Criticality of compressible rotating flows

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The effect of compressibility on the criticality of swirling subsonic flows is investigated. This study extends previous works by Rusak and Lee [J. Fluid Mech. **461**, 301 (2002); **501**, 25 (2004)] on the critical swirl of subsonic vortex flows in a circular straight pipe. We derive an asymptotic solution in the case of an isothermal plug-flow with solid-body rotation. In the limit of low Mach number $M_0 \ll 1$, it is shown that the critical swirl increases with M_0 as $S_c \sim S_{c,0}/(1-M_0^2)^{1/2}$, where $S_{c,0}$ is the critical swirl of the incompressible flow. This result still holds when varying the thermodynamic properties of the flow or when considering different vortex models as the Batchelor vortex. Physically, compressibility is found to slow down phase and group velocities of axisymmetric Kelvin waves, thus decreasing the rotation contribution to flow criticality. It is shown that compressibility damps the stretching mechanism which contributes to the wave propagation in the incompressible limit. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2427090]

Under certain circumstances, vortex flows are known to undergo a brutal disorganization, the so-called vortex breakdown. Its study is relevant to many engineering applications such as aircrafts, tornadoes, and combustion chambers. Among theoretical analyses, the critical-state theory of Benjamin[1] relates the occurrence of axisymmetric vortex breakdown to the existence of infinitely long standing axisymmetric waves on the vortex. Such waves are relevant to infinitely long straight pipes or long pipes with periodic inlet and outlet conditions. A supercritical vortex supports only downstream propagating inertial waves, whereas upstream and downstream propagating waves may exist in a subcritical vortex. The transition between both states may appear at a critical swirl $S=S_c$, where S compares rotation with advection (see below). Rusak and Lee[2] extended this concept to compressible pipe flows and derived a general equation which defines the critical swirl for given velocity and thermodynamic fields. By applying these results to isothermal solid-body rotation and Batchelor vortices,[3] they numerically showed that S_c is an increasing function of the Mach number. The first objective of this Brief Communication is to give an analytical expression of the criticality conditions as a function of the Mach number. Then, we will show that the study of the inertial Kelvin waves provides a physical interpretation of the phenomenon. Finally, results are compared with different models of thermodynamic field and velocity distributions.

Linear stability formulation: Let us consider a compressible swirling flow characterized by its velocity $\mathbf{u}$, pressure p, density ρ, and temperature T. In the following, we choose the value w_0 of the axial velocity on the axis of the flow as the reference velocity scale of the problem, and the characteristic transverse length scale R_c of the flow as the reference length scale. Pressure, density and temperature are made nondimensional by using the values p_0, ρ_0, and T_0 of the swirling flow on its axis. The swirl parameter $S=v_0/w_0$ which compares the intensity of rotation to that of advection is defined as the ratio between v_0, which is the maximum value of the orthoradial swirling velocity, and w_0.

The flow is assumed to be axisymmetric and governed by the compressible Euler equations (in a nondimensional form):

$$d_t\rho + \rho\nabla\cdot\mathbf{u} = 0, \quad \rho d_t\mathbf{u} + \nabla p/(\gamma M_0^2) = 0, \quad d_t s = 0, \quad (1)$$

where $M_0 = w_0/(\gamma p_0/\rho_0)^{1/2}$. Here, d_t represents the material derivative and $\mathbf{u}=(u,v,w)$ denotes the velocity vector expressed in a cylindrical frame (r,θ,z), where the z axis corresponds to the vortex centerline and the r axis to its radius. The fluid is a perfect gas for which thermodynamic properties are related by the state equation $p=\rho T$. The specific heat ratio is $\gamma=C_p/C_v=1.4$ and $s=\ln(p^{1/\gamma}/\rho)$ denotes the entropy.

