Worksheet n°5: Feedback control

Let a physical system be governed by the following equations

$$\frac{dw}{dt} = Aw + Bu, y = Cw + g$$

$$A = \begin{bmatrix} 1 & -10 \\ 10 & 1 \end{bmatrix}, B = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, C = (1 \quad 1)$$

where u is a control signal subject to a noise v.
Octave:

```
A=[[1 -10];[10 1]]

B=[-1; -1]

C=[1  1]

eig(A)   % eigenvalues of A

sys=ss(A,[B zeros(size(B))],C, [0 1])      % two inputs u and g, one output y

figure(1)

impulse(sys)     % plot impulse response
```

A/ Temporal domain

a/ State-feedback and pole placement.

We look for a control law under the form: $u = Kw + v$. The closed-loop system is:

$$\frac{dw}{dt} = Aw + Bu = (A + BK)w + Bv$$

$$y = Cw + g$$

Find K so that the closed-loop eigenvalues (eigenvalues of $A + BK$) are located at $\lambda = -1 \pm i10$.
Octave:

```
K=[9/5 11/5]              % gains of controller

eig(A+B*K)                % eigenvalues of closed-loop system

syssf=ss(A+B*K,[B zeros(size(B))],C,[0 1])  % two inputs v and g, one output y

figure(2)

impulse(syssf(1,1),'b')   % impulse response of closed-loop system from input 1 (v) to output
                          % 1 (y)

hold on

K2=[10 10]                % other control gains
```

```
eig(A+B*K2)                 % closed-loop eigenvalues

syssf2=ss(A+B*K2,[B zeros(size(B))],C,[0 1])   % two inputs v and g, one output y

impulse(syssf2(1,1),'r')  % impulse response of closed-loop system from input 1 (v) to output
                          % (1) y

%%

dt=0.01           % sampling time

t=0:dt:1000;      % time samples

S=0.01;           % PSD of white noise

variance=S/dt;  % variance of white noise

v=sqrt(variance)*randn(size(t));          % white noise of PSD (S)

[yy,tt]=lsim(syssf(1,1),v,t);        % generate output y from noise v with close-loop system

figure(3)

plot(tt,v,'g')                       % plot noise

hold on

plot(tt,yy,'k');                % plot output y of closed-loop system

std(yy)    % standard deviation of output

norm(syssf(1,1))*sqrt(S)  % 2-norm of impulse response times sqrt(PSD)

%%

[yy2,tt2]=lsim(syssf2(1,1),v,t); % generate output y from noise v with close-loop system

plot(tt2,yy2,'m');    % plot output y of closed-loop system

std(yy2)    % standard deviation of output

norm(syssf2(1,1))*sqrt(S)    % 2-norm of impulse response times sqrt(PSD)

 %%
```

b/ Observer-feedback and pole placement.

Introducing an estimated state for the control input ($u = Kw_e + v$), the governing equation reads:

$$\dot{w} = Aw + BKw_e + Bv, \qquad y = Cw + g$$

A dynamic observer may be obtained by determining L such that

$$\dot{w}_e = Aw_e + BKw_e - L(y - y_e), \qquad y = Cw + g, \qquad y_e = Cw_e.$$

The estimation error $e = w - w_e$ is governed by $\dot{e} = \dot{w} - \dot{w}_e = (A + LC)e + (B \quad L)\binom{v}{g}$.

L needs to be chosen so that $A+LC$ be stable. Find L so that the eigenvalues of $A+LC$ are located at $\lambda = -1 \pm i10$.

Octave:

```
L=[-11/5; -9/5]  % observer gains

eig(A+L*C)       % eigenvalues of equation governing error

sysdo=ss(A+B*K+L*C,-L,eye(size(A)),zeros(2,1)) % one input y, two outputs w1 and w2

figure(4)

impulse(sysdo,'b')          % impulse response of observer

hold on

L2=[-9; -9]                 % alternative gains

eig(A+L2*C)                 % eigenvalues of equation governing error

sysdo2=ss(A+B*K+L2*C,-L2,eye(size(A)),zeros(2,1)) % one input y, two outputs w1 and w2

impulse(sysdo2,'r')         % impulse response of observer
```

c/ The compensator (combining the estimator and the controller) is given by:

$$\dot{w}_e = (A + BK + LC)w_e - Ly$$
$$u = Kw_e + v$$

Octave:

```
sysc=ss(A+B*K+L*C,[-L zeros(size(L))],K,[0 1])     % compensator, two inputs (y,v), one output
                                                   % u

figure(5)

impulse(sysc)               % impulse response

eig(A+B*K+L*C)              % eigenvalues of compensator
```

d/ The closed-loop system is:

$$\begin{pmatrix} \dot{w} \\ w_e \end{pmatrix} = \overbrace{\begin{pmatrix} A & BK \\ -LC & A+BK+LC \end{pmatrix}}^{A_{cl}} \begin{pmatrix} w \\ w_e \end{pmatrix} + \overbrace{\begin{pmatrix} B & 0 \\ 0 & -L \end{pmatrix}}^{B_{cl}} \begin{pmatrix} v \\ g \end{pmatrix}, \qquad \begin{pmatrix} u \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & K \\ C & 0 \end{pmatrix}}_{C_{cl}} \begin{pmatrix} w \\ w_e \end{pmatrix} + \begin{pmatrix} v \\ g \end{pmatrix}$$

Octave:

```
Acl=[[A B*K];[-L*C A+B*K+L*C]]

Bcl=[[B zeros(size(B))];[ zeros(size(B))  -L]]

Ccl=[[zeros(size(C)) K];[C zeros(size(C))]]

syscl=ss(Acl,Bcl,Ccl,eye(2,2))      % closed-loop system, two inputs v and g, two outputs u
                                    % and y
```

```
eig(Acl)                          % closed-loop eigenvalues
%%
figure(6)
[y,tim]=impulse(syscl);           % impulse responses of closed-loop system
plot(tim(:),y( :,2,1), 'b')       % plot impulse response from input 1 (v) to output 2 (y)
hold on
%%
Acl2=[[A B*K2];[-L2*C A+B*K2+L2*C]]   % closed-loop system with alternative gains
Bcl2=[[B zeros(size(B))];[ zeros(size(B))  -L2]]
Ccl2=[[zeros(size(C)) K2];[C zeros(size(C))]]
syscl2=ss(Acl2,Bcl2,Ccl2,eye(2,2))
eig(Acl2)
%%
[y2,tim2]=impulse(syscl2,tim) ;   % impulse responses of closed-loop system
plot(tim2(:),y2( :,2,1),'r')      % plot impulse response from input 1 (v) to output 2 (y)
%%
[y3,tim3]=impulse(syssf(1,1),tim);    % impulse response of state-feedback controller from
                                      % v to y
plot(tim3(:),y3(:),'k')           % plot impulse response of state-feedback controller from
                                  % input v to output y
%%
norm(syscl(2,1))        % norm of impulse response from input 1 (v) to output 2 (y)
norm(syscl2(2,1))       % same but with alternative compensator
norm(syssf(1,1))                  % same with state-feedback controller
%%
 [yy3,tt3]=lsim(syscl(2,1),v,t);  % generate output y from noise v with close-loop system
figure(3)
plot(tt3,yy3,'b');                % plot output y of closed-loop system
```

```
hold on;
std(yy3)                          % standard deviation of output
norm(syscl(2,1))*sqrt(S)          % 2-norm of impulse response times sqrt(PSD)
%%
[yy4,tt4]=lsim(syscl2(2,1),v,t);  % same with alternative compensator
plot(tt4,yy4,'r');                % same with alternative compensator
std(yy4)                          % standard deviation of output
norm(syscl2(2,1))*sqrt(S)         % 2-norm of impulse response times sqrt(PSD)
%%
pkg load all;
[spectra,freq]=pwelch(yy3,1000,0.5,1000,1/dt);% compute frequency spectrum of output y
    % MATLAB:  [spectra,freq]=pwelch(yy3,1000,500,1000,1/dt);
figure(7)
loglog(freq*2*pi,abs(spectra))  % plot frequency spectrum
```

B/ Frequency space

a/ Physical system.
Take the Laplace transform of the equations governing the physical system:

$$\frac{dw}{dt} = Aw + Bu, \qquad y = Cw + g$$

and determine the transfer functions between (u, g) and y.

The transfer-function between u and y is denoted $P(s)$. What are the poles and zeros of this transfer function?

Compare the result with Octave:

```
tf(sys)
P=tf(sys(1,1))   % open-loop transfer function P(s) between u and y.
pole(P)          % poles of P(s)
zero(P)          % zeros of P(s)
figure(8)
bode(P)          % Plot magnitude and phase of transfer function between u and y.
```

b/ What is the transfer-function $K(s)$ of the compensator (defined in A/c/) between y and u ?

Octave:

```
tf(sysc)
K=tf(sysc(1,1))  % transfer-function K(s) of compensator between y and u.
pole(K)          % poles of K(s)
zero(K)          % zeros of K(s)
figure(9)
bode(K)          % Plot magnitude and phase of transfer function between y and u.
```

c/ Closed-loop system

```
cltf=tf(syscl)    % 4 transfer functions of closed-loop system
get(cltf)         % properties of cltf
cltf(1,1).num     % numerator of closed-loop transfer function from v to u
cltf(1,1).den     % denominator of closed-loop transfer function from v to u
pole(cltf(1,1))   % poles of closed-loop transfer function from v to u
pole(cltf(1,2))   % poles of closed-loop transfer function from g to u
pole(cltf(2,1))   % poles of closed-loop transfer function from v to y
pole(cltf(2,2))   % poles of closed-loop transfer function from g to y
zero(1-P*K)       % should be compared to previous poles
figure(10)
bode(cltf(1,1))   % bode plot from from v to u   % control cost assessment
figure(11)
bode(cltf(2,1))           % bode plot from from v to y
                          % performance assessment, compare to figure 7
```

d/ Robustness

Find largest a satisfying $0 < a < 1$, such that $1 - aPK$ exhibits a zero with a positive real part.

Find smallest a satisfying $a > 1$, such that $1 - aPK$ exhibits a zero with a positive real part.

Find smallest ϕ satisfying $\phi > 0$, such that $1 - e^{i\phi}PK$ exhibits a zero with a positive real part.