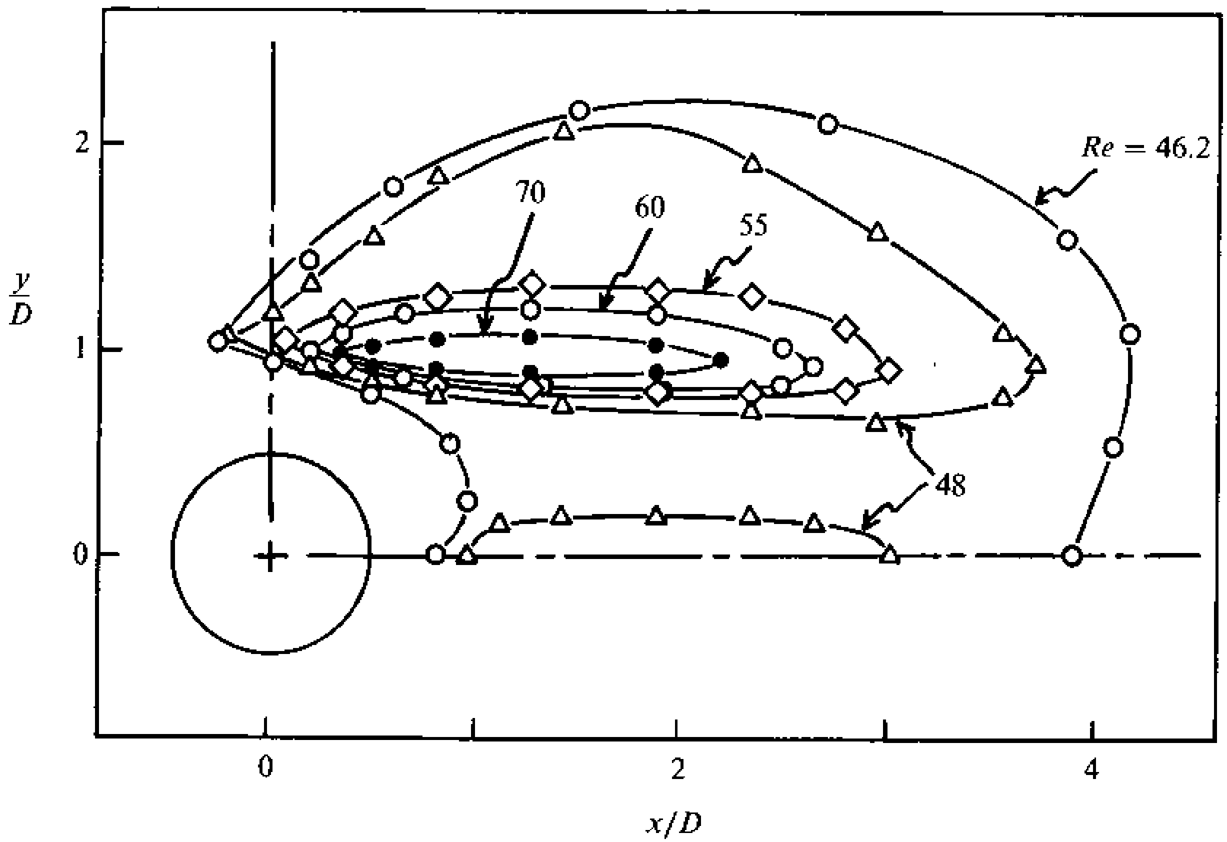
Gradients



Strykowski & Sreenivasan JFM 1990

Outline

- Flow stabilization with global mode control
- Gradient-based optimization
- Gradient with Lagrangian method
  - General result
  - Application to simple examples
- Sensitivity of eigenvalue to base-flow modifications
  - General result
  - Application to cylinder flow
- Sensitivity of eigenvalue to steady forcing
  - General result
  - Application to cylinder flow

Eigenvalue sensitivity

$$\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w = f$$

Incompressible Navier-Stokes equations for:

$$w = \begin{pmatrix} u \\ p \end{pmatrix}, f = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{N}(w_1, w_2) = \begin{pmatrix} u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\ 0 \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} -\nu\Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}$$

Base-flow

$w(t) = w_0 + \epsilon w_1(t) + \cdots$

Order ϵ^0:

$$\frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = f$$

Non-linear equilibrium point:

Global mode

Order ϵ:

$$\mathcal{B}\partial_t w_1 + \mathcal{N}_{w_0} w_1 + \mathcal{L} w_1 = 0$$

We look for w_1 under the form :

$$w_1 = e^{\lambda t}\widehat{w} + \text{c.c}$$

This leads to the following eigenproblem:

$$\lambda\mathcal{B}\widehat{w} + \mathcal{N}_{w_0}\widehat{w} + \mathcal{L}\widehat{w} = 0$$

Open-loop control problem

Let us consider a situation where there is one unstable global mode: for example, cylinder flow at $Re = 100$.
Without control: the base-flow w_0 and the global mode $\hat{w}$ are determined by:

$$\begin{cases} \frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = 0 \\ \lambda\mathcal{B}\hat{w} + \mathcal{N}_{w_0}\hat{w} + \mathcal{L}\hat{w} = 0 \end{cases}$$

Here $\sigma = \mathrm{Re}(\lambda) > 0$. We would like to stabilize this flow ($\sigma < 0$).

With steady forcing f (think of a control cylinder):

$$\begin{cases} \frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = f \\ \lambda\mathcal{B}\hat{w} + \mathcal{N}_{w_0}\hat{w} + \mathcal{L}\hat{w} = 0 \end{cases}$$

$$\Rightarrow \lambda = \lambda(w_0) = \lambda(w_0(f)) = \lambda(f)$$

Control problem: find smallest f which achieves stabilization: $\sigma(f) < 0$.

