Exam

We consider the Kuramoto-Sivashinski equation:

$$\partial_t w + w\partial_x w = -\eta\partial_{xx} w - \mu\partial_{xxxx} w + f(x,t) \qquad (*)$$

where $w(x,t)$ and $f(x,t)$ are (periodic in space) real-valued functions such that $w(x+L,t) = w(x,t)$ and $f(x+L,t) = f(x,t)$. In the following, $k = 2\pi/L$ designates the wavenumber based on a given spatial period L. η and μ are positive real constants. In the following, $< w_1, w_2 >$ is the scalar-product:

$$< w_1, w_2 > = \frac{1}{L}\int_0^L \overline{w_1(x)} w_2(x) dx$$

where $\overline{(\cdot)}$ represents the conjugate of a complex number.

Note that question n°9 can be done independently of the questions 1 to 8.

1/ What do the different terms in the Kuramoto-Sivashinski equation represent? (1 point)

2/ In the case $f(x,t) = 0$, show that $w = U$ with U as a real constant is a fixed point of the governing equations. (0.5 point)

3/ In this section, we assume $f(x,t) = 0$ and study the linear dynamics around the fixed point $w = U$. We therefore consider $w(x,t) = U + w'(x,t)$ with the amplitude of $w'(x,t)$ being small.

a) Write the equation governing $w'(x,t)$ under the form $\partial_t w' = \mathcal{L}w'$. What are the eigenvalues λ_n and eigenvectors $\widehat{w}_n$ of $\mathcal{L}$? Note that both quantities (eigenvalues and eigenvectors) are complex. In particular, what is the eigenvalue related to the eigenvector $\widehat{w}(x) = e^{ikx}$? (2 points)
b) Show that the flow is marginally stable for $\eta = \eta_c = \mu k^2$. What is the frequency ω_c of this mode? Represent schematically the eigenvalue spectrum for η slightly above η_c. (2 points)

4/ Adjoint global mode

a) Using the scalar product $<\cdot,\cdot>$, determine the operator $\tilde{\mathcal{L}}$ adjoint to $\mathcal{L}$. (2 points)
b) Show that $\widetilde{w}(x) = e^{ikx}$ is an eigenvector of $\tilde{\mathcal{L}}$. What is the eigenvalue associated to this eigenvector? Note that the normalization constant has been chosen so that: $< \widetilde{w}, \widehat{w} > = 1$. (1 point)
c) Is the system non-normal? (0.5 point)

5/ Amplitude equations in the case of near-resonant forcing (9 points)

We choose η in the vicinity of η_c such that:

$$\eta = \eta_c + \delta',$$

where $\delta' = \epsilon\delta$ with $0 < \epsilon \ll 1$, $\delta = O(1)$. We choose a forcing such that:

$$f(x,t) = \left(E' f(x) e^{i\omega_f t} + \text{c.c.}\right)$$

where $E' = \epsilon^{\frac{3}{2}} E$, $E = O(1)$ is the forcing amplitude (positive real). The forcing frequency ω_f is chosen in the vicinity of the natural frequency ω_c of the flow:

$$\omega_f = \omega_c + \Omega'$$

where $\Omega' = \epsilon\Omega$, $\Omega = O(1)$.

The solution of the Kuramoto-Sivashinski equation is sought under the form:

$$w = U + \epsilon^{\frac{1}{2}} w_{\frac{1}{2}}(t,\tau) + \epsilon w_1(t,\tau) + \epsilon^{\frac{3}{2}} w_{\frac{3}{2}}(t,\tau) + \cdots$$

where $\tau = \epsilon t$ is a slow time-scale.

a) What are the equations governing $w_{\frac{1}{2}}$, w_1 and $w_{\frac{3}{2}}$? (3 points)

b) Show that $w_{\frac{1}{2}}(t,\tau) = \left(A(\tau) e^{i\omega_c t}\hat{w}(x) + \text{c.c.}\right)$ is an acceptable solution for $w_{\frac{1}{2}}$. (1 point)

c) Determine an exact solution for w_1. (3 points)

d) Show that the solution $w_{\frac{3}{2}}(t,\tau)$ is bounded only if:

$$\frac{dA}{d\tau} = \alpha\delta A - \beta A|A|^2 + \gamma E e^{i\Omega\tau}$$

with α, β and γ three complex constants. What are the analytical expressions of these constants?

