Worksheet n°2: Lagrangian technique, sensitivity to base-flow modifications, sensitivity to steady forcing

1/ Lagrangian technique

Consider the following problem:

- State: $w = w(x)$ on $0 \leq x \leq 1$
- Control: $f = f(x)$ on $0 \leq x \leq 1$
- Constraints:

$$w\partial_x w = \alpha w + \nu \partial_{xx} w + f$$

$$w(0) = 1$$

$$\partial_x w(1) = 0$$

- Objective functional:

$$\Im(f) = \Im(w(f), f) = \int_0^1 ((w - w_0)^2 + l^2 f^2) dx$$

where w_0 is a target field and l a scalar constant.

Compute $\frac{d\mathcal{J}}{df} = \nabla_f \mathcal{J}$ such that $\delta\mathcal{J} = < \nabla_f \mathcal{J}, \delta f >$ with $\langle a, b \rangle = \int_0^1 ab dx$.

2/ Sensitivity to base-flow modifications

2a) Theory

Consider the following problem:

- State: $[\hat{w}, \lambda]$
- Control: w_0
- Constraint: $\lambda \mathcal{B}\hat{w} + \mathcal{N}_{w_0}\hat{w} + \mathcal{L}\hat{w} = 0$
- Objective: $\Im(w_0) = \lambda$

Compute $\frac{d\mathcal{J}}{dw_0} = \nabla_{w_0}\lambda$ such that $\delta\lambda = < \nabla_{w_0}\lambda, \delta w_0 >$ with $< w_1, w_2 > = \iint (u_1^* u_2 + v_1^* v_2) dxdy$

Solution:

$$\nabla_{w_0}\lambda = -\widetilde{\mathcal{N}}_{\hat{w}}\widetilde{w}$$

$$\text{with: } \lambda^* \mathcal{B}\widetilde{w} + \widetilde{\mathcal{N}}_{w_0}\widetilde{w} + \tilde{L}\widetilde{w} = 0$$

$$\text{normalization condition: } \iint [\widetilde{w}^* \mathcal{B}\hat{w}] dxdy = 1$$

$$\text{and: } \widetilde{\mathcal{N}}_{\hat{w}} = \begin{bmatrix} \partial_x \hat{u} & \partial_y \hat{u} & 0 \\ \partial_x \hat{v} & \partial_y \hat{v} & 0 \\ 0 & 0 & 0 \end{bmatrix}^* + \begin{bmatrix} -\hat{u}^*\partial_x - \hat{v}^*\partial_y & 0 & 0 \\ 0 & -\hat{u}^*\partial_x - \hat{v}^*\partial_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2b) Implementation

In folder Mesh:

FreeFem++ mesh.edp // generate mesh

In folder BF:

FreeFem++ init.edp // generate initial guess solution, here zero flowfield

FreeFem++ newton.edp // compute base-flow

In folder Eigs:

FreeFem++ eigen.edp // compute unstable direct global mode

FreeFem++ eigenadj.edp // compute unstable adjoint global mode

FreeFem++ norm.edp // compute scaled unstable adjoint global mode

Complete program sensbf.edp (look for "???" in this file) and represent real and imaginary parts of $\nabla_{w_0}\lambda$.

Observe that these vector fields are not divergence-free.

2c) Divergence-free gradient

The symmetric divergence-free gradient $\nabla_{w_0}\lambda\big|_P$ may be expressed as: $\nabla_{w_0}\lambda\big|_P = \begin{pmatrix} \partial_y\psi \\ -\partial_x\psi \end{pmatrix}$ with $\psi = 0$ on the symmetry line. Find the equations governing ψ by writing that:

$$< \nabla_{w_0}\lambda\big|_P, \delta u_0 > = < \nabla_{w_0}\lambda, \delta u_0 >$$

for all divergence-free $\delta u_0 = \begin{pmatrix} \partial_y\phi \\ -\partial_x\phi \end{pmatrix}$.

Solution:

$$-\partial_{xx}\psi - \partial_{yy}\psi = \partial_x(\nabla_{v_0}\lambda) - \partial_y(\nabla_{u_0}\lambda)$$

$$\psi = 0 \text{ on symmetry line}$$

$$-n_x\partial_x\psi - n_y\partial_y\psi = \left(\frac{d\lambda}{dv_0}\right) n_x - \left(\frac{d\lambda}{du_0}\right) n_y \text{ on all other boundaries}$$

Complete program sensbf-incomp.edp and represent real and imaginary parts of ψ and $\nabla_{w_0}\lambda\big|_P$. Compare to $\nabla_{w_0}\lambda$.

3/ Sensitivity to steady forcing

3a) Theory

Consider the following problem:

- State: $[w_0, \widehat{w}, \lambda]$
- Control: f
- Constraints: $\begin{cases} \frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = f \\ \lambda\mathcal{B}\widehat{w} + \mathcal{N}_{w_0}\widehat{w} + \mathcal{L}\widehat{w} = 0 \end{cases}$
- Objective: $\Im(f) = \lambda\big(w_0(f)\big)$

Compute $\nabla_f \lambda$ such that $\delta\lambda = < \nabla_f \lambda, \delta f >$.

Solution:

$$\nabla_f \lambda = \widetilde{w}_0$$

$$\widetilde{\mathcal{N}}_{w_0}\widetilde{w}_0 + \tilde{L}\widetilde{w}_0 = \nabla_{w_0}\lambda$$

3b) Implementation

Complete program sensforc.edp and represent real and imaginary parts of $\nabla_f \lambda$.