The present study is based on a small perturbation technique. Each quantity is considered as the superposition of a basic state and an infinitesimal perturbation $\mathbf{q}(r,z,t)=\underline{\mathbf{q}}(r)+\epsilon\mathbf{q}'(r,z,t)$, where $\epsilon \ll 1$ is a little parameter. The invariance of the basic flow under z and t translations allows us to decompose the fluctuating quantities into normal modes $\mathbf{q}'=\hat{\mathbf{q}}(r)e^{i(kz-\omega t)}$, where $\hat{\mathbf{q}}=[\hat{u},\hat{v},\hat{w},\gamma M_0^2\hat{\rho},\gamma M_0^2\hat{p},\gamma M_0^2\hat{T}]$ is the perturbation amplitude vector and $\underline{\mathbf{q}}=[0,V,W,\bar{\rho},\bar{p},\bar{T}]$ denotes the basic flow. The coefficient γM_0^2 is introduced to obtain nondegenerate equations in the limit $M_0=0$. In a temporal study, k denotes the real axial wave number and $\omega=\omega_r+i\omega_i$ is the complex pulsation with ω_r the frequency and ω_i the temporal amplification rate of the disturbance. Upon substituting the normal mode decomposition into Eqs. (1) and linearizing them around the basic flow, one obtains the linearized Euler equations. After elementary manipulations, it is possible to reduce the system to a single second order differential equation for the function $\phi(r)=\bar{\rho}r\hat{u}$.

We first consider a basic flow corresponding to an isothermal plug-flow with solid-body rotation. The nondimensional expressions for the velocity components (V,W) and for the thermodynamic quantities read

$$V=Sr, \quad W=1, \quad \bar{p}=\bar{\rho}=e^{\gamma M_0^2 S^2 r^2/2}, \quad \bar{T}=1. \quad (2)$$

In such a case, the equation for ϕ reduces to

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$$\frac{d^2\phi}{dr^2} - \left[\frac{1}{r} + \underbrace{\gamma M_0^2 S^2 r}_{\text{coupling}}\right]\frac{d\phi}{dr} + \left[\underbrace{\frac{4S^2}{(1-c)^2}}_{\text{rotation}} - k^2 - \underbrace{4M_0^2 S^2}_{\text{coupling}} + \underbrace{M_0^2(1-c)^2k^2}_{\text{acoustic}} + \underbrace{\frac{(\gamma-1)M_0^2S^4r^2}{(1-c)^2}}_{\text{coupling}}\right]\phi = 0, \tag{3}$$

where $c=\omega/k$ stands for the phase velocity of the disturbance. Different contributions to wave dynamics are identified in Eq. (3): the term *rotation* is the restoring effect of rotation, the term *acoustic* represents the acoustic contribution, whereas terms labelled *coupling* represent coupling between rotation and acoustic effects.

Equation (3) along with the boundary conditions $\phi(0)=\phi(1)=0$ constitute an eigenvalue problem either for ω in a temporal stability analysis or for S in a criticality analysis. In the former approach, for given wave number k and swirl S, nontrivial solutions ϕ exist only for some values of the pulsation ω. In the latter approach, stationary solutions of infinite extent (i.e., $c=k=0$) may exist only for some values of S.

Flow criticality: According to Benjamin,[1] the long-wave limit $k=0$ allows us to distinguish supercritical flows that cannot support standing waves $c=0$ from subcritical flows where standing waves may exist. Upon setting $k=c=0$ in Eq. (3), we obtain

$$\frac{d^2\phi_c}{dr^2} - [r^{-1} + \gamma M_0^2 S^2 r]\frac{d\phi_c}{dr} + [4(1-M_0^2)S^2 + (\gamma-1)M_0^2S^4r^2]\phi_c = 0, \tag{4}$$

with boundary conditions $\phi_c(0)=\phi_c(1)=0$. The subscript c refers to critical conditions.[1] Equation (4) was first derived in Ref. 2 with a term for pipe length correction. The solution of Eq. (4) with the condition $\phi_c(0)=0$ reads $\phi_c(r)=\exp(\gamma M_0^2S^2r^2/4)M_{\kappa,\mu}(\lambda r^2)$, where $M_{\kappa,\mu}$ represents the Whittaker function[4] with coefficients defined by

