1/

2/

3a) Solution:

$$\partial_t w' + U\partial_x w' + w'\partial_x w' = -\eta\partial_{xx}w' - \mu\partial_{xxxx}w'$$

In linear form:

$$\partial_t w' = -U\partial_x w' - \eta\partial_{xx}w' - \mu\partial_{xxxx}w' = \mathcal{L}w'$$

Normal modes:

$$\widehat{w} = e^{inkx}$$

$$\mathcal{L}\widehat{w} = \lambda\widehat{w}$$

$$\lambda_{n=-\infty\cdots\infty} = -inkU + \eta n^2k^2 - \mu n^4k^4$$

Therefore, if $\eta = \eta_c$:

$$\sigma_n = \ \eta n^2k^2 - \mu n^4k^4$$

$$\omega_n = -inkU$$

$$\sigma_0 = 0, \omega_0 = 0, \widehat{w}_0 = 1$$

$$\sigma_1 = 0, \omega_1 = -ikU, \widehat{w}_1 = e^{ikx}$$

$$\sigma_n = \mu n^2k^4 \ - \mu n^4k^4 = \mu n^2k^4\,(1-n^2) < 0 \text{ for all n≥2}$$

$$\omega_n = -inkU$$

4a) The adjoint operator is defined as:

$$\langle w_1, \mathcal{L}w_2\rangle = \langle\tilde{\mathcal{L}}w_1, w_2\rangle$$

$$\mathcal{L}w = -U\partial_x w - \eta\partial_{xx}w - \mu\partial_{xxxx}w$$

$$\int_0^L \overline{w_1(x)}\mathcal{L}w_2(x)dx = \int_0^L \overline{w_1(x)}(-U\partial_x w_2 - \eta\partial_{xx}w_2 - \mu\partial_{xxxx}w_2)dx$$

$$= \int_0^L \left(-U\overline{w_1(x)}\partial_x w_2 - \eta\overline{w_1(x)}\partial_{xx}w_2 - \mu\overline{w_1(x)}\partial_{xxxx}w_2\right)dx$$

$$= \int_0^L \left(U\overline{\partial_x w_1}w_2 + \eta\overline{\partial_x w_1}\partial_x w_2 + \mu\overline{\partial_x w_1}\partial_{xxx}w_2\right)dx$$

$$+ [-U\overline{w_1}w_2 - \eta\overline{w_1}\partial_x w_2 - \mu\overline{w_1}\partial_{xxx}w_2]_0^L$$

$$= \int_0^L \left(\overline{U\partial_x w_1}w_2 - \overline{\eta\partial_{xx}w_1}w_2 - \overline{\mu\partial_{xxxx}w_1}w_2\right)dx$$

$$+ \Big[-U\overline{w_1}w_2 - \eta\overline{w_1}\partial_x w_2 - \mu\overline{w_1}\partial_{xxx}w_2 + \eta\overline{\partial_x w_1}w_2 + \mu\overline{\partial_x w_1}\partial_{xx}w_2$$

$$- \mu\overline{\partial_{xx}w_1}\partial_x w_2 + \mu\overline{\partial_{xxx}w_1}w_2\Big]_0^L$$

Hence:

$$\tilde{\mathcal{L}} = U\partial_x w - \eta\partial_{xx}w - \mu\partial_{xxxx}w$$

4b) Eigenvalue/eigenvector:

$$\widetilde{w} = \xi e^{ikx}$$

$$\tilde{\mathcal{L}}\widetilde{w} = \lambda\widetilde{w}$$

$$Uik + \eta k^2 - \mu k^4 = \lambda$$

Normalization condition:

$$\langle \widetilde{w}, \widehat{w} \rangle = 1 \Longrightarrow \frac{1}{L}\int_0^L \overline{\xi e^{\imath kx}} e^{ikx}dx = \bar{\xi} = 1$$

$$\widetilde{w} = e^{ikx}$$

5a)

$$\partial_t w' + U\partial_x w' + w'\partial_x w' = -\eta_c\partial_{xx}w' - \epsilon\delta\partial_{xx}w' - \mu\partial_{xxxx}w' + \epsilon^{\frac{3}{2}}Ee^{i\omega_0 t + i\Omega' t}\hat{f}(x)$$

$$w' = U + \epsilon^{\frac{1}{2}}w_{\frac{1}{2}}(t, \tau = \epsilon t) + \epsilon w_1(t,\tau) + \epsilon^{\frac{3}{2}}w_{3/2}(t,\tau) + \cdots +$$