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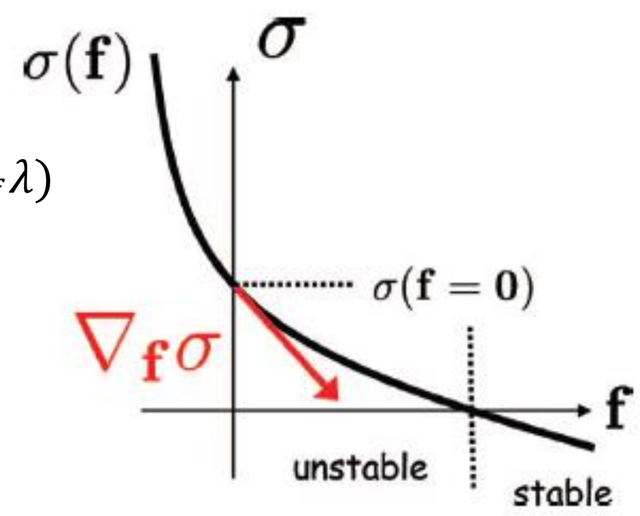
Gradient-based method

- First order Taylor expansion:

$$\delta\lambda = \frac{d\lambda}{df}\delta f = < \nabla_f \lambda, \delta f >$$

Amplification rate: $\delta\sigma = < \nabla_f \sigma, \delta f >$ with $\nabla_f \sigma = \mathrm{Re}(\nabla_f \lambda)$



- Steepest ascent: $\delta f = \epsilon \nabla_f \sigma \Rightarrow \delta\sigma = \epsilon \left\| \nabla_f \sigma \right\|^2$

- Steepest descent: $\delta f = -\epsilon \nabla_f \sigma \Rightarrow \delta\sigma = -\epsilon \left\| \nabla_f \sigma \right\|^2$

- Iterative technique (idea similar to Newton method):

=> $\epsilon > 0$ chosen so that $\delta\sigma = -\sigma$: $-\sigma = -\epsilon \left\| \nabla_f \sigma \right\|^2 \Rightarrow \epsilon = \frac{\sigma}{\left\| \nabla_f \sigma \right\|^2}$

=> Steady forcing update: $\delta f = -\frac{\sigma}{\left\| \nabla_f \sigma \right\|^2} \nabla_f \sigma$

Computation of gradient

How to compute $\nabla_f \lambda$?

- Finite differences: $< \nabla_f \lambda, \delta f > = \lim_{\epsilon \to 0} \frac{\lambda(f+\epsilon\delta f)-\lambda(f)}{\epsilon}$. To fully determine $\nabla_f \lambda$, evaluate derivative for all degrees of freedom of f. Method only tractable if f displays a small number of dofs.

- When f displays a large number of dofs => Lagrangian formulation

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Example

State : $w = w(x)$ Control : $f = f(x)$ $\langle a, b\rangle = \int_0^1 ab dx$

$$F(w,f) = w\partial_x w - \alpha w - \nu\partial_{xx} w - f, G(w) = \{w(0) - 1, \partial_x w(1) - 0\}$$

$$F(w + \epsilon\delta w, f + \epsilon\delta f) = F(w,f) + \epsilon \underbrace{(\delta w\partial_x w + w\partial_x \delta w - \alpha\delta w - \nu\partial_{xx}\delta w)}_{\left.\frac{\partial F}{\partial w}\right|_{(w,f)} \delta w} + \epsilon \underbrace{(-\delta f)}_{\left.\frac{\partial F}{\partial f}\right|_{(w,f)} \delta f}$$

$$\mathfrak{I}(w,f) = \int_0^1 ((w - w_0)^2 + l^2 f^2) dx$$

$$\mathfrak{I}(w + \epsilon\delta w, f + \epsilon\delta f) = \mathfrak{I}(w,f) + \epsilon \underbrace{\int_0^1 2(w - w_0)\delta w dx}_{<\left.\frac{\partial \mathfrak{I}}{\partial w}\right|_{(w,f)}, \delta w>} + \epsilon \underbrace{\int_0^1 2l^2 f\delta f dx}_{<\left.\frac{\partial \mathfrak{I}}{\partial f}\right|_{(w,f)}, \delta f>}$$

Lagrangian formulation: general form

Theorem: Let us introduce the following function (called the Langrangian):

$$\mathcal{L}(w, f, \widetilde{w}) = \mathfrak{J}(w, f) - \langle \widetilde{w}, F(w, f) \rangle$$

with $\widetilde{w}$ being a Lagrange multiplier or adjoint state to be defined. Here $(w, f, \widetilde{w})$ are considered as independent variables for $\mathcal{L}$.

Then, denoting $\widetilde{(\cdot)}$ the adjoint of a linear operator with respect to $<\cdot,\cdot>$:

$$\frac{\partial \mathcal{L}}{\partial \widetilde{w}} = -F(w, f)$$

$$\frac{\partial \mathcal{L}}{\partial w} = \frac{\partial \mathfrak{J}}{\partial w} - \widetilde{\left(\frac{\partial F}{\partial w}\right)} \widetilde{w}$$

$$\frac{\partial \mathcal{L}}{\partial f} = \frac{\partial \mathfrak{J}}{\partial f} - \widetilde{\left(\frac{\partial F}{\partial f}\right)} \widetilde{w}$$

Lagrangian formulation: general form

Theorem (continued):

If $(w, f, \widetilde{w})$ are such that $\frac{\partial \mathcal{L}}{\partial \widetilde{w}} = 0$ and $\frac{\partial \mathcal{L}}{\partial w} = 0$,

then:

$$F(w, f) = 0$$

$$\widetilde{\left(\frac{\partial F}{\partial w}\right)} \widetilde{w} = \frac{\partial \mathfrak{J}}{\partial w}$$

$$\frac{d\mathfrak{J}}{df} = \frac{\partial \mathcal{L}}{\partial f} = \frac{\partial \mathfrak{J}}{\partial f} - \widetilde{\left(\frac{\partial F}{\partial f}\right)} \widetilde{w}$$

The first equation means that w and f satisfy the governing equation.

The second equation defines the adjoint state $\widetilde{w}$ as a function of w and f.

The last equation determines the gradient of the objective functional as a function of $\widetilde{w}$.

Lagrangian formulation: general form

$$\mathcal{L}(w+\varepsilon\delta w, f+\epsilon\delta f, \widetilde{w}+\varepsilon\delta\widetilde{w}) = \mathcal{L}(w,f,\widetilde{w}) + \epsilon\left[\left\langle\frac{\partial\mathcal{L}}{\partial w},\delta w\right\rangle + \left\langle\frac{\partial\mathcal{L}}{\partial f},\delta f\right\rangle + \left\langle\frac{\partial\mathcal{L}}{\partial\widetilde{w}},\delta\widetilde{w}\right\rangle\right]$$

$$\left\langle\frac{\partial\mathcal{L}}{\partial w},\delta w\right\rangle = \lim_{\varepsilon\to 0}\frac{\mathcal{L}(w+\varepsilon\delta w, f,\widetilde{w}) - \mathcal{L}(w,f,\widetilde{w})}{\varepsilon}$$

$$\left\langle\frac{\partial\mathcal{L}}{\partial u},\delta f\right\rangle = \lim_{\varepsilon\to 0}\frac{\mathcal{L}(w, f+\epsilon\delta f\ ,\widetilde{w}) - \mathcal{L}(w,f,\widetilde{w})}{\varepsilon}$$

$$\left\langle\frac{\partial\mathcal{L}}{\partial\widetilde{w}},\delta\widetilde{w}\right\rangle = \lim_{\varepsilon\to 0}\frac{\mathcal{L}(w,u,\widetilde{w}+\varepsilon\delta\widetilde{w}) - \mathcal{L}(w,f,\widetilde{w})}{\varepsilon}$$

