Exam

We consider the Ginzburg-Landau equation:

$$\partial_t w = Lw + cw|w|^2 + f(x,t)$$

where

$$L = -U\partial_x + \mu(x) + \gamma\partial_{xx},$$

$$\mu(x) = i\omega_0 + \mu_0 - \mu_2\frac{x^2}{2}.$$

Here $U, \gamma, \omega_0, \mu_0, \mu_2$ and c are positive real constants. The state $w(x,t)$ is a complex variable on $-\infty < x < +\infty$ such that $|w| \to 0$ as $x \to \pm\infty$. The function $f(x,t)$ is a given external (complex valued) forcing. In the following, $< w_1, w_2 >$ is the scalar-product:

$$< w_1, w_2 >= \int_{-\infty}^{+\infty} \overline{w_1(x)} w_2(x) dx$$

where $\overline{(\cdot)}$ represents the conjugate.

1/What do the different terms in the Ginzburg Landau equation represent?

2/ Linear dynamics.

In this section, we study the linear dynamics without external forcing. We therefore consider the equation:

$$\partial_t w = Lw.$$

a/ Show that

$$\phi(x) = n_\phi e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}} \text{ with } \chi = \left(\frac{\mu_2}{2\gamma}\right)^{\frac{1}{4}} \text{ and } n_\phi = \frac{\sqrt{\chi}}{\pi^{\frac{1}{4}} e^{\frac{1}{8}\frac{U^2}{\gamma^2\chi^2}}}$$

is an eigenvector of L. What is the eigenvalue associated to this eigenvector?

Note that the constant n_ϕ has been selected so that ϕ is unit norm:

$$< \phi, \phi >= 1.$$

b/ Show that the flow is unstable if the constant μ_0 is chosen such that:

$$\mu_0 > \mu_c,$$

where $\mu_c = \frac{U^2}{4\gamma} + \sqrt{\frac{\gamma\mu_2}{2}}$.

c/ Determine the operator $\tilde{L}$ adjoint to L. We will consider for this the scalar product $<\cdot,\cdot>$.

d/ Show that:

$$\psi(x) = n_\psi e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}} \text{ with } n_\psi = \frac{\sqrt{\chi}}{\pi^{\frac{1}{4}} e^{\frac{1}{8}\frac{U^2}{\gamma^2\chi^2}}},$$

is an eigenvector of $\tilde{L}$. What is the eigenvalue associated to this eigenvector?

Note that the normalization constant n_ψ has been chosen so that:

$$<\psi,\psi> = 1.$$

Can you qualitatively represent $\phi(x)$ and $\psi(x)$ on a same graph?

e/ We note that:

$$<\psi,\phi> = e^{-\frac{1}{4\sqrt{2}}\frac{U^2}{\gamma^{\frac{3}{2}}\mu_2^{\frac{1}{2}}}}$$

What does $<\psi,\phi>$ represent? What is the effect of the advection velocity U and viscosity γ on this coefficient? What is the value of $<\psi,\bar{\phi}>$?

3/ Amplitude equations.

We choose μ_0 in the vicinity of μ_c such that:

$$\mu_0 = \mu_c + \delta',$$

where $\delta' = \epsilon\delta$ with $0 < \epsilon \ll 1, \delta = O(1)$.

The operator L may therefore be written as $L = L_c + \epsilon\delta$, where L_c is the operator L obtained for $\delta' = 0$, that is $\mu_0 = \mu_c$.

a/ Show that $(-i\omega_0 I + L_c)\phi = 0$.

b/ We choose a forcing such that:

$$f(x,t) = E'\delta(x - x_f)e^{i\omega_f t}$$

where $E' = \epsilon^{\frac{3}{2}}E$, $E = O(1)$ is the forcing amplitude (positive real) and $\delta(x - x_f)$ is the dirac function at $x = x_f$ (we remind the reader that $\int_{-\infty}^{+\infty}\delta(x - x_f)w(x)dx = w(x_f)$ for any function w). The forcing frequency ω_f is chosen in the vicinity of the natural frequency ω_0 of the flow:

$$\omega_f = \omega_0 + \Omega'$$

where $\Omega' = \epsilon\Omega, \Omega = O(1)$.

We additionally consider that the operator L may be perturbed by an arbitrary perturbation operator $\Delta L' = \epsilon \Delta \mathrm{L}$ (which will be defined later). Hence, the full perturbed operator L reads:

$$L = L_c + \epsilon\delta + \epsilon\Delta L.$$

The solution is sought under the form:

$$w = \epsilon^{\frac{1}{2}} w_{\frac{1}{2}}(t,\tau) + \epsilon^{\frac{3}{2}} w_{\frac{3}{2}}(t,\tau) + \cdots$$

where $\tau = \epsilon t$ is a slow time-scale.