Order $\epsilon^{\frac{1}{2}}$:

$$\partial_t w_{\frac{1}{2}} + U\partial_x w_{\frac{1}{2}} = -\eta_c \partial_{xx} w_{\frac{1}{2}} - \mu \partial_{xxxx} w_{\frac{1}{2}}$$

Order ϵ:

$$\partial_t w_1 + U\partial_x w_1 + w_{1/2}\partial_x w_{1/2} = -\eta_c \partial_{xx} w_1 - \mu \partial_{xxxx} w_1$$

Order $\epsilon^{\frac{3}{2}}$:

$$\begin{aligned}\partial_t w_{3/2} + \partial_\tau w_{1/2} + U\partial_x w_{3/2} + w_{1/2}\partial_x w_1 + w_1 \partial_x w_{1/2} \\ = -\eta_c \partial_{xx} w_{\frac{3}{2}} - \mu \partial_{xxxx} w_{\frac{3}{2}} - \delta \partial_{xx} w_{\frac{1}{2}} + E e^{i\omega_c t + i\Omega\tau} \hat{f}(x)\end{aligned}$$

5b)

$$w_{1/2}(x,t) = A(\tau) e^{i\omega_c t} \widehat{w} + c.c.$$

5c)

$$\begin{aligned}\partial_t w_1 + U\partial_x w_1 + \eta_c \partial_{xx} w_1 + \mu \partial_{xxxx} w_1 = -w_{\frac{1}{2}} \partial_x w_{\frac{1}{2}} \\ = -\left(A e^{i\omega_c t}\widehat{w} + \bar{A} e^{-i\omega_c t}\overline{\widehat{w}}\right)\partial_x\left(A e^{i\omega_c t}\widehat{w} + \bar{A} e^{-i\omega_c t}\overline{\widehat{w}}\right)\end{aligned}$$

$$= -ikA^2 e^{2i\omega_c t} e^{2ikx} + ik\bar{A}^2 e^{-2i\omega_c t} e^{-2ikx}$$

Solution:

$$w_1 = A^2 e^{2i\omega_c t} \widehat{w}_{AA} + cc$$

$$2i\omega_c \widehat{w}_{AA} + U\partial_x \widehat{w}_{AA} + \eta_c \partial_{xx} \widehat{w}_{AA} + \mu \partial_{xxxx} \widehat{w}_{AA} = -ike^{2ikx}$$

$$\widehat{w}_{AA} = \zeta e^{2ikx}$$

$$(2i\omega_c + 2ikU - 4k^2\eta_c + 16k^4\mu)\zeta = -ik$$

$$(-4k^2\mu k^2 + 16k^4\mu)\zeta = -ik$$

$$\zeta = -\frac{i}{12k^3\mu}$$

Comment on phase w_1: no mean-flow

5d)

$$\begin{aligned}\partial_t w_{3/2} + U\partial_x w_{3/2} - \eta_c \partial_{xx} w_{\frac{3}{2}} - \mu \partial_{xxxx} w_{\frac{3}{2}} \\ = -\partial_\tau w_{\frac{1}{2}} - \delta \partial_{xx} w_{\frac{1}{2}} - w_{\frac{1}{2}} \partial_x w_1 - w_1 \partial_x w_{\frac{1}{2}} + E e^{i\omega_c t + i\Omega\tau} \hat{f}(x)\end{aligned}$$

Resonant terms:

$$= -\frac{dA}{d\tau} e^{i\omega_c t}\widehat{w} + \delta k^2 A e^{i\omega_c t}\widehat{w} - \bar{A}e^{-i\omega_c t}\overline{\widehat{w}}A^2 e^{2i\omega_c t}(2ik)\widehat{w}_{AA} - A^2 e^{2i\omega_c t}\widehat{w}_{AA}\,\bar{A}e^{-i\omega_c t}(-ik)\overline{\widehat{w}} + Ee^{i\omega_c t}e^{i\Omega\tau}\hat{f}(x) + c.c.$$

Scalar-product with adjoint for solvability condition:

$$\langle \widetilde{w}, -\frac{dA}{d\tau}\widehat{w} + \delta k^2 A\widehat{w} - A^2\bar{A}\overline{\widehat{w}}\widehat{w}_{AA} ik + Ee^{i\Omega\tau}\hat{f}(x)\rangle = 0$$

$$\frac{dA}{d\tau} = \delta k^2 A - A^2\bar{A}\langle \widetilde{w}, \overline{\widehat{w}}\widehat{w}_{AA}\rangle ik + Ee^{i\Omega\tau}\langle \widetilde{w}, \hat{f}(x)\rangle$$

$$\langle \widetilde{w}, \overline{\widehat{w}}\widehat{w}_{AA}\rangle = \langle e^{ikx}, e^{-ikx}\zeta e^{2ikx}\rangle = \zeta$$

$$\frac{dA}{d\tau} = \delta k^2 A - A^2\bar{A}\zeta ik + Ee^{i\Omega\tau}\langle \widetilde{w}, \hat{f}(x)\rangle$$

$$\alpha = k^2 > 0$$

$$\beta = \zeta ik = -\frac{i}{12k^3\mu}ik = \frac{1}{12k^2\mu} > 0$$

$$\gamma = \langle \widetilde{w}, \hat{f}(x)\rangle$$

We have a super-critical Hopf-bifurcation.