$$\kappa = \frac{-2i(1-M_0^2)}{M_0\sqrt{4(\gamma-1)-\gamma^2M_0^2}}; \quad \mu = \frac{1}{2};$$

$$\lambda = i\frac{S^2M_0\sqrt{4(\gamma-1)-\gamma^2M_0^2}}{2}.$$

The dispersion relation is obtained by applying the outer boundary condition $\phi_c(1)=0$ to the solution, so the eigenvalues are the zeros of the function $M_{\kappa,\mu}(\lambda)$. It is convenient to write the Whittaker function in terms of the confluent hypergeometric function M: $M_{\kappa,\mu}(x)=e^{-x/2}x^{\mu+1/2}M(1/2+\mu-\kappa,1+2\mu,x)$. In the limit $M_0 \ll 1$, it is possible to connect the solution for ϕ_c with Bessel functions of the first kind J_n by using the formula $\lim_{a\to\infty}M(a,b,-x/a)/\Gamma(b)=x^{1-b/2}J_{b-1}(2\sqrt{x})$ (see Ref. 4). One thus obtains the asymptotic dispersion relation. In the limit $M_0 \ll 1$, the asymptotic critical swirl reads

$$S_{c,\mathrm{as}} = S_{c,0}(1-M_0^2)^{-1/2}, \tag{5}$$

where $S_{c,0}=j_{1,1}/2$ is the critical swirl of the incompressible plug-flow with solid-body rotation.[5] $j_{1,1}$ denotes the first zero of J_1. Note that expression (5) looks like the Prandtl-Glauert transformation used to take compressibility into account when evaluating aerodynamic coefficients in subsonic flows. Nevertheless, no similarity law was found from Eq. (4) and Eq. (5) cannot be derived directly.

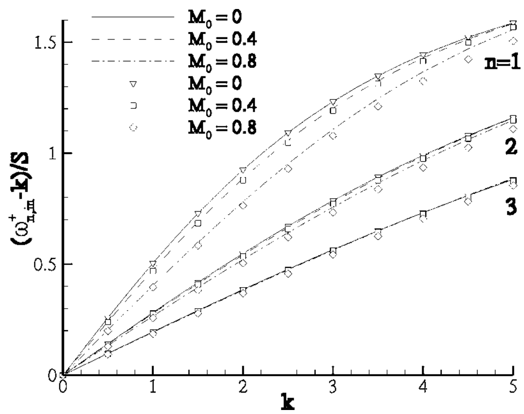

FIG. 2. Three first inertial temporal branches for different Mach numbers ($S=2$). Lines correspond to numerical simulations and symbols correspond to asymptotic solutions (7).

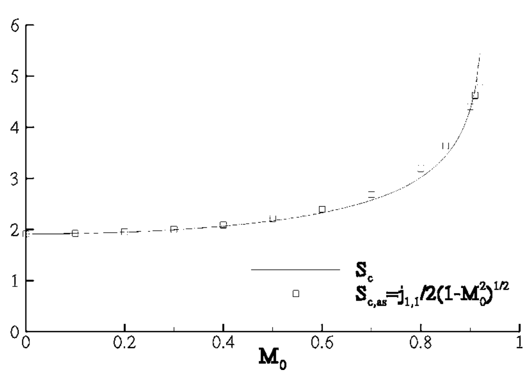

FIG. 1. Critical swirl as a function of the characteristic Mach number: numerical results from Ref. 2 (line) compared with the asymptotic prediction (5) (symbols).