Lagrangian formulation: general form

Lagrangian:

$$\mathcal{L}(w,f,\widetilde{w}) = \Im(w,f) - \langle \widetilde{w}, F(w,f) \rangle$$

Variation with respect to adjoint state :

$$\lim_{\varepsilon \to 0} \frac{\mathcal{L}(w,f,\widetilde{w}+\varepsilon\delta\widetilde{w}) - \mathcal{L}(w,f,\widetilde{w})}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\Im(w,f) - \langle \widetilde{w}+\varepsilon\delta\widetilde{w}, F(w,f) \rangle - \Im(w,f) + \langle \widetilde{w}, F(w,f) \rangle}{\varepsilon}$$

$$= \langle \delta\widetilde{w}, -F(w,f) \rangle$$

Hence : $\dfrac{\partial \mathcal{L}}{\partial \widetilde{w}} = -F(w,f)$

Lagrangian formulation: general form

Lagrangian:

$$\mathcal{L}(w,f,\widetilde{w}) = \Im(w,f) - \langle \widetilde{w}, F(w,f) \rangle$$

Variation with respect to state :

$$\lim_{\varepsilon \to 0} \frac{\mathcal{L}(w+\varepsilon\delta w,f,\widetilde{w}) - \mathcal{L}(w,f,\widetilde{w})}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\Im(w+\varepsilon\delta w,f) - \langle \widetilde{w}, F(w+\varepsilon\delta w,f) \rangle - \Im(w,f) + \langle \widetilde{w}, F(w,f) \rangle}{\varepsilon}$$

$$= \left\langle \frac{\partial \Im}{\partial w}, \delta w \right\rangle - \left\langle \widetilde{w}, \frac{\partial F}{\partial w}\delta w \right\rangle = \left\langle \frac{\partial \Im}{\partial w}, \delta w \right\rangle - \left\langle \widetilde{\left(\frac{\partial F}{\partial w}\right)} \widetilde{w}, \delta w \right\rangle$$

$$= \left\langle \frac{\partial \Im}{\partial w} - \widetilde{\left(\frac{\partial F}{\partial w}\right)} \widetilde{w}, \delta w \right\rangle$$

Hence: $\frac{\partial \mathcal{L}}{\partial w} = \frac{\partial \Im}{\partial w} - \widetilde{\left(\frac{\partial F}{\partial w}\right)} \widetilde{w}$

Lagrangian formulation: general form

Lagrangian:

$$\mathcal{L}(w,f,\widetilde{w}) = \mathfrak{I}(w,f) - \langle \widetilde{w}, F(w,f)\rangle$$

Variation with respect to control :

$$\lim_{\varepsilon\to 0}\frac{\mathcal{L}(w,f+\varepsilon\delta f,\widetilde{w}) - \mathcal{L}(w,f,\widetilde{w})}{\varepsilon}$$
$$= \lim_{\varepsilon\to 0}\frac{\mathfrak{I}(w,f+\varepsilon\delta f) - \langle \widetilde{w}, F(w,f+\varepsilon\delta f)\rangle - \mathfrak{I}(w,f) + \langle \widetilde{w}, F(w,f)\rangle}{\varepsilon}$$
$$= \left\langle \frac{\partial \mathfrak{I}}{\partial f}, \delta f\right\rangle - \left\langle \widetilde{w}, \frac{\partial F}{\partial f}\delta f\right\rangle = \left\langle \frac{\partial \mathfrak{I}}{\partial f}, \delta f\right\rangle - \left\langle \widetilde{\left(\frac{\partial F}{\partial f}\right)}\widetilde{w}, \delta f\right\rangle$$
$$= \left\langle \frac{\partial \mathfrak{I}}{\partial f} - \widetilde{\left(\frac{\partial F}{\partial f}\right)}\widetilde{w}, \delta f\right\rangle$$

Hence: $\frac{\partial \mathcal{L}}{\partial f} = \frac{\partial \mathfrak{I}}{\partial f} - \widetilde{\left(\frac{\partial F}{\partial f}\right)}\widetilde{w}$

Lagrangian formulation: general form

$$\mathfrak{J}(f) = \mathfrak{J}(w(f), f) = \mathcal{L}\big(w(f), f, \widetilde{w}(f)\big) + \langle \widetilde{w}(f), F(w(f), f) \rangle$$

with $w(f)$ and $\widetilde{w}(f)$ defined from $\frac{\partial \mathcal{L}}{\partial w} = \frac{\partial \mathcal{L}}{\partial \widetilde{w}} = 0.$

$$\frac{d\mathfrak{J}(f)}{df} = \frac{d\mathfrak{J}(w(f), f)}{df}$$

$$= \left\langle \overbrace{\frac{\partial \mathcal{L}}{\partial w}}^{0}, \frac{dw}{df} \right\rangle + \frac{\partial \mathcal{L}}{\partial f} + \left\langle \overbrace{\frac{\partial \mathcal{L}}{\partial \widetilde{w}}}^{0}, \frac{d\widetilde{w}}{df} \right\rangle + \left\langle \frac{d\widetilde{w}}{df}, \underbrace{F(w(f), f)}_{0} \right\rangle + \left\langle \widetilde{w}(f), \underbrace{\frac{\partial F}{\partial w}\frac{dw}{df} + \frac{\partial F}{\partial f}}_{0} \right\rangle = \frac{\partial \mathcal{L}}{\partial f}$$

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$$F(w + \epsilon\delta w, f + \epsilon\delta f) = F(w,f) + \epsilon \underbrace{(\delta w\partial_x w + w\partial_x\delta w - \alpha\delta w - \nu\partial_{xx}\delta w)}_{\left.\frac{\partial F}{\partial w}\right|_{(w,f)}\delta w} + \epsilon \underbrace{(-\delta f)}_{\left.\frac{\partial F}{\partial f}\right|_{(w,f)}\delta f}$$

$$\mathfrak{I}(w,f) = \int_0^1 ((w - w_0)^2 + l^2 f^2) dx$$

$$\mathfrak{I}(w + \epsilon\delta w, f + \epsilon\delta f) = \mathfrak{I}(w,f) + \epsilon \underbrace{\int_0^1 2(w - w_0)\delta w dx}_{<\left.\frac{\partial \mathfrak{I}}{\partial w}\right|_{(w,f)},\delta w>} + \epsilon \underbrace{\int_0^1 2l^2 f\delta f dx}_{<\left.\frac{\partial \mathfrak{I}}{\partial f}\right|_{(w,f)},\delta f>}$$

