Worksheet n°1 : mesh, base-flow, global modes, adjoint global modes

0/ Very short reminder on finite elements

Let us solve the following problem:

$$u - (\partial_{xx}u + \partial_{yy}u) = f$$

$$u = d \text{ on } \Gamma_d$$

$$au + b\partial_n u = c \text{ on } \Gamma_m$$

We consider test functions $\breve{u}$ satisfying $\breve{u} = 0$ on Γ_d. After multiplying the governing equation by the test-function, we take an integral over the complete domain:

$$\iint \breve{u}\big(u - (\partial_{xx}u + \partial_{yy}u\big)dxdy = \iint \breve{u}f dxdy$$

Integrating by parts, we obtain:

$$\iint \big(\breve{u}u + \partial_x\breve{u}\partial_x u + \partial_y\breve{u}\partial_y u\big)dxdy - \int \big(\breve{u}n_x\partial_x u + \breve{u}n_y\partial_y u\big)ds = \iint \breve{u}f dxdy$$

The boundary term is zero on Γ_d because of $\breve{u} = 0$. Therefore, taking into account the boundary condition on Γ_m, we have:

$$\iint \big(\breve{u}u + \partial_x\breve{u}\partial_x u + \partial_y\breve{u}\partial_y u\big)dxdy - \int_{\Gamma_m} \breve{u}\left(c - \frac{a}{b}u\right)ds = \iint \breve{u}f dxdy$$

Rearranging:

$$\iint \big(\breve{u}u + \partial_x\breve{u}\partial_x u + \partial_y\breve{u}\partial_y u\big)dxdy + \int_{\Gamma_m} \frac{a}{b}\breve{u}u ds = \iint \breve{u}f dxdy + \int_{\Gamma_m} \breve{u}c ds$$

Using for example P2 elements for u and $\breve{u}$, we obtain the following discretized form (taking into account that $u = d$ on Γ_d):

$$Au = b$$

1/ Generate mesh

In folder Mesh:

FreeFem++ mesh.edp

2/ Base-flow

The base-flow is solution of the following non-linear equation:

$$\frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = 0, \quad \mathcal{N}(w_1, w_2) = \begin{pmatrix} u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\ 0 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} -\nu\Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}$$

with the following boundary conditions:

$$(u_0 = 1, v_0 = 0) \text{ on } \Gamma_{in}$$

$$(u_0 = 0, v_0 = 0) \text{ on } \Gamma_{wall}$$

$$\big(-p_0 n_x + \nu(n_x\partial_x u_0 + n_y\partial_y u_0) = 0, -p_0 n_y + \nu(n_x\partial_x v_0 + n_y\partial_y v_0) = 0\big) \text{ on } \Gamma_{out}$$

$$\big(\partial_y u_0 = 0, v_0 = 0\big) \text{ on } \Gamma_{lat}$$

The Newton iteration is based on successive solutions of:

$$(\mathcal{N}_w + \mathcal{L})\delta w = -\frac{1}{2}\mathcal{N}(w,w) - \mathcal{L}w \text{ where } \mathcal{N}_w\delta w = \begin{pmatrix} \delta u \cdot \nabla u + u \cdot \nabla \delta u \\ 0 \end{pmatrix}$$

with boundary conditions such that $w + \delta w$ satisfy the above mentioned boundary conditions.

Hence:

$$\delta u \partial_x u + \delta v \partial_y u + u \partial_x \delta u + v \partial_y \delta u + \partial_x \delta p - \nu\left(\partial_{xx}\delta u + \partial_{yy}\delta u\right) = -u\partial_x u - v\partial_y u - \partial_x p + \nu\left(\partial_{xx} u + \partial_{yy} u\right)$$

$$\delta u \partial_x v + \delta v \partial_y v + u \partial_x \delta v + v \partial_y \delta v + \partial_y \delta p - \nu\left(\partial_{xx}\delta v + \partial_{yy}\delta v\right) = -u\partial_x v - v\partial_y v - \partial_y p + \nu\left(\partial_{xx} v + \partial_{yy} v\right)$$

$$-\partial_x \delta u - \partial_y \delta v = \partial_x u + \partial_y v$$

with:

$$(\delta u = 1 - u, \delta v = -\mathrm{v}) \text{ on } \Gamma_{in}$$

$$(\delta u = -u, \delta v = -v) \text{ on } \Gamma_{wall}$$

$$(-\delta p n_x + \nu\left(n_x \partial_x \delta u + n_y \partial_y \delta u\right) = p n_x - \nu\left(n_x \partial_x u + n_y \partial_y u\right), -\delta p n_y + \nu\left(n_x \partial_x \delta v + n_y \partial_y \delta v\right) = p n_y - \nu\left(n_x \partial_x v + n_y \partial_y v\right) \text{ on } \Gamma_{out})$$

$$(\partial_y \delta u = -\partial_y u, \delta v = -v) \text{ on } \Gamma_{lat}$$