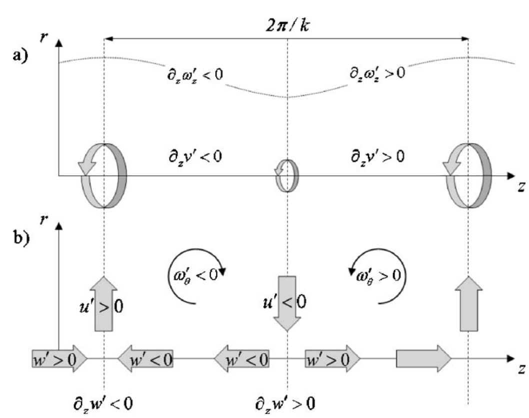

FIG. 3. Physical interpretation of the Kelvin wave propagation: the evolution of the axial vorticity component of the initial perturbations (line); thick arrows refer to velocity components.

The asymptotic results are now compared with a direct computation of the eigenvalues of Eq. (3). The numerical procedure is based on a shooting method and Eq. (3) is integrated via a classical fourth-order Runge-Kutta scheme. Figure 1 shows the critical swirl of the flow as a function of M_0. Both results, asymptotic and numerical, are in good agreement for a wide range of Mach numbers and Eq. (5) could be useful for engineering applications involving compressible swirling flows. Direct numerical simulations of pipe flows[6] and experiments on delta wings[7] also confirm that axisymmetric breakdown occurrence may be delayed by compressibility effects.

Normal mode analysis: To give a physical support to our results, it is necessary to analyze the dynamics of the Kelvin waves associated with the flow criticality. For that purpose, the swirl S is now a prescribed parameter and, for a given wave number k, we look for possible eigenvalues ω. If the same procedure as above is applied to Eq. (3) with boundary conditions, the following dispersion relation is obtained in the limit $M_0 \ll 1$, with the additional assumption $kM_0 \ll 1$:

$$((k-\omega)^2 - 4S^2)(M_0^2(k-\omega)^2 - k^2) - j_{1,n}^2(k-\omega)^2 = 0, \quad (6)$$

where $j_{1,n}$ is the nth zero of J_1. This relation gives the pulsation as a function of k and flow parameters. Four branches of solutions are readily found:

$$\omega_{n,\mathrm{ac}}^{\pm}(k) = k \pm \frac{\sqrt{j_{1,n}^2 + k^2 + 4S^2M_0^2 + \sqrt{(j_{1,n}^2+k^2)^2 + 8S^2M_0^2(j_{1,n}^2 + 2S^2M_0^2 - k^2)}}}{\sqrt{2}M_0},$$

$$\omega_{n,\mathrm{in}}^{\pm}(k) = k \pm \frac{\sqrt{j_{1,n}^2 + k^2 + 4S^2M_0^2 - \sqrt{(j_{1,n}^2+k^2)^2 + 8S^2M_0^2(j_{1,n}^2 + 2S^2M_0^2 - k^2)}}}{\sqrt{2}M_0}. \quad (7)$$

The two first modes correspond to acoustic waves, whereas the two other modes correspond to inertial waves. Each mode is linearly stable ($\omega_i=0$) and oscillates in the flow. By setting $S=0$ or $M_0=0$ in (6), one recovers the limiting cases of either a subsonic flow without swirl or an incompressible solid-body rotation flow respectively. The first term k in Eq. (7) corresponds to the effect of uniform convection in the basic flow (2) which acts as a Doppler shift on ω. The second term corresponds either to the acoustic or to the inertial wave modified by compressibility and swirl effects.

Acoustic branches $\omega_{n,\mathrm{ac}}^{\pm}$ are beyond the scope of the present study. Concerning inertial waves $\omega_{n,\mathrm{in}}^{\pm}$, the smallest group velocity is obtained for the branch $n=1$ at $k=0$, that is

$$v_{g,\min} = \left.\frac{d\omega_{1,\mathrm{in}}^{-}}{dk}\right|_{k=0} = 1 - 2S(4M_0^2S^2 + j_{1,1}^2)^{-1/2}. \quad (8)$$