Example

Lagrangian:

$$\mathcal{L}(w, f, \tilde{w}) = \mathfrak{J}(w, f) - \langle \tilde{w}, F(w, f) \rangle$$

Variation with respect to state :

$$< \frac{\partial \mathcal{L}}{\partial w}, \delta w >= \left\langle \frac{\partial \mathfrak{J}}{\partial w}, \delta w \right\rangle - \left\langle \tilde{w}, \frac{\partial F}{\partial w} \delta w \right\rangle$$

$$= \int_0^1 [2(w - w_0) - \tilde{w}\partial_x w + \alpha \tilde{w}] \delta w dx \underbrace{- \int_0^1 \tilde{w}(w \partial_x \delta w - \nu \partial_{xx} \delta w) dx}_{(*)}$$

Example

$$(*) = -[\widetilde{w}w\delta w - \nu\widetilde{w}\partial_x\delta w]_0^1 + \int_0^1 (\partial_x(w\widetilde{w})\delta w - \nu\partial_x\widetilde{w}\partial_x\delta w)dx$$

$$=-[\widetilde{w}w\delta w - \nu\widetilde{w}\partial_x\delta w]_0^1 + \int_0^1 \partial_x(w\widetilde{w})\delta w dx \; - [\nu\partial_x\widetilde{w}\delta w]_0^1 + \int_0^1 \nu\partial_{xx}\widetilde{w}\delta w dx$$

To kill boundary integral: $[\widetilde{w}w\delta w - \nu\widetilde{w}\partial_x\delta w + \nu\partial_x\widetilde{w}\delta w]_0^1$=0

$$w(0) = 1, \delta w(0) = 0, \partial_x\delta w \neq 0 \Rightarrow \widetilde{w}(0) = 0$$

$$\partial_x w(1) = 0, \partial_x\delta w(1) = 0, \delta w(1) \neq 0 \Rightarrow \nu\partial_x\widetilde{w}(1) + \widetilde{w}(1)w(1) = 0$$

Hence:

$$\frac{\partial\mathcal{L}}{\partial w} = 2(w - w_0) + w\partial_x\widetilde{w} + \alpha\widetilde{w} + \nu\partial_{xx}\widetilde{w}$$

$$\widetilde{w}(0) = 0$$

$$\nu\partial_x\widetilde{w}(1) + \widetilde{w}(1)w(1) = 0$$

Example

Lagrangian:

$$\mathcal{L}(w, f, \widetilde{w}) = \mathfrak{J}(w, f) - \langle \widetilde{w}, F(w, f) \rangle$$

Variation with respect to control:

$$< \frac{\partial \mathcal{L}}{\partial f}, \delta f >= \left\langle \frac{\partial \mathfrak{J}}{\partial f}, \delta f \right\rangle - \left\langle \widetilde{w}, \frac{\partial F}{\partial f} \delta f \right\rangle = \int_0^1 (2l^2 f + \widetilde{w})\, \delta f dx$$

Hence: $\frac{\partial \mathcal{L}}{\partial f} = 2l^2 f + \widetilde{w}$

Example

Conclusion:

$$\frac{\partial \mathcal{L}}{\partial \widetilde{w}} = 0 \Rightarrow w\partial_x w - \alpha w - \nu \partial_{xx} w - f = 0, w(0) = 1, \partial_x w(1) = 0$$

$$\frac{\partial \mathcal{L}}{\partial w} = 0 \Rightarrow \begin{cases} -w\partial_x \widetilde{w} - \alpha \widetilde{w} - \nu \partial_{xx} \widetilde{w} = 2(w - w_0) \\ \widetilde{w}(0) = 0 \\ -\nu \partial_x \widetilde{w}(1) - \widetilde{w}(1) w(1) = 0 \end{cases}$$

Gradient:

$$\frac{d\mathfrak{I}}{df} = \frac{\partial \mathcal{L}}{\partial f} = 2l^2 f + \widetilde{w}$$

The Ginzburg-Landau eq. (cont'd)

7/ Open-loop control that modifies the stability characteristics of the flow $\mu(x)$.

We consider an open-loop control that achieves a modification of $\mu(x)$. The eigenvalue λ of the most unstable global mode is a function of $\mu(x)$.

Compute $\nabla_\mu \lambda(x)$, such that $\delta\lambda = \left\langle \nabla_\mu \lambda, \delta\mu \right\rangle = \int_{-\infty}^{+\infty} \overline{\nabla_\mu \lambda(x)}\ \delta\mu(x) dx$

Where should the open-loop control modify $\mu(x)$ so as to achieve the strongest eigenvalue-shift?

The Ginzburg-Landau eq. (cont'd)

State:

$$[\hat{w}, \lambda]$$

Control:

$$\mu$$

Constraint:

$$\lambda \hat{w} + \mathcal{L} \hat{w} = 0$$

Objective:

$$\lambda(\mu)$$

Variation: $\delta\lambda = \langle \nabla_\mu \lambda, \delta\mu \rangle$. $\nabla_\mu \lambda$?

The Ginzburg-Landau eq. (cont'd)

Lagrangian:

$$\mathcal{L}([\hat{w},\lambda],[\tilde{w}],[\mu]) = \lambda - \langle \tilde{w}, \lambda\hat{w} + \mathcal{L}\hat{w}\rangle$$

Variation with respect to the state => definition of adjoint:

$$\lim_{\varepsilon\to 0}\frac{\mathcal{L}([\hat{w}+\epsilon\delta\hat{w},\lambda+\epsilon\delta\lambda],[\tilde{w}],[\mu]) - \mathcal{L}([\hat{w},\lambda],[\tilde{w}],[\mu])}{\varepsilon}$$

$$= \delta\lambda - \langle \tilde{w}, \delta\lambda\hat{w} + \lambda\delta\hat{w} + \mathcal{L}\delta\hat{w}\rangle$$

$$= \delta\lambda(1 - \langle \tilde{w},\hat{w}\rangle) - \langle \tilde{w}, \lambda\delta\hat{w} + \mathcal{L}\delta\hat{w}\rangle$$

$$= \delta\lambda(1 - \langle \tilde{w},\hat{w}\rangle) - \left\langle \lambda^*\tilde{w} + \tilde{\mathcal{L}}\tilde{w}, \delta\hat{w}\right\rangle$$

$$= \left\langle \frac{\partial\mathcal{L}}{\partial[\hat{w},\lambda]}, [\delta\hat{w},\delta\lambda]\right\rangle$$

$$\frac{\partial\mathcal{L}}{\partial[\hat{w},\lambda]} = \left[\lambda^*\tilde{w} + \tilde{\mathcal{L}}\tilde{w}, 1 - \langle \tilde{w},\hat{w}\rangle\right]$$