What is the equation governing $w_{\frac{1}{2}}$? What is the equation governing $w_{\frac{3}{2}}$?

c/ Show that $w_{\frac{1}{2}}(t,\tau) = A(\tau)e^{i\omega_0 t}\phi(x)$ is an acceptable solution for $w_{\frac{1}{2}}$.

Show that the solution $w_{\frac{3}{2}}(t,\tau)$ is bounded only if:

$$\frac{dA}{d\tau} = \delta A + A\frac{< \psi, \Delta L\phi >}{< \psi, \phi >} + c\frac{< \psi, \phi|\phi|^2 >}{< \psi, \phi >}A|A|^2 + \frac{\overline{\psi(x_f)}}{< \psi, \phi >}Ee^{\mathrm{i}\Omega\tau}.$$

d/ Considering $B'(t) = \epsilon^{\frac{1}{2}}A(\tau)e^{i\omega_0 t}$, show that the leading order solution of the problem is given by:

$$w(x,t) = B'(t)\phi(x)$$

where:

$$\frac{dB'}{dt} = (i\omega_0 + \delta')B' + \frac{< \psi, \Delta L'\phi >}{< \psi, \phi >}B' + c\frac{< \psi, \phi|\phi|^2 >}{< \psi, \phi >}B'|B'|^2 + \frac{\overline{\psi(x_f)}}{< \psi, \phi >}E'e^{\mathrm{i}\omega_f t}$$

4/ Frequency response.

In the case ($\delta' < 0, \Delta L' = 0$ and $c = 0$), show that the transfer function of the flow is:

$$\frac{\hat{B}'}{E'}(\omega_f) = \frac{\overline{\psi(x_f)}}{<\psi,\phi>} \frac{1}{i(\omega_f - \omega_0) - \delta'}$$

where:

$$B' = e^{i\omega_f t}\hat{B}'.$$

Represent qualitatively the magnitude of the transfer function $\left|\frac{\hat{B}'}{E'}\right|$ as a function of the forcing frequency ω_f.

Where should the forcing be located (x_f) to obtain the strongest response in magnitude? How does the magnitude of the response evolve as the advection velocity of the system U is increased?

5/ Open-loop control that modifies the stability characteristics of the flow $\mu(x)$

In the case ($E' = 0, c = 0$), we consider an open-loop control that achieves a modification of L such that:

$$\Delta L' = \delta\mu(x)$$

Show that the eigenvalue of the system (with control) verifies:

$$\lambda = i\omega_0 + \delta' + \frac{1}{<\psi,\phi>} \int_{-\infty}^{+\infty} \overline{\psi(x)}\ \delta\mu(x)\phi(x)dx$$

Note that $i\omega_0 + \delta'$ is the eigenvalue of the system without control, the last term therefore being the eigenvalue shift due to the open-loop control.

Where should the open-loop control modify $\mu(x)$ so as to achieve the strongest eigenvalue-shift?

6/ Closed-loop control

In the case ($E' = 0, c = 0$), we consider the following perturbation operator:

$$\Delta L' w = K\delta(x - x_a)w(x_s)$$

Can you comment this expression? In particular, what do x_a, x_s and K represent?

Show that the eigenvalue of the system (with closed-loop control) verifies:

$$\lambda = i\omega_0 + \delta' + \frac{K\overline{\psi}(x_a)\phi(x_s)}{<\psi,\phi>}$$

How should the actuator and sensor locations be chosen to maximize the eigenvalue-shift? How should K be chosen to render the closed-loop system marginally stable?

7/Open-loop control with harmonic forcing

a/ In the case $\Delta L' = 0$, show that the leading-order solution of the flowfield may be given by:

$$w = C'(t)e^{i\omega_f t}\phi(x)$$

where:

$$\frac{dC'}{dt} = (-i\Omega' + \delta')C' + c\frac{<\psi, \phi|\phi|^2>}{<\psi, \phi>}C'|C'|^2 + \frac{\overline{\psi(x_f)}}{<\psi, \phi>}E'$$

Hint: note that C' verifies $C' = \epsilon^{\frac{1}{2}}A(\tau)e^{-i\Omega\tau}$

b/ Numerical simulations of the equation governing C' show that there exists a threshold amplitude E_c', such that:

$$\text{If } E' > E_c' \text{ then } C' \to C_0' \text{ as } t \to \infty,$$

where C_0' is a complex constant.

What is the frequency of the flowfield in this case? Can you comment this result?

How should the forcing location x_f be chosen to minimize the threshold amplitude E_c'?