Show that the weak form of these equations is (with $\breve{w}$ as the test-function satisfying $\breve{u} = \breve{v} = 0$ on Γ_{in} and Γ_{wall} and $\breve{v} = 0$ on Γ_{lat})

$$\iint \left(\breve{u}\left(\delta u \partial_x u + \delta v \partial_y u + u \partial_x \delta u + v \partial_y \delta u\right) + \breve{v}\left(\delta u \partial_x v + \delta v \partial_y v + u \partial_x \delta v + v \partial_y \delta v\right) - \delta p(\partial_x \breve{u} + \partial_y \breve{v}) + \nu\left(\partial_x \breve{u} \partial_x \delta u + \partial_y \breve{u} \partial_y \delta u + \partial_x \breve{v} \partial_x \delta v + \partial_y \breve{v} \partial_y \delta v\right) - \breve{p}(\partial_x \delta u + \partial_y \delta v) \right) dxdy = \iint \left(-\breve{u}\left(u\partial_x u + v\partial_y u\right) - \breve{v}\left(u \partial_x v + v \partial_y v\right) + p\left(\partial_x \breve{u} + \partial_y \breve{v}\right) - \nu\left(\partial_x \breve{u} \partial_x u + \partial_y \breve{u} \partial_y u + \partial_x \breve{v} \partial_x v + \partial_y \breve{v} \partial_y v\right) + \breve{p}(\partial_x u + \partial_y v)\right) dxdy$$

After discretization (taking into account all the Dirichlet boundary-conditions), we obtain:

$$A\delta w = b$$

In folder BF:

vi param.txt // target Reynolds number, here Re=100

FreeFem++ init.edp // generate initial guess solution, here zero flowfield

FreeFem++ newton.edp // compute base-flow

FreeFem++ plotUvvp.edp // show base-flow at Re=100

3/ Global modes

The global modes are the structures such that:

$$\lambda\mathcal{B}\widehat{w} + (\mathcal{N}_{w_0} + \mathcal{L})\widehat{w} = 0, \quad \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

where $(\mathcal{N}_{w_0} + \mathcal{L})$ is the linearized Navier-Stokes operator:

$$(\mathcal{N}_{w_0} + \mathcal{L})\widehat{w} = \begin{pmatrix} \hat{u}\partial_x u_0 + \hat{v}\partial_y u_0 + u_0\partial_x\hat{u} + v_0\partial_y\hat{u} + \partial_x\hat{p} - \nu(\partial_{xx}\hat{u} + \partial_{yy}\hat{u}) \\ \hat{u}\partial_x v_0 + \hat{v}\partial_y v_0 + u_0\partial_x\hat{v} + v_0\partial_y\hat{v} + \partial_y\hat{p} - \nu(\partial_{xx}\hat{v} + \partial_{yy}\hat{v}) \\ -(\partial_x\hat{u} + \partial_y\hat{v}) \end{pmatrix}$$

$(\mathcal{N}_{w_0} + \mathcal{L})$ acts on a subspace of functions $\widehat{w}$ satisfying the following boundary conditions $\widehat{(*)}$

$$(\hat{u} = 0, \hat{v} = 0) \text{ on } \Gamma_{in} \text{ and } \Gamma_{wall}$$

$$(-\hat{p}n_x + \nu(n_x\partial_x\hat{u} + n_y\partial_y\hat{u}) = 0, -\hat{p}n_y + \nu(n_x\partial_x\hat{v} + n_y\partial_y\hat{v}) = 0) \text{ on } \Gamma_{out}$$

$$(\partial_y\hat{u} = 0, \hat{v} = 0) \text{ on } \Gamma_{lat}$$

Show that the weak form of these equations is (with $\breve{w}$ as the test-function satisfying $\breve{u} = \breve{v} = 0$ on Γ_{in} and Γ_{wall} and $\breve{v} = 0$ on Γ_{lat}):

