Correction

2b/

$$\partial_x\phi = n_\phi\left(\frac{U}{2\gamma}-\chi^2 x\right)e^{\frac{U}{2\gamma}x-\frac{\chi^2x^2}{2}}$$

$$\partial_{xx}\phi = n_\phi\left(-\chi^2+\frac{U^2}{4\gamma^2}+\chi^4x^2-\frac{U}{\gamma}\chi^2 x\right)e^{\frac{U}{2\gamma}x-\frac{\chi^2x^2}{2}}$$

$$\begin{aligned}L\phi &= -U\partial_x\phi + i\omega_0\phi+\mu_0\phi-\mu_2\frac{x^2}{2}\phi+\gamma\partial_{xx}\phi\\ &= \left[-U\left(\frac{U}{2\gamma}-\chi^2x\right)+i\omega_0+\mu_0-\mu_2\frac{x^2}{2}\right.\\ &\left.+\gamma\left(-\chi^2+\frac{U^2}{4\gamma^2}+\chi^4x^2-\frac{U}{\gamma}\chi^2x\right)\right]n_\phi e^{\frac{U}{2\gamma}x-\frac{\chi^2x^2}{2}} = \underbrace{\left(i\omega_0+\mu_0-\frac{U^2}{4\gamma}-\gamma\chi^2\right)}_{\lambda}\phi\end{aligned}$$

Hence:

$$\lambda = i\omega_0+\mu_0-\frac{U^2}{4\gamma}-\sqrt{\frac{\gamma\mu_2}{2}}$$

2c/

$$< u, Lv > = < \tilde{L}u, v >$$

$$\int_{-\infty}^{\infty}\bar{u}(-U\partial_x v+\mu(x)v+\gamma\partial_{xx}v)dx = \int_{-\infty}^{\infty}(-\bar{u}U\partial_x v+\bar{u}\mu(x)v+\gamma\bar{u}\partial_{xx}v)dx =$$

$$= [-\bar{u}Uv+\gamma\bar{u}\partial_x v]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}(U\partial_x\bar{u}v+\bar{u}\mu(x)v-\gamma\partial_x\bar{u}\partial_x v)dx$$

$$= [-\bar{u}Uv+\gamma\bar{u}\partial_x v-\gamma\partial_x\bar{u}v]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}(U\partial_x\bar{u}+\bar{u}\mu(x)+\gamma\partial_{xx}\bar{u})v dx$$

$$= [-\bar{u}Uv+\gamma\bar{u}\partial_x v-\gamma\partial_x\bar{u}v]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\overline{\left(U\partial_x u+u\overline{\mu(x)}+\gamma\partial_{xx}u\right)}v dx$$

Hence:

$$\tilde{L} = U\partial_x+\overline{\mu(x)}+\gamma\partial_{xx}$$

2d/

$$\tilde{L}\psi = \lambda^*\psi$$

$$\partial_x \psi = (-\frac{U}{2\gamma} - \chi^2 x) e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$$

$$\partial_{xx}\psi = \left(-\chi^2 + \frac{U^2}{4\gamma^2} + \chi^4 x^2 + \frac{U}{\gamma}\chi^2 x\right) e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$$

$$\tilde{L}\psi = U\partial_x\psi + \overline{\mu(x)}\psi + \gamma\partial_{xx}\psi = \left(-\frac{U^2}{4\gamma} - i\omega_0 + \mu_0 - \gamma\chi^2\right) e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}} = \bar{\lambda}\psi$$

3a/

$$\mu(x) = i\omega_0 + \mu_c + \epsilon\delta - \mu_2 \frac{x^2}{2}$$

$$L = \underbrace{-U\partial_x + i\omega_0 + \mu_c - \mu_2 \frac{x^2}{2} + \gamma\partial_{xx}}_{L_c} + \epsilon\delta + \epsilon\Delta L$$

$$L_c\phi = i\omega_0\phi$$

3b/ Then:

$$\partial_t w = \epsilon^{\frac{1}{2}}\partial_t w_{\frac{1}{2}} + \epsilon^{\frac{3}{2}}\left(w_{\frac{3}{2}} + \partial_\tau w_{\frac{1}{2}}\right)$$

Throwing everything inside:

$$\epsilon^{\frac{1}{2}}\partial_t w_{\frac{1}{2}} + \epsilon^{\frac{3}{2}}\left(\partial_t w_{\frac{3}{2}} + \partial_\tau w_{\frac{1}{2}}\right)$$
$$= L_c\left(\epsilon^{\frac{1}{2}} w_{\frac{1}{2}} + \epsilon^{\frac{3}{2}} w_{\frac{3}{2}}\right) + \epsilon\delta\epsilon^{\frac{1}{2}} w_{\frac{1}{2}} + \epsilon\Delta L\epsilon^{\frac{1}{2}} w_{\frac{1}{2}} + c\epsilon^{\frac{3}{2}} w_{\frac{1}{2}}\left|w_{\frac{1}{2}}\right|^2 + \epsilon^{\frac{3}{2}} E\delta(x - x_f) e^{i\omega_0 t} e^{\mathrm{i}\Omega\tau}$$

Order $\epsilon^{\frac{1}{2}}$:

$$\partial_t w_{\frac{1}{2}} = L_c w_{\frac{1}{2}}$$

$$w_{\frac{1}{2}} = A(\tau) e^{i\omega_0 t}\phi$$

Order $\epsilon^{\frac{3}{2}}$:

$$\partial_t w_{\frac{3}{2}} + \partial_\tau w_{\frac{1}{2}} = L_c w_{\frac{3}{2}} + \delta w_{\frac{1}{2}} + \Delta L w_{\frac{1}{2}} + c w_{\frac{1}{2}}\left|w_{\frac{1}{2}}\right|^2 + E\delta(x - x_f) e^{i\omega_0 t} e^{\mathrm{i}\Omega\tau}$$