The Ginzburg-Landau eq. (cont'd)

Lagrangian:

$$\mathcal{L}([\hat{w},\lambda],[\tilde{w}],[\mu]) = \lambda - \langle \tilde{w}, \lambda\hat{w} + L\hat{w} \rangle$$

Variation with respect to control:

$$\lim_{\varepsilon\to 0} \frac{\mathcal{L}([\hat{w},\lambda],[\tilde{w}],[\mu+\epsilon\delta\mu]) - \mathcal{L}([\hat{w},\lambda],[\tilde{w}],[\mu])}{\varepsilon}$$

$$= -\langle \tilde{w}, -\delta\mu\hat{w} \rangle = \langle \hat{w}^*\tilde{w}, \delta\mu \rangle$$

$$= \left\langle \frac{\partial \mathcal{L}}{\partial \mu}, \delta\mu \right\rangle$$

So that:

$$\frac{\partial \mathcal{L}}{\partial \mu} = \hat{w}^*\tilde{w}$$

The Ginzburg-Landau eq. (cont'd)

Conclusion:
The gradient of $\lambda(\mu)$ is given by

$$\nabla_\mu \lambda = \widehat{w}^* \widetilde{w}$$

where:

$$\lambda^* \widetilde{w} + \tilde{\mathcal{L}} \widetilde{w} = 0$$
$$1 - \langle \widetilde{w}, \widehat{w} \rangle = 0$$

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Sensitivity to base-flow modifications

State:

$$[\widehat{w}, \lambda]$$

Control:

$$w_0$$

Constraint:

$$\lambda \mathcal{B} \widehat{w} + \mathcal{N}_{w_0} \widehat{w} + \mathcal{L} \widehat{w} = 0$$

Objective:

$$\lambda(w_0)$$

Scalar-product $<>$ for definition of gradient $\delta\lambda = < \nabla_{w_0} \lambda, \delta w_0 >$:

$$< w_1, w_2 > = \iint (u_1^* u_2 + v_1^* v_2 + p_1^* p_2) dx dy$$

Sensitivity to base-flow modifications

Lagrangian:

$$\mathcal{L}([\hat{w},\lambda],[\tilde{w}],[w_0]) = \lambda - \left\langle \tilde{w}, \lambda\mathcal{B}\hat{w} + \mathcal{N}_{w_0}\hat{w} + \mathcal{L}\hat{w} \right\rangle$$

Scalar-product for state:

$$< [w_1,\lambda_1],[w_2,\lambda_2] > = < w_1,w_2 > + \lambda_1^* \lambda_2$$

Scalar-product for adjoint-state:

$$< w_1, w_2 >$$

Scalar-product for control:

$$< w_1, w_2 >$$

Sensitivity to base-flow modifications

Lagrangian:

$$\mathcal{L}([\hat{w},\lambda],[\tilde{w}],[w_0]) = \lambda - \left\langle \tilde{w}, \lambda\mathcal{B}\hat{w} + \mathcal{N}_{w_0}\hat{w} + \mathcal{L}\hat{w} \right\rangle$$

Variation with respect to the state => definition of adjoint:

$$\lim_{\varepsilon\to 0} \frac{\mathcal{L}([\hat{w}+\epsilon\delta\hat{w},\lambda+\epsilon\delta\lambda],[\tilde{w}],[w_0]) - \mathcal{L}([\hat{w},\lambda],[\tilde{w}],[w_0])}{\varepsilon}$$

$$= \delta\lambda - < \tilde{w}, \delta\lambda\mathcal{B}\hat{w} + \lambda\mathcal{B}\delta\hat{w} + \mathcal{N}_{w_0}\delta\hat{w} + \mathcal{L}\delta\hat{w} >$$

$$= \delta\lambda(1 - < \tilde{w}, \mathcal{B}\hat{w} >) - < \tilde{w}, \lambda\mathcal{B}\delta\hat{w} + \mathcal{N}_{w_0}\delta\hat{w} + \mathcal{L}\delta\hat{w} >$$

$$= \delta\lambda(1 - < \tilde{w}, \mathcal{B}\hat{w} >) - < \lambda^*\mathcal{B}\tilde{w} + \widetilde{\mathcal{N}}_{w_0}\tilde{w} + \mathcal{L}\tilde{w}, \delta\hat{w} >$$

$$= < \frac{\partial\mathcal{L}}{\partial[\hat{w},\lambda]}, [\delta\hat{w},\delta\lambda] >$$

$$\frac{\partial\mathcal{L}}{\partial[\hat{w},\lambda]} = \left[\lambda^*\mathcal{B}\tilde{w} + \widetilde{\mathcal{N}}_{w_0}\tilde{w} + \tilde{L}\tilde{w}, 1 - < \tilde{w}, \mathcal{B}\hat{w} >\right]$$

Sensitivity to base-flow modifications

Lagrangian:

$$\mathcal{L}([\hat{w},\lambda],[\tilde{w}],[w_0]) = \lambda - \left\langle \tilde{w}, \lambda\mathcal{B}\hat{w} + \mathcal{N}_{w_0}\hat{w} + \mathcal{L}\hat{w} \right\rangle$$

Variation with respect to control:

$$\lim_{\varepsilon\to 0} \frac{\mathcal{L}([\hat{w},\lambda],[\tilde{w}],[w_0+\epsilon\delta w_0]) - \mathcal{L}([\hat{w},\lambda],[\tilde{w}],[w_0])}{\varepsilon}$$

$$= -\langle \tilde{w}, \mathcal{N}(\delta w_0, \hat{w}) \rangle = -\langle \tilde{w}, \mathcal{N}_{\hat{w}}\delta w_0 \rangle = -\left\langle \widetilde{\mathcal{N}}_{\hat{w}}\tilde{w}, \delta w_0 \right\rangle$$

$$= < \frac{\partial \mathcal{L}}{\partial w_0}, \delta w_0 >$$

So that:

$$\frac{\partial \mathcal{L}}{\partial w_0} = -\widetilde{\mathcal{N}}_{\hat{w}}\tilde{w}$$