6c)

7) $C' = B'e^{-i\omega_f t} \Rightarrow \frac{dC'}{dt} = \frac{dB'}{dt}e^{-i\omega_f t} - i\omega_f B'e^{-i\omega_f t} = (i\omega_c + \alpha\delta')B'e^{-i\omega_f t} - \beta B'e^{-i\omega_f t}|B'|^2 + \gamma E'e^{i\omega_f t}e^{-i\omega_f t} - i\omega_f B'e^{-i\omega_f t} = (i\omega_c + \alpha\delta')C' - \beta C'|C'|^2 + \gamma E' - i\omega_f C' = (-i\Omega' + \alpha\delta')C' - \beta C'|C'|^2 + \gamma E'$

8a) Order $\epsilon^{\frac{1}{2}}$:

$$\partial_t w_{\frac{1}{2}} + U\partial_x w_{\frac{1}{2}} = -\eta_c \partial_{xx} w_{\frac{1}{2}} - \mu\partial_{xxxx} w_{\frac{1}{2}} + Ee^{i\omega_f}\hat{f}(x)$$

Order ϵ:

$$\partial_t w_1 + U\partial_x w_1 + w_{1/2}\partial_x w_{1/2} = -\eta_c\partial_{xx} w_1 - \mu\partial_{xxxx} w_1$$

Order $\epsilon^{\frac{3}{2}}$:

$$\partial_t w_{3/2} + \partial_\tau w_{1/2} + U\partial_x w_{3/2} + w_{1/2}\partial_x w_1 + w_1\partial_x w_{1/2}$$

$$= -\eta_c \partial_{xx} w_{\frac{3}{2}} - \mu \partial_{xxxx} w_{\frac{3}{2}} - \delta \partial_{xx} w_{\frac{1}{2}}$$

8b) $w_{\frac{1}{2}} = \left(A(\tau)e^{i\omega_c t}\widehat{w}(x) + \text{c.c.}\right) + \left(Ee^{i\omega_f t}\widehat{w}_E(x) + \text{c.c.}\right)$

With:

$$i\omega_f \widehat{w}_E + U\partial_x \widehat{w}_E + \eta_c \partial_{xx}\widehat{w}_E + \mu\partial_{xxxx}\widehat{w}_E = \hat{f}(x)$$

In the case:

$$\hat{f}(x) = e^{inkx}, \widehat{w}_E = \theta_n e^{inkx}$$

$$\theta_n = \frac{1}{i\omega_f - \lambda_n}$$

8c)$-w_{\frac{1}{2}}\partial_x w_{\frac{1}{2}} =$
$-\left(Ae^{i\omega_c t}e^{ikx} + \bar{A}e^{-i\omega_c t}e^{-ikx} + Ee^{i\omega_f t}\theta_n e^{inkx} + Ee^{i\omega_f t}\overline{\theta_n}e^{-inkx}\right)\partial_x\left(Ae^{i\omega_c t}e^{ikx} + \bar{A}e^{-i\omega_c t}e^{-ikx} + Ee^{i\omega_f t}\theta_n e^{inkx} + Ee^{i\omega_f t}\overline{\theta_n}e^{-inkx}\right) = -ik\left(Ae^{i\omega_c t}e^{ikx} + \bar{A}e^{-i\omega_c t}e^{-ikx} + Ee^{i\omega_f t}\theta_n e^{inkx} + Ee^{-i\omega_f t}\overline{\theta_n}e^{-inkx}\right)\left(Ae^{i\omega_c t}e^{ikx} - \bar{A}e^{-i\omega_c t}e^{-ikx} + Ene^{i\omega_f t}\theta_n e^{inkx} - Ene^{-i\omega_f t}\overline{\theta_n}e^{-inkx}\right) = -ik\left(A^2e^{2i\omega_c t}e^{2ikx} + \text{c.c}\right) - ik\left(E^2 n\theta_n^2 e^{2i\omega_f t}e^{2inkx} + \text{c.c}\right) - ik\left(AEn\theta_n e^{i(\omega_c+\omega_f)t}e^{i(n+1)kx} + \text{c.c.}\right) - ik\left(AEn\overline{\theta_n}e^{i(\omega_c-\omega_f)t}e^{-i(n-1)kx} + \text{c.c.}\right)$