Hence, for $S < S_{c,\mathrm{as}}$, where $S_{c,\mathrm{as}}$ is defined by Eq. (5), the group velocity is positive and axisymmetric Kelvin waves propagate only downstream. At $S=S_{c,\mathrm{as}}$, a standing wave of infinite extent may appear. Beyond this swirl value, $S > S_{c,\mathrm{as}}$, $v_{g,\min} < 0$ and some waves propagate upstream. In the limit $M_0 \ll 1$, the critical-state concept[1] agrees with the zero group velocity criterion[8] as expected. As a consequence, a subcritical ($v_{g,\min} < 0$) incompressible flow may become supercritical ($v_{g,\min} > 0$) when increasing M_0.

Figure 2 displays the three first upper temporal branches $n=1$, 2, and 3 of the inertial waves where we compare asymptotic with numerical results. Only the rotation contribution, i.e., $\omega_{n,\mathrm{in}}^{\pm} - k$, is sketched (the lower branches are symmetrical to positive ones with respect to the k axis and are not shown). Again, the asymptotic solution is seen to accurately describe the Kelvin waves dynamics. For a fixed k, the frequency decreases when M_0 is increased, this effect being reduced for slowest branches $n=2,3$. Consequently, the amplitudes of the phase and group velocities decrease with increasing M_0. Hence, the increase in critical swirl number S_c with M_0 is attributed to a decrease of the Kelvin wave frequencies. This damping effect reduces the rotation contribution with respect to the downstream advection by Doppler shift. Let us now describe the physical mechanism responsible of this frequency reduction.

A physical interpretation: We consider the role of compressibility in the axisymmetric Kelvin waves propagation mechanism. This mechanism has been described in Ref. 9 in the case of an incompressible flow.

Since the axial velocity acts only as a Doppler shift on ω, one can set $W=0$ in Eq. (2) without loss of generality. The linearized conservation equations of the azimuthal and axial vorticity fluctuations components read

$$\partial_t \omega_\theta' = 2S\partial_z v' - (d\bar{p}/dr)\partial_z T'/\gamma M_0^2 \bar{\rho}^2, \quad (9a)$$

$$\partial_t \omega_z' = 2S\partial_z w' - 2S\nabla \cdot \mathbf{u}'. \quad (9b)$$

In the incompressible limit, the last terms in Eqs. (9a) and (9b) vanish. One considers an axial vorticity perturbation $\omega_z' = (\partial_r r v')/r$, with a given wave number k, superimposed on the basic flow (2) as schematized in Fig. 3. Extrema of v' and ω_z' coincide. Axial gradients $\partial_z v'$ lead to the production of azimuthal vorticity fluctuations ω_θ' via tilting of the basic flow vorticity in the (z,θ) plane [Eq. (9a)]. Having $\omega_\theta' = \partial_z u' - \partial_r w'$, one gets radial gradients of axial velocity, which means axial stretching of the basic vorticity when ω_z' is minimum and contraction when ω_z' is maximum (see Fig.

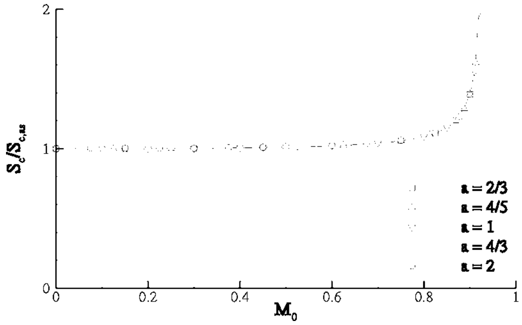

FIG. 4. Critical swirl, divided by the asymptotic prediction, as a function of the Mach number M_0. The critical swirl value of the incompressible vortex is $S_{c,0} \sim 1.56$ whatever the value of a.

TABLE I. Critical swirls for different thermodynamic models of the solid-body rotation flow (numerical results).