Sensitivity to base-flow modifications

Conclusion:
The gradient of $\lambda(w_0)$ is given by

$$\nabla_{w_0}\lambda = -\widetilde{\mathcal{N}}_{\widehat{w}}\widetilde{w}$$

where:

$$\lambda^* \mathcal{B}\widetilde{w} + \widetilde{\mathcal{N}}_{w_0}\widetilde{w} + \tilde{L}\widetilde{w} = 0$$
$$1 - < \widetilde{w}, \mathcal{B}\widehat{w} > = 0$$

Sensitivity to base-flow modifications

Let $(\lambda, \widehat{w})$ be an eigenvalue/eigenvector :

$$\lambda \mathcal{B} \widehat{w} + \mathcal{N}_{w_0} \widehat{w} + \mathcal{L} \widehat{w} = 0$$

λ is a function of w_0. The gradient of the function $\lambda(w_0)$ such that $\delta\lambda =< \nabla_{w_0}\lambda, \delta w_0 >$ is given by:

$$\nabla_{w_0}\lambda = -\widetilde{\mathcal{N}}_{\widehat{w}} \widetilde{w}$$

with

$$\lambda^* \mathcal{B} \widetilde{w} + \widetilde{\mathcal{N}}_{w_0} \widetilde{w} + \tilde{L} \widetilde{w} = 0$$

and the normalization condition:

$$< \widetilde{w}, \mathcal{B} \widehat{w} > = 1$$

Outline

- Flow stabilization with eigenvalue mode control
- Gradient-based optimization
- Gradient with Lagrangian method
  - General result
  - Application to simple examples
- Sensitivity of eigenvalue to base-flow modifications
  - General result
  - Application to cylinder flow
- Sensitivity of eigenvalue to steady forcing
  - General result
  - Application to cylinder flow

Sensitivity to base-flow modifications

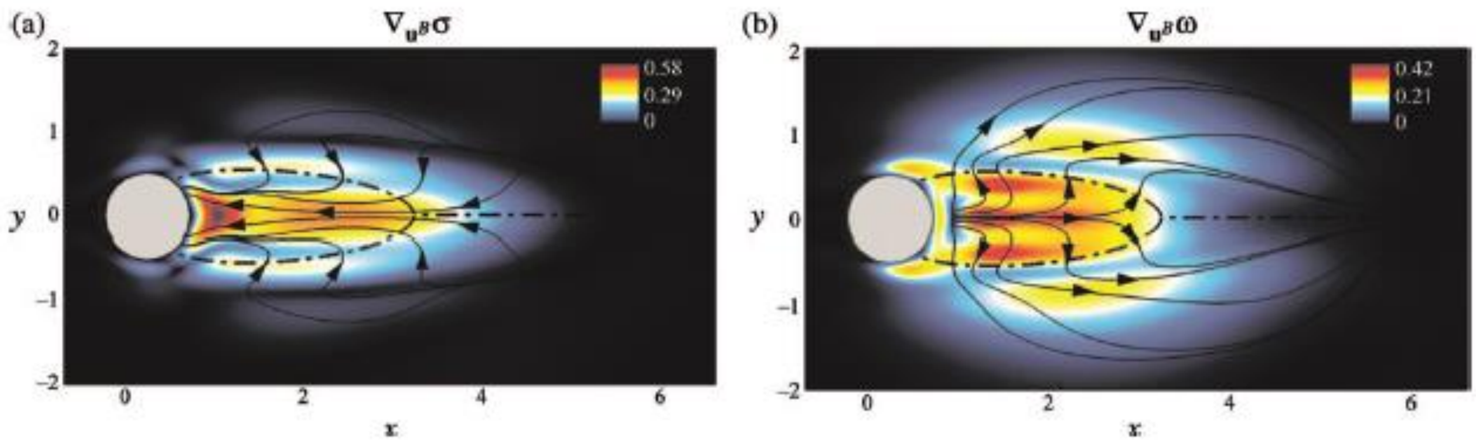

Fig. 10 Flow around a cylinder at Re=47 and sensitivities associated with a modification of the base-flow. (*a*) Sensitivity of the amplification rate. (*b*) Sensitivity of the frequency. Adapted from Ref. [135].

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Outline

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Sensitivity to steady forcing

State:

$$[w_0, \widehat{w}, \lambda]$$

Control:

$$f$$

Constraints:

$$\begin{cases} \frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = f \\ \lambda\mathcal{B}\widehat{w} + \mathcal{N}_{w_0}\widehat{w} + \mathcal{L}\widehat{w} = 0 \end{cases}$$

Objective:

$$\lambda(f) = \lambda(w_0(f))$$

Scalar-product for definition of gradient $\delta\lambda = < \nabla_f \lambda, \delta f >$:

$$< w_1, w_2 > = \iint (u_1^* u_2 + v_1^* v_2 + p_1^* p_2) dxdy$$

Sensitivity to steady forcing

Lagrangian:

$$\mathcal{L}([w_0,\widehat{w},\lambda],[\widetilde{w}_0,\widetilde{w}],[f])$$
$$=\lambda-\left\langle[\widetilde{w}_0,\widetilde{w}],\left[\frac{1}{2}\mathcal{N}(w_0,w_0)+\mathcal{L}w_0-f,\lambda\mathcal{B}\widehat{w}+\mathcal{N}_{w_0}\widehat{w}+\mathcal{L}\widehat{w}\right]\right\rangle$$
$$=\lambda-\left\langle\widetilde{w}_0,\frac{1}{2}\mathcal{N}(w_0,w_0)+\mathcal{L}w_0-f\right\rangle-\left\langle\widetilde{w},\lambda\mathcal{B}\widehat{w}+\mathcal{N}_{w_0}\widehat{w}+\mathcal{L}\widehat{w}\right\rangle$$

Scalar product for state:

$$<[w_1,\widetilde{w}_1,\lambda_1],[w_2,\widetilde{w}_2,\lambda_2]>=<w_1,w_2>+<\widetilde{w}_1,\widetilde{w}_2>+\lambda_1^*\lambda_2$$

Scalar product for adjoint state:

$$<[w_1,\widetilde{w}_1],[w_2,\widetilde{w}_2]>=<w_1,w_2>+<\widetilde{w}_1,\widetilde{w}_2>$$

Sensitivity to steady forcing

Lagrangian:

$$\mathcal{L}([w_0,\hat{w},\lambda],[\tilde{w}_0,\tilde{w}],[f])$$
$$= \lambda - \left\langle \tilde{w}_0, \frac{1}{2}\mathcal{N}(w_0,w_0) + \mathcal{L}w_0 - f \right\rangle - \left\langle \tilde{w}, \lambda\mathcal{B}\hat{w} + \mathcal{N}_{w_0}\hat{w} + \mathcal{L}\hat{w} \right\rangle$$