4/ Model cylinder by considering $\delta f = -u_0$. At the end of sensforc.edp, compute control map: $\delta\lambda = -\nabla_f \lambda \cdot u_0$. Represent real and imaginary parts. Interpret results.

5/ We would like to check the validity of $\nabla_f \lambda$. For this, we consider a steady forcing of the form: $f = \epsilon e^{-\frac{(x-0.6)^2+(y-0.6)^2}{0.2^2}}$, $g = 0$.

Compare the curves $\lambda(\epsilon)$ and $\lambda(\epsilon = 0) + < \nabla_f \lambda, f(\epsilon) >$ for various values of ϵ.

For this, in GradientsCheck:

```
FreeFem++ forcing.edp       // generates steady forcing for given epsilon
vi newton.edp               // ADD EFFECT OF STEADY FORCING IN NEWTON ITERATION
FreeFem++ newton.edp        // compute new base-flow which takes into account forcing
FreeFem++ eigen.edp         // computes new eigenvalue
vi eigshift.edp             // ADD EVALUATION OF EIGENVALUE SHIFT BY GRADIENT
FreeFem++ eigshift.edp      // compare eigenvalue shift given by gradient and finite
                            // differences
```

6/ Appendix

Operators:

$$\mathcal{N}(w_1, w_2) = \begin{pmatrix} u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\ 0 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} -\nu\Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Base-flow :

$$\frac{1}{2}\mathcal{N}(w_0,w_0)+\mathcal{L}w_0=f$$

with:

$$(u_0=1, v_0=0) \text{ on } \Gamma_{in}$$

$$(u_0=0, v_0=0) \text{ on } \Gamma_{wall}$$

$$\left(-p_0 n_x+\nu\left(n_x\partial_x u_0+n_y\partial_y u_0\right)=0, -p_0 n_y+\nu\left(n_x\partial_x v_0+n_y\partial_y v_0\right)=0\right) \text{ on } \Gamma_{out}$$

$$\left(\partial_y u_0=0, v_0=0\right) \text{ on } \Gamma_{lat}$$

Direct global modes:

$$\lambda\mathcal{B}\widehat{w}+\left(\mathcal{N}_{w_0}+\mathcal{L}\right)\widehat{w}=0,$$

with:

$$\left(\mathcal{N}_{w_0}+\mathcal{L}\right)\widehat{w}=\begin{pmatrix}\hat{u}\partial_x u_0+\hat{v}\partial_y u_0+u_0\partial_x\hat{u}+v_0\partial_y\hat{u}+\partial_x\hat{p}-\nu\left(\partial_{xx}\hat{u}+\partial_{yy}\hat{u}\right)\\ \hat{u}\partial_x v_0+\hat{v}\partial_y v_0+u_0\partial_x\hat{v}+v_0\partial_y\hat{v}+\partial_y\hat{p}-\nu\left(\partial_{xx}\hat{v}+\partial_{yy}\hat{v}\right)\\ -\left(\partial_x\hat{u}+\partial_y\hat{v}\right)\end{pmatrix}$$

$$(\hat{u}=0, \hat{v}=0) \text{ on } \Gamma_{in} \text{ and } \Gamma_{wall}$$

$$\left(-\hat{p}n_x+\nu\left(n_x\partial_x\hat{u}+n_y\partial_y\hat{u}\right)=0, -\hat{p}n_y+\nu\left(n_x\partial_x\hat{v}+n_y\partial_y\hat{v}\right)=0\right) \text{ on } \Gamma_{out}$$

$$\left(\partial_y\hat{u}=0, \hat{v}=0\right) \text{ on } \Gamma_{lat}$$

Adjoint global modes:

$$\lambda\mathcal{B}\widetilde{w}+\left(\widetilde{\mathcal{N}_{w_0}}+\tilde{\mathcal{L}}\right)\widetilde{w}=0$$

with:

$$\left(\widetilde{\mathcal{N}}_{w_0}+\tilde{\mathcal{L}}\right)\widetilde{w}=\begin{pmatrix}-u_0\partial_x\tilde{u}-v_0\partial_y\tilde{u}+\tilde{u}\partial_x u_0+\tilde{v}\partial_x v_0+\partial_x\tilde{p}-\nu\left(\partial_{xx}\tilde{u}+\partial_{yy}\tilde{u}\right)\\ -u_0\partial_x\tilde{v}-v_0\partial_y\tilde{v}+\tilde{u}\partial_y u_0+\tilde{v}\partial_y v_0+\partial_y\tilde{p}-\nu\left(\partial_{xx}\tilde{v}+\partial_{yy}\tilde{v}\right)\\ -\left(\partial_x\tilde{u}+\partial_y\tilde{v}\right)\end{pmatrix}$$

$$(\tilde{u}=0, \tilde{v}=0) \text{ on } \Gamma_{in} \text{ and } \Gamma_{wall}$$

$$\begin{aligned}\Big(-\tilde{p}n_x+\nu\partial_x\tilde{u}n_x+\nu\partial_y\tilde{u}n_y=-\tilde{u}u_0n_x-\tilde{u}v_0n_y, -\tilde{p}n_y+\nu\partial_x\tilde{v}n_x+\nu\partial_y\tilde{v}n_y\\ =-\tilde{v}u_0n_x-\tilde{v}v_0n_y\Big) \text{ on } \Gamma_{out}\end{aligned}$$