M_0	$\bar{p}=\bar{\rho}$[a]	$\bar{p}=\bar{\rho}^{1/\gamma}$[b]	$\bar{p}=\bar{\rho}(1-(\gamma-1)M_0^2U_\theta^2/2)$[c]
0	1.92	1.92	1.92
0.4	2.07	2.09	2.05
0.6	2.33	2.37	2.36

[a]Isothermal.
[b]Homoentropic.
[c]Homoenthalpic.

3). This restoring mechanism reverses the initial gradient of ω_z' [Eq. (9b)], thus leading to the propagation of the perturbation.[9]

Then, we consider a low compressible flow $M_0 \ll 1$ and long inertial waves $k \ll 1$, with $c=\omega/k=O(1)$ (see Fig. 4). We now evaluate the magnitudes of different terms in Eq. (9). The last term in Eq. (9a) is a baroclinic contribution and is invoked in Ref. 10 to account for the increase in critical swirl with M_0. Its order of magnitude is however $O(kM_0^2)$ and it could be neglected with respect to other terms which are $O(k)$. The velocity field divergence in Eq. (9b) can be transformed via the continuity equation into $\nabla\cdot\mathbf{u}'=-[\partial_t\rho' + (d_r\bar{\rho})u']/\bar{\rho}$. The temporal derivative of the density is $O(\omega M_0^2)$ and appears to be negligible with respect to the convective term. This term is $O(M_0^2)$ to be compared to the incompressible terms in Eq. (9b) which are $O(k)$. The convective term is thus the leading compressible effect and must be retained. Now, as already noted, the perturbation of azimuthal vorticity is associated to axial and radial velocities through $\omega_\theta'=\partial_z u' - \partial_r w'$. This means that regions where ω_z' is maximum or minimum lead to a radial outflow $u'>0$ or inflow $u'<0$ respectively (see Fig. 4). We conclude that compressibility damps the inertial waves propagation mechanism by producing an effect which is opposite to that of the stretching term in Eq. (9b).

Discussion: We now briefly discuss the application of this analysis to different flow fields. First, Table I summarizes some critical swirl values S_c obtained when changing the thermodynamic properties. In all instances, S_c remains an increasing function of the Mach number. Changing thermodynamics leads to comparable results. The asymptotic prediction (5) thus remains acceptable whatever the thermodynamic model.

Second, we consider an isothermal Batchelor vortex defined by $W(r)=a+(1-a)e^{-r^2}$ and $V(r)=S(1-e^{-r^2})/r$, where $a=w_\infty/w_0$ is the ratio between velocities away and on the axis. This model is representative of realistic unbounded columnar vortices.[3] The critical swirl S_c is numerically evaluated by using the shooting method. Figure 4 presents the evolution of S_c, divided by the asymptotic prediction $S_{c,as}=S_{c,0}/(1-M_0^2)^{1/2}$, as a function of M_0. The criticality conditions are well predicted by the asymptotic model up to $M_0=0.7$ where the difference with respect to the numerical results is lower than 5%. For larger M_0, results diverge. Besides, the ratio $S_c/S_{c,as}$ does not depend on the parameter a, indicating that the waves depend only on the axial velocity on the axis even if compressibility is taken into account (the inertial waves propagate in the core of the vortex and thus behave independently of the external flow). Finally, note that the results obtained for $a=1$ are different from those obtained in Ref. 2 where S_c was found to become singular in the neighborhood of the value $M_0=0.69$.

These last results prove that the present conclusions, i.e., Eq. (5) and subsequent interpretations, are poorly dependent on the swirling flow model and confirm the relevance of the Kelvin waves to describe the dynamics in supercritical vortex flows, before the transition to breakdown occurs. According to Refs. 11 and 12, vortex breakdown is a result of the interaction of azimuthal vorticity waves with relatively fixed inlet state in an incompressible vortex. When $S>S_{c,0}$, azimuthal disturbances move upstream and accumulate at the inlet condition thus initiating the instability process. A similar situation is strongly expected to occur in compressible pipe flows[10] and accounts for the study of compressibility effects on the waves dynamics.

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