Variation with respect to the state => definition of adjoint

$$\lim_{\varepsilon\to 0} \frac{\mathcal{L}([w_0+\epsilon\delta w_0,\hat{w}+\epsilon\delta\hat{w},\lambda+\epsilon\delta\lambda],[\tilde{w}_0,\tilde{w}],[f]) - \mathcal{L}([w_0,\hat{w},\lambda],[\tilde{w}_0,\tilde{w}],[f])}{\varepsilon}$$

$$= \delta\lambda - \langle \tilde{w}_0, \mathcal{N}(w_0,\delta w_0) + \mathcal{L}\delta w_0 \rangle$$
$$- \langle \tilde{w}, \delta\lambda\mathcal{B}\hat{w} + \lambda\mathcal{B}\delta\hat{w} + \mathcal{N}(\delta w_0,\hat{w}) + \mathcal{N}(w_0,\delta\hat{w}) + \mathcal{L}\delta\hat{w} \rangle$$

$$= (1 - < \tilde{w},\mathcal{B}\hat{w} >)\delta\lambda - \left\langle \tilde{w}_0, \mathcal{N}_{w_0}\delta w_0 + \mathcal{L}\delta w_0 \right\rangle - \left\langle \tilde{w}, \lambda\mathcal{B}\delta\hat{w} + \mathcal{N}_{\hat{w}}\delta w_0 + \mathcal{N}_{w_0}\delta\hat{w} + \mathcal{L}\delta\hat{w} \right\rangle$$

$$= (1 - < \tilde{w},\mathcal{B}\hat{w} >)\delta\lambda - \left\langle (\widetilde{\mathcal{N}}_{w_0} + \mathcal{L})\tilde{w}_0 + \widetilde{\mathcal{N}_{\hat{w}}}\tilde{w}, \delta w_0 \right\rangle - \left\langle (\lambda^*\mathcal{B} + \widetilde{\mathcal{N}}_{w_0} + \mathcal{L})\tilde{w}, \delta\hat{w} \right\rangle$$

$$= < \frac{\partial\mathcal{L}}{\partial[\mathrm{w}_0,\hat{w},\lambda]}, [\delta w_0,\delta\hat{w},\delta\lambda] >$$

Sensitivity to steady forcing

$$\frac{\partial \mathcal{L}}{\partial[\mathrm{w}_0, \hat{w}, \lambda]} = \left[\widetilde{\mathcal{N}}_{w_0}\widetilde{w}_0 + \mathcal{L}\widetilde{w}_0 + \widetilde{\mathcal{N}}_{\hat{w}}\widetilde{w}, \lambda^* \mathcal{B}\widetilde{w} + \widetilde{\mathcal{N}}_{w_0}\widetilde{w} + \mathcal{L}\widetilde{w}, 1 - < \widetilde{w}, \mathcal{B}\hat{w} > \right]$$

Sensitivity to steady forcing

Lagrangian:

$$\mathcal{L}([w_0,\hat{w},\lambda],[\tilde{w}_0,\tilde{w}],[f])$$
$$= \lambda - \left\langle \tilde{w}_0, \frac{1}{2}\mathcal{N}(w_0,w_0) + \mathcal{L}w_0 - f \right\rangle - \langle \tilde{w}, \lambda\mathcal{B}\hat{w} + \mathcal{N}_{w_0}\hat{w} + \mathcal{L}\hat{w} \rangle$$

Variation with respect to the control:

$$\lim_{\varepsilon\to 0} \frac{\mathcal{L}([w_0,\hat{w},\lambda],[\tilde{w}_0,\tilde{w}],[f+\epsilon\delta f]) - \mathcal{L}([w_0,\hat{w},\lambda],[\tilde{w}_0,\tilde{w}],[f])}{\varepsilon}$$

$$= -\langle \tilde{w}_0, -\delta f \rangle$$

$$= < \frac{\partial \mathcal{L}}{\partial f}, \delta f >$$

$$\frac{\partial \mathcal{L}}{\partial f} = \tilde{w}_0$$

Sensitivity to steady forcing

Conclusion:
The gradient of $\lambda(f)$ is given by

$$\nabla_f \lambda = \widetilde{w}_0$$

where:

$$\widetilde{\mathcal{N}}_{w_0}\widetilde{w}_0 + \mathcal{L}\widetilde{w}_0 = -\widetilde{\mathcal{N}}_{\widehat{w}}\widetilde{w}$$
$$\lambda^* \mathcal{B}\widetilde{w} + \widetilde{\mathcal{N}}_{w_0}\widetilde{w} + \mathcal{L}\widetilde{w} = 0$$
$$1 - < \widetilde{w}, \mathcal{B}\widehat{w} > = 0$$

Sensitivity to steady forcing

Let f be a steady forcing acting on the base-flow:

$$\frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = f$$

Let $(\lambda, \hat{w})$ be an eigenvalue/eigenvector:

$$\lambda\mathcal{B}\hat{w} + \mathcal{N}_{w_0}\hat{w} + \mathcal{L}\hat{w} = 0$$

The base-flow w_0 is a function of f while λ is a function of w_0. The gradient of the function $\lambda(f) = \lambda(w_0(f))$, defined such that $\delta\lambda =< \nabla_f \lambda, \delta f >$, is given by:

$$\nabla_f \lambda = \tilde{w}_0$$

where:

$$\tilde{\mathcal{N}}_{w_0}\tilde{w}_0 + \mathcal{L}\tilde{w}_0 = -\tilde{\mathcal{N}}_{\hat{w}}\tilde{w}$$

$$\lambda^*\mathcal{B}\tilde{w} + \tilde{\mathcal{N}}_{w_0}\tilde{w} + \mathcal{L}\tilde{w} = 0$$

and the normalization condition $< \tilde{w}, \mathcal{B}\hat{w} >= 1$.

