R(L)$ is sufficiently small, a flow that is initially near-critical at the inlet is forced back to a supercritical state in the most downstream part of the nozzle; say, between some abscissa $z=z_0$ and the exit plane $z=L$. Since the wave excitation responsible for the wall deceleration occurs only in the near-critical regime, the recirculation is then bound to be trapped in the interval $0<z<z_0$.

To further justify this reasoning, we compare the numerically simulated flow in a situation of incipient trapped recirculation with the formulas of Batchelor[4] [Eqs. (7.5.22) and (7.5.23), p. 548]. These formulas stem from an assumption of parallel flow and therefore provide an accurate approximation wherever the flow is locally supercritical and far from the critical regime, and the geometry has moderate axial gradients [$|R'(z)|\ll 1$]. Considering such a geometry [here with $L=5$ and $R(L)=0.5$], we therefore use this comparison as a diagnosis of the local nearness of the flow to criticality. This is done in Fig. 3, which plots the wall axial velocities obtained for $\omega-\omega_c=\omega_R-\omega_c=0.0594$. The parallel approximation is seen to be valid for $z_0\approx 2.0<z<L$. In this zone, the flow has therefore returned to supercritical [note that one has $R(z)<0.83$ there] and is locally determined by the value of $R(z)$. The recirculation occurs at $z\approx 1.69<z_0$, in a zone where the formulas of Batchelor are seen to diverge. Thus, it is indeed confined in the zone of near-critical flow, where the contraction is still weak enough to trigger a nonlinear response of the flow. Incidentally, our analysis also shows that the formulas of Batchelor, when used to characterize the flow downstream of a contraction, may be applied for increasing ω only until the near-critical regime is reached at the inlet, as the trapped recirculation then invalidates Eq. (3) for larger values of ω.

Since the trapped wall recirculation obtained from our analysis was found by subjecting a near-critical inflow to a sufficiently large contraction, it is expected that it will also be encountered for other types of inflow. Besides, it would be of foremost importance to investigate if this phenomenon results in boundary layer separation when the viscosity and a no-slip condition at the wall are taken into account. Work is in progress along these lines.

[1] F. Gallaire and J.-M. Chomaz, "The role of boundary conditions in a simple model of incipient vortex breakdown," Phys. Fluids **16**, 274 (2004).
[2] Z. Rusak and C. C. Meder, "Near-critical swirling flow in a slightly contracting pipe," AIAA J. **42**, 2284 (2004).
[3] R. Grimshaw and Z. Yi, "Resonant generation of finite-amplitude waves by the uniform flow of a uniformly rotating fluid past an obstacle," Mathematika **40**, 30 (1993).
[4] G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1967).
[5] S. Wang and Z. Rusak, "On the stability of an axisymmetric rotating flow in a pipe," Phys. Fluids **8**, 1007 (1996).
[6] F. Gallaire, J.-M. Chomaz, and P. Huerre, "Closed-loop control of vortex breakdown: a model study," J. Fluid Mech. **511**, 67 (2004).
[7] A. Szeri and P. Holmes, "Nonlinear stability of axisymmetric swirling flows," Philos. Trans. R. Soc. London, Ser. A **326**, 327 (1988).
[8] J. D. Buntine and P. G. Saffman, "Inviscid swirling flows and vortex breakdown," Proc. R. Soc. London, Ser. A **449**, 139 (1995).
[9] S. Wang and Z. Rusak, "The dynamics of a swirling flow in a pipe and transition to axisymmetric vortex breakdown," J. Fluid Mech. **340**, 177 (1997).
[10] T. B. Benjamin, "Theory of the vortex breakdown phenomenon," J. Fluid Mech. **14**, 593 (1962).