Comment on the signs of α and β. Comment on γ. (3 points)

6) Considering $B'(t) = \epsilon^{\frac{1}{2}} A(\tau) e^{i\omega_c t}$, we assume (do not show this result) that the leading order solution of the problem may be rewritten as:

$$w(x,t) = U + \left(B'(t)\hat{w}(x) + \text{c.c.}\right)$$

where:

$$\frac{dB'}{dt} = (i\omega_c + \alpha\delta')B' - \beta B'|B'|^2 + \gamma E' e^{i\omega_f t}.$$

In the case $E' = 0$, represent schematically the bifurcation diagram ($|B'|$ as a function of δ'). In particular, provide all amplitudes $|B'|$ of various states as a function of δ'. What is the frequency of the flowfield in each state? (2 points)

7/Open-loop control with harmonic forcing

a) Show that the leading-order solution of the flowfield may be given by:

$$w = U + \left(C'(t)e^{i\omega_f t}\hat{w}(x) + \text{c.c.}\right)$$

where:

$$\frac{dC'}{dt} = (-i\Omega' + \alpha\delta')C' - \beta C'|C'|^2 + \gamma E'$$

Hint: consider the amplitude equation governing $B'(t)$ and note that C' verifies $C' = B'e^{-i\omega_f t}$. (1 point)

b) Numerical simulations of the equation governing C' show that there exists a threshold amplitude E'_c, such that:

If $E' > E'_c$ then $C' \to C'_0$ as $t \to \infty$,

where C'_0 is a complex constant.

What is the frequency of the flowfield in this case? Can you comment this result?

How should the forcing be chosen to minimize the threshold amplitude E'_c? (2 points)

8/ Amplitude equations in the case of non-resonant forcing. We now choose a forcing such that:

$$f(x,t) = \left(E' f(x) e^{i\omega_f t} + \text{c.c.}\right)$$

where the forcing frequency ω_f is chosen not close to the natural frequency ω_c of the flow.

Briefly (do not make any computations, look at the results in your course!) explain the changes to be made with respect to question 5: in particular, what scaling should be chosen for the forcing amplitude E' and what amplitude equation do you expect? (2 points)

9/ Optimal control

In the following, we consider the following space-time scalar-product (T is a given time):

$$\{w_1, w_2\} = \int_0^T \langle w_1, w_2 \rangle \, dt$$

We consider the objective functional:

$$\mathcal{J}'(f) = \mathcal{J}(w(f), f)$$

with $\mathcal{J}(w,f) = (1-\alpha)\{w(x,t), w(x,t)\} + \alpha\langle w(x,t=T), w(x,t=T)\rangle + l^2\{f(x,t), f(x,t)\}$.

Here $0 \leq \alpha \leq 1$ and l^2 are two tunable parameters and $w(f)$ represents the solution of equation (*), while $f(0 \leq x \leq L, 0 \leq t \leq T)$ is the spatio-temporal control function that appears in this equation. The initial condition of (*) is fixed as $w(x,t=0) = w_I$.

a) What do the two parameters α and l^2 represent? (1 point)

b) We consider the Lagrangian:

$$\mathcal{L}(w, \widetilde{w}, f) = \mathcal{J}(w,f) - \{\widetilde{w}, \partial_t w + w\partial_x w + \eta\partial_{xx} w + \mu\partial_{xxxx} w - f\}$$

i) Determine $\frac{\partial \mathcal{L}}{\partial \widetilde{w}}$ such that: $\lim_{\epsilon\to 0} \frac{\left(\mathcal{L}(w,\widetilde{w}+\epsilon\delta\widetilde{w},f) - \mathcal{L}(w,\widetilde{w},f)\right)}{\epsilon} = \left\{\frac{\partial \mathcal{L}}{\partial \widetilde{w}}, \delta\widetilde{w}\right\}$ (1 point)

ii) Determine $\frac{\partial \mathcal{L}}{\partial w}$ such that: $\lim_{\epsilon\to 0} \frac{\left(\mathcal{L}(w+\epsilon\delta w,\widetilde{w},f) - \mathcal{L}(w,\widetilde{w},f)\right)}{\epsilon} = \left\{\frac{\partial \mathcal{L}}{\partial w}, \delta w\right\}$ (4 points)

iii) Determine $\frac{\partial \mathcal{L}}{\partial f}$ such that: $\lim_{\epsilon\to 0} \frac{\left(\mathcal{L}(w,\widetilde{w},f+\epsilon\delta f) - \mathcal{L}(w,\widetilde{w},f)\right)}{\epsilon} = \left\{\frac{\partial \mathcal{L}}{\partial f}, \delta f\right\}$ (2 points)

c) Determine the equations to be solved to obtain $\mathrm{d}\mathcal{J}'/df$. (1 point)

d) Explain how you could use $\mathrm{d}\mathcal{J}'/df$ in a closed-loop framework if you assume that you know w at all positions and all times (you are God!). (1 point)