$$\iint \Big(\breve{u}(-\hat{u}\partial_x u_0 - \hat{v}\partial_y u_0 - u_0\partial_x\hat{u} - v_0\partial_y\hat{u}) + (\partial_x\breve{u})\hat{p} - \nu(\partial_x\breve{u}\partial_x\hat{u} + \partial_y\breve{u}\partial_y\hat{u}) + \breve{v}(-\hat{u}\partial_x v_0 - \hat{v}\partial_y v_0 - u_0\partial_x\hat{v} - v_0\partial_y\hat{v}) + (\partial_y\breve{v})\hat{p} - \nu(\partial_x\breve{v}\partial_x\hat{v} + \partial_y\breve{v}\partial_y\hat{v}) + \breve{p}(\partial_x\hat{u} + \partial_y\hat{v})\Big)dxdy = \lambda\iint (\breve{u}\hat{u} + \breve{v}\hat{v})dxdy$$

With a finite element-discretization:

$$A\widehat{w} = \lambda B\widehat{w}$$

In folder Eigs:

FreeFem++ eigen.edp:

4/ Definition of adjoint operator.

The adjoint operator $(\widetilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}})$ is the operator satisfying for all $\widehat{w}$ and $\widetilde{w}$ the following relations:

$$< \widetilde{w}, (\mathcal{N}_{w_0} + \mathcal{L})\widehat{w} > = < (\widetilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}})\widetilde{w}, \widehat{w} >$$

Here $\widehat{w}$ is in the subspace satisfying the boundary conditions $\widehat{(*)}$.

Determine the adjoint operator $(\widetilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}})$and the boundary conditions $\widetilde{(*)}$ that $\widetilde{w}$ satisfies.

Solution:

$$(\widetilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}})\widetilde{w} = \begin{pmatrix} -u_0\partial_x\tilde{u} - v_0\partial_y\tilde{u} + \tilde{u}\partial_x u_0 + \tilde{v}\partial_x v_0 + \partial_x\tilde{p} - \nu(\partial_{xx}\tilde{u} + \partial_{yy}\tilde{u}) \\ -u_0\partial_x\tilde{v} - v_0\partial_y\tilde{v} + \tilde{u}\partial_y u_0 + \tilde{v}\partial_y v_0 + \partial_y\tilde{p} - \nu(\partial_{xx}\tilde{v} + \partial_{yy}\tilde{v}) \\ -(\partial_x\tilde{u} + \partial_y\tilde{v}) \end{pmatrix}$$

$$(\tilde{u} = 0, \tilde{v} = 0) \text{ on } \Gamma_{in} \text{ and } \Gamma_{wall}$$

$$(-\tilde{p}n_x + \nu\partial_x\tilde{u}n_x + \nu\partial_y\tilde{u}n_y = -\tilde{u}u_0n_x - \tilde{u}v_0n_y, -\tilde{p}n_y + \nu\partial_x\tilde{v}n_x + \nu\partial_y\tilde{v}n_y = -\tilde{v}u_0n_x - \tilde{v}v_0n_y) \text{ on } \Gamma_{out}$$

$$(\partial_y\tilde{u} = 0, \tilde{v} = 0) \text{ on } \Gamma_{lat}$$

5/ The adjoint global modes are solution of the following eigen-problem :

$$\lambda \mathcal{B}\widetilde{w} + \left(\widetilde{\mathcal{N}_{w_0}} + \tilde{\mathcal{L}}\right)\widetilde{w} = 0$$

with the above mentioned boundary conditions.

Show that the weak form of these equations is:

$$\begin{aligned}\iint \Big(& \breve{u}\left(u_0\partial_x\tilde{u} + v_0\partial_y\tilde{u} - \tilde{u}\partial_x u_0 - \tilde{v}\partial_x v_0\right) + (\partial_x\breve{u})\tilde{p} - \nu\left(\partial_x\breve{u}\partial_x\tilde{u} + \partial_y\breve{u}\partial_y\tilde{u}\right) + \breve{v}\left(u_0\partial_x\tilde{v} + v_0\partial_y\tilde{v} - \tilde{u}\partial_y u_0 - \tilde{v}\partial_y v_0\right) \\ &+ \left(\partial_y\breve{v}\right)\tilde{p} - \nu\left(\partial_x\breve{v}\partial_x\tilde{v} + \partial_y\breve{v}\partial_y\tilde{v}\right) + \breve{p}\left(\partial_x\tilde{u} + \partial_y\tilde{v}\right)\Big)\, dxdy \\ &- \int_{\Gamma_{out}} \breve{u}\left(\tilde{u}u_0 n_x + \tilde{u}v_0 n_y\right)ds - \int_{\Gamma_{out}} \breve{v}\left(\tilde{v}u_0 n_x + \tilde{v}v_0 n_y\right)ds = \lambda \iint \left(\breve{u}\tilde{u} + \breve{v}\tilde{v}\right)dxdy\end{aligned}$$

After discretization, we obtain:

$$\tilde{A}\widetilde{w} = \lambda B\widetilde{w}$$

Complete program eigenadj.edp (look for ??? in this file) to compute the adjoint global modes.