9/

$$\mathcal{L}(w,\widetilde{w},f) = (1-\alpha)\{w,w\} + \alpha\langle w(T),w(T)\rangle + l^2\{f,f\}$$
$$- \{\widetilde{w}, \partial_t w + w\partial_x w + \eta\partial_{xx} w + \mu\partial_{xxxx} w - f\}$$

Variation with respect to the state:

$$\lim_{\epsilon\to 0}\frac{\big(\mathcal{L}(w+\epsilon\delta w,\widetilde{w},f) - \mathcal{L}(w,\widetilde{w},f)\big)}{\epsilon}$$
$$= (1-\alpha)\{\delta w,w\} + (1-\alpha)\{w,\delta w\} + \langle 2\alpha w(T),\delta w(T)\rangle$$
$$- \{\widetilde{w}, \partial_t\delta w + w\partial_x\delta w + \delta w\partial_x w + \eta\partial_{xx}\delta w + \mu\partial_{xxxx}\delta w\}$$
$$= \{2(1-\alpha)w + \partial_t\widetilde{w} + \partial_x(\widetilde{w}w) - \widetilde{w}\partial_x w - \eta\partial_{xx}\widetilde{w} - \mu\partial_{xxxx}\widetilde{w}, \delta w\}$$
$$+ \langle 2\alpha w(T),\delta w(T)\rangle - \langle\widetilde{w}(T),\delta w(T)\rangle + \langle\widetilde{w}(0),\delta w(0)\rangle$$
$$+ [-\widetilde{w}w\delta w - \eta\widetilde{w}\partial_x\delta w - \mu\widetilde{w}\partial_{xxx}\delta w + \eta\partial_x\widetilde{w}\delta w + \mu\partial_x\widetilde{w}\partial_{xx}\delta w - \mu\partial_{xx}\widetilde{w}\partial_x\delta w$$
$$+ \mu\partial_{xxx}\widetilde{w}\delta w]_0^L =$$
$$= \{2(1-\alpha)w + \partial_t\widetilde{w} + w\partial_x\widetilde{w} - \eta\partial_{xx}\widetilde{w} - \mu\partial_{xxxx}\widetilde{w}, \delta w\}$$
$$+ \langle 2\alpha w(T) - \widetilde{w}(T),\delta w(T)\rangle$$
$$+ \langle\widetilde{w}(0),\delta w(0)\rangle + [-\widetilde{w}w\delta w - \eta\widetilde{w}\partial_x\delta w - \mu\widetilde{w}\partial_{xxx}\delta w + \eta\partial_x\widetilde{w}\delta w + \mu\partial_x\widetilde{w}\partial_{xx}\delta w$$
$$- \mu\partial_{xx}\widetilde{w}\partial_x\delta w + \mu\partial_{xxx}\widetilde{w}\delta w]_0^L$$

The functions w, $\widetilde{w}$ and δw are periodic in x. Hence all boundary terms at $x = 0$ and $x = L$ vanish.

Also, since $w(t=0) = w_I$, $\delta w(t=0) = 0$.

We choose: $2\alpha w(T) - \widetilde{w}(T) = 0$ so as to kill the temporal boundary term at $t = T$.

Therefore:

$$\frac{\partial\mathcal{L}}{\partial w} = 2(1-\alpha)w + \partial_t\widetilde{w} + w\partial_x\widetilde{w} - \eta\partial_{xx}\widetilde{w} - \mu\partial_{xxxx}\widetilde{w}$$

$$\widetilde{w}(T) = 2\alpha w(T)$$

Adjoint is defined as:

$$-\partial_t\widetilde{w} - w\partial_x\widetilde{w} + \eta\partial_{xx}\widetilde{w} + \mu\partial_{xxxx}\widetilde{w} = 2(1-\alpha)w$$

$$\widetilde{w}(T) = 2\alpha w(T)$$

$$\lim_{\epsilon\to 0}\frac{\big(\mathcal{L}(w,\widetilde{w},f+\epsilon\delta f) - \mathcal{L}(w,\widetilde{w},f)\big)}{\epsilon} = 2l^2\{f,\delta f\} - \{\widetilde{w},-\delta f\} = \{2l^2 f + \widetilde{w},\delta f\}$$

Hence:

$$\frac{\partial\mathcal{L}}{\partial f} = 2l^2 f + \widetilde{w}$$