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Cylinder flow

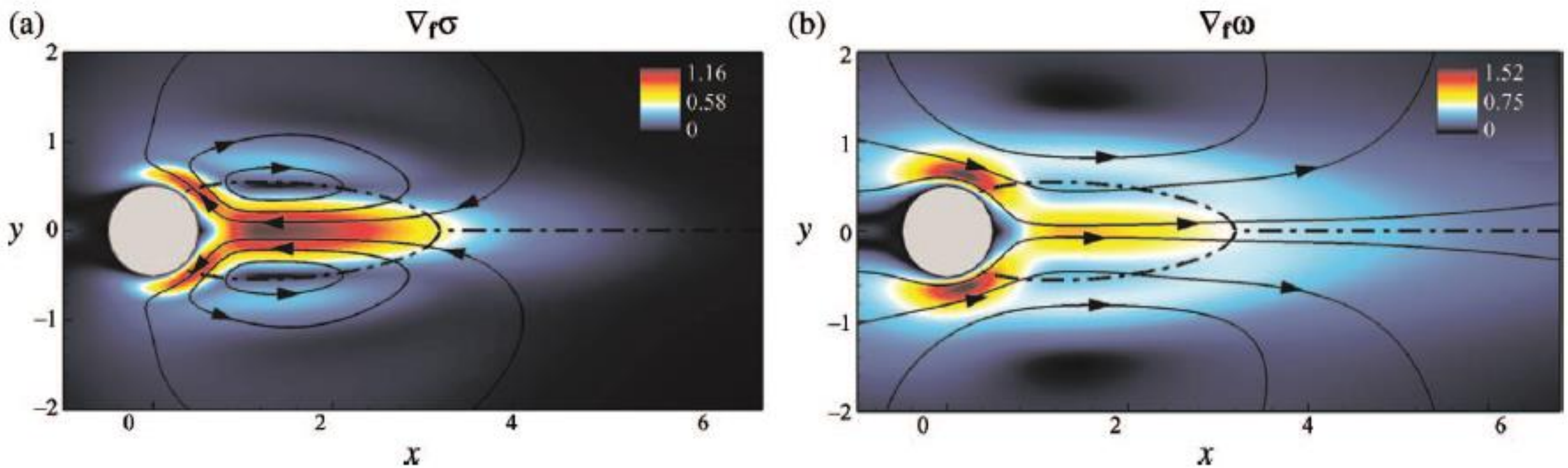

Fig. 12 Flow around a cylinder at Re=47 and sensitivities associated with a steady forcing of the base-flow. (*a*) Sensitivity of the amplification rate. (*b*) Sensitivity of the frequency. Adapted from Ref. [135].

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Cylinder flow: control maps

Control cylinder modeled by pure drag force: $\delta f = \begin{pmatrix} -u_0 \\ 0 \end{pmatrix}$

Eigenvalue shift: $\delta\lambda = < \nabla_f \lambda, \delta f > = < \nabla_f \lambda, \begin{pmatrix} -u_0 \\ 0 \end{pmatrix} >$

$$\delta\sigma + i\delta\omega = < \nabla_f \sigma, \begin{pmatrix} -u_0 \\ 0 \end{pmatrix} > - i < \nabla_f \omega, \begin{pmatrix} -u_0 \\ 0 \end{pmatrix} >$$

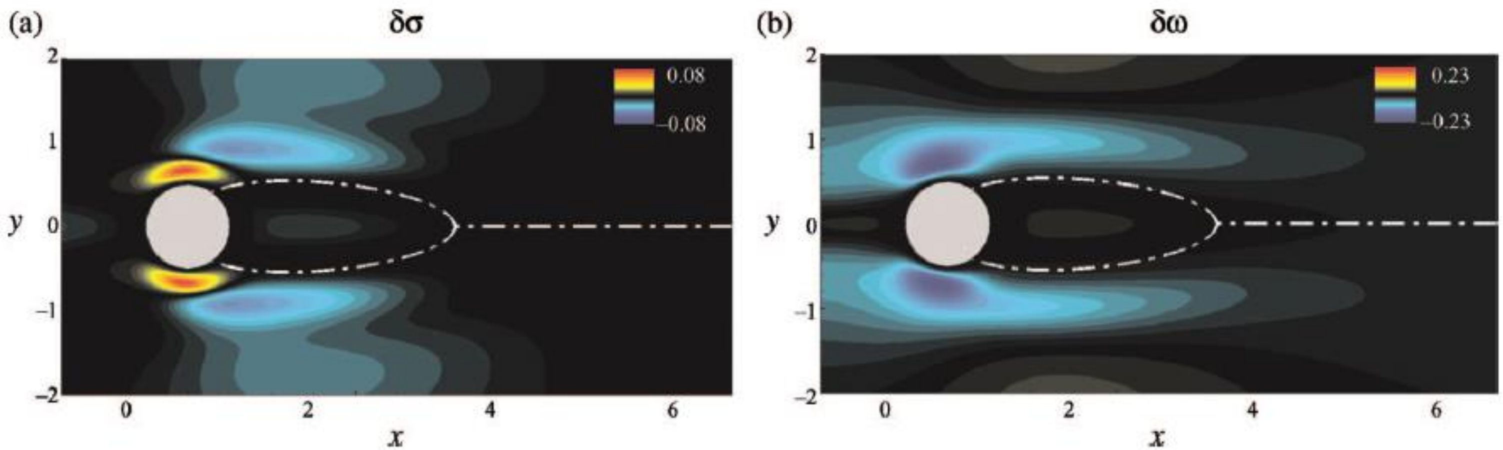

Fig. 13 Flow around a cylinder at Re=47. (*a*) Variation of the amplification rate with respect to the placement of a control cylinder of infinitesimal size located at the current point. (*b*) Associated variation of the frequency. Adapted from Ref. [135].

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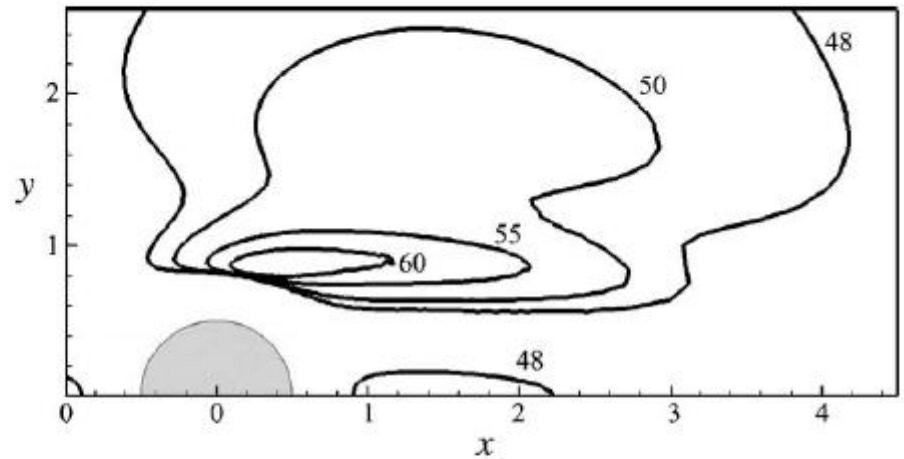