6/ Compute the angle between the direct and adjoint global modes to evaluate the non-normality of the mode. Check bi-orthogonality of direct and adjoint global modes.

7/ Modify program eigen.edp to solve the eigen-problem:

$$A^*\widetilde{w}' = \mu B\widetilde{w}'$$

where A^* designates the transconjugate of matrix A. Compare $\widetilde{w}'$ and $\widetilde{w}$.

Show that: $(\mu^* - \lambda)\widetilde{w}'^* B\widehat{w} = 0$. Interpret the results.

8/ DNS simulations. We consider the Navier-Stokes equations in perturbative form: $w(t) = w_0 + w'$(t):

$$\begin{cases}\partial_t u' + u'\cdot\nabla u_0 + u_0\cdot\nabla u' + u'\cdot\nabla u' & = & -\nabla p' + \nu\Delta u' \\ \nabla\cdot u' & = & 0\end{cases}$$

A first –order semi-implicit discretization in time yields:

$$\begin{cases}\dfrac{u^{n+1} - u^n}{\Delta t} + u^{n+1}\cdot\nabla u_0 + u_0\cdot\nabla u^{n+1} + u^n\cdot\nabla u^n & = & -\nabla p^{n+1} + \nu\Delta u^{n+1} \\ \nabla\cdot u^{n+1} & = & 0\end{cases}$$

This may be re-arranged into:

$$\begin{cases}\dfrac{u^{n+1}}{\Delta t} + u^{n+1}\cdot\nabla u_0 + u_0\cdot\nabla u^{n+1} + \nabla p^{n+1} - \nu\Delta u^{n+1} & = & \dfrac{u^n}{\Delta t} - u^n\cdot\nabla u^n \\ \nabla\cdot u^{n+1} & = & 0\end{cases}$$

Show that the weak form with $\breve{w}$ as the test-function is:

$$\iint \left(\breve{u} \left(\frac{u^{n+1}}{\Delta t} + u^{n+1}\, \partial_x u_0 + v^{n+1} \partial_y u_0 + u_0 \partial_x u^{n+1} + v_0 \partial_y u^{n+1} \right) - (\partial_x \breve{u}) p^{n+1} + \nu \left(\partial_x \breve{u} \partial_x u^{n+1} + \partial_y \breve{u} \partial_y u^{n+1} \right) \right.$$
$$+ \breve{v} \left(\frac{v^{n+1}}{\Delta t} + u^{n+1} \partial_x v_0 + v^{n+1} \partial_y v_0 + u_0 \partial_x v^{n+1} + v_0 \partial_y v^{n+1} \right) - \left(\partial_y \breve{v} \right) p^{n+1}$$
$$\left. + \nu \left(\partial_x \breve{v} \partial_x v^{n+1} + \partial_y \breve{v} \partial_y v^{n+1} \right) + \breve{p} \left(\partial_x u^{n+1} + \partial_y v^{n+1} \right) \right) dxdy$$
$$= \iint \left(\frac{\breve{u} u^n}{\Delta t} - \breve{u}(u^n \cdot \nabla u^n) + \frac{\breve{v} v^n}{\Delta t} - \breve{v}(v^n \cdot \nabla v^n) \right) dxdy$$

After spatial discretization, we obtain:

$$A w^{n+1} = b$$

In folder DNS,

FreeFem++ init.edp // Initial condition = real part of unit energy eigenvector in ../Eigs
FreeFem++ dns.edp // Launch linearized DNS simulation
Octave plotlinlog('out_0.txt',1,2,1) // plot perturbation energy as a function of time
Octave plotlinlin('out_0.txt',1,3,1) // plot u-velocity in wake as a function of time
FreeFem++ plotUvvp.edp // Plot flowfield after 100 time steps

9/ Perform a linearized DNS simulation with a unit energy adjoint flowfield as initial condition. Compare perturbation energy as a function of time with results obtained in 8/ Relate this result to the angle computed in 6/

10/ Perform a non-linear simulation to observe saturation.

11/ Vary the Reynolds number, find critical Reynolds number with stability analyses and observe saturation amplitudes with non-linear simulations as a function of Reynolds number in the range $40 < Re < 100$.