3c/

$$\partial_t w_{\frac{3}{2}} = L_c w_{\frac{3}{2}} - \frac{dA}{d\tau} e^{i\omega_0 t}\phi + \delta A e^{i\omega_0 t}\phi + A e^{i\omega_0 t}\Delta L\phi + cA|A|^2 e^{i\omega_0 t}\phi|\phi|^2 + E\delta(x-x_f)e^{i\omega_0 t}e^{\mathrm{i}\Omega\tau}$$

Compatibility condition:

$$< \psi, -\frac{dA}{d\tau} e^{i\omega_0 t}\phi + \delta A e^{i\omega_0 t}\phi + A e^{i\omega_0 t}\Delta L\phi + cA|A|^2 e^{i\omega_0 t}\phi|\phi|^2 + E\delta(x-x_f)e^{i\omega_0 t}e^{\mathrm{i}\Omega\tau} > = 0$$

$$< \psi, -\frac{dA}{d\tau}\phi + \delta A\phi + A e^{i\omega_0 t}\Delta L\phi + cA|A|^2\phi|\phi|^2 + E\delta(x-x_f)e^{\mathrm{i}\Omega\tau} > = 0$$

$$-\frac{dA}{d\tau} < \psi,\phi > + \delta A < \psi,\phi > + A < \psi, \Delta L\phi > + cA|A|^2 < \psi, \phi|\phi|^2 > + E e^{\mathrm{i}\Omega\tau} < \psi, \delta(x-x_f) > = 0$$

$$\frac{dA}{d\tau} = \delta A + A\frac{< \psi, \Delta L\phi >}{< \psi,\phi >} + c\frac{< \psi, \phi|\phi|^2 >}{< \psi,\phi >} A|A|^2 + \frac{\overline{\psi(x_f)}}{< \psi,\phi >} E e^{\mathrm{i}\Omega\tau}$$

3d/

$$\frac{dB'}{dt} = \epsilon^{\frac{1}{2}}\left(\frac{dA}{d\tau}\epsilon + i\omega_0 A\right)e^{i\omega_0 t} = \left(\frac{1}{A}\frac{dA}{d\tau}\epsilon + i\omega_0\right)B'$$

$$= \left(i\omega_0 + \delta\epsilon + \frac{< \psi, \Delta L\phi >}{< \psi,\phi >}\epsilon + c\frac{< \psi, \phi|\phi|^2 >}{< \psi,\phi >}|A|^2\epsilon + \frac{\frac{\overline{\psi(x_f)}}{< \psi,\phi >} E e^{\mathrm{i}\Omega\tau}}{B' e^{-i\omega_0 t}\epsilon^{-\frac{1}{2}}}\epsilon \right) B'$$

$$\frac{dB'}{dt} = i\omega_0 B' + \delta' B' + \frac{< \psi, \Delta L'\phi >}{< \psi,\phi >}B' + c\frac{< \psi, \phi|\phi|^2 >}{< \psi,\phi >}B'|B'|^2 + \frac{\overline{\psi(x_f)}}{< \psi,\phi >}E' e^{\mathrm{i}\omega_f \mathrm{t}}$$

4/ =>Response strongest for $\omega_f = \omega_0$

=> Forcing should be located at x_f where adjoint is strongest

=> Response is stronger if $< \psi,\phi >$ large => non-normal

5/ Conclusion: the eigenvalue shift is strongest if:

- $\delta\mu(x)$ is in the region where $\overline{\psi(x)}\phi(x)$ is strong, i.e. in the overlap region of the direct and adjoint
- the system is non-normal: $< \psi,\phi >$ is small.

6/

$$\lambda = i\omega_0 + \delta' + \frac{< \psi, \Delta L\phi >}{< \psi,\phi >} = i\omega_0 + \delta' + \frac{< \psi, K\delta(x-x_a)\phi(x_s) >}{< \psi,\phi >} = i\omega_0 + \delta' + \frac{K\overline{\psi}(x_a)\phi(x_s)}{< \psi,\phi >}$$

To stabilize the flow, we choose:

$$\frac{K\overline{\psi}(x_a)\phi(x_s)}{<\psi,\phi>} = -\delta' => K = \frac{<\psi,\phi>}{\overline{\psi}(x_a)\phi(x_s)}$$

7a/

$$\frac{dC'}{dt} = \epsilon^{\frac{1}{2}}\left(-i\Omega + \frac{1}{A}\frac{dA}{d\tau}\right)Ae^{-i\Omega\tau}\epsilon = \epsilon^{\frac{1}{2}}\left(-i\Omega + \delta + c\frac{<\psi,\phi|\phi|^2>}{<\psi,\phi>}|A|^2 + \frac{\overline{\psi(x_f)}}{<\psi,\phi>}\frac{E}{A}e^{\mathrm{i}\Omega\tau}\right)Ae^{-i\Omega\tau}\epsilon$$

$$= \left(-i\Omega' + \delta' + c\frac{<\psi,\phi|\phi|^2>}{<\psi,\phi>}|C'|^2 + \epsilon\frac{\overline{\psi(x_f)}}{<\psi,\phi>}\frac{E}{C'\epsilon^{-\frac{1}{2}}e^{\mathrm{i}\Omega\tau}}e^{\mathrm{i}\Omega\tau}\right)C'$$

$$\frac{dC'}{dt} = -i\Omega'C' + \delta'C' + c\frac{<\psi,\phi|\phi|^2>}{<\psi,\phi>}C'|C'|^2 + \frac{\overline{\psi(x_f)}}{<\psi,\phi>}E'$$

$$w = C'(t)e^{i\omega_f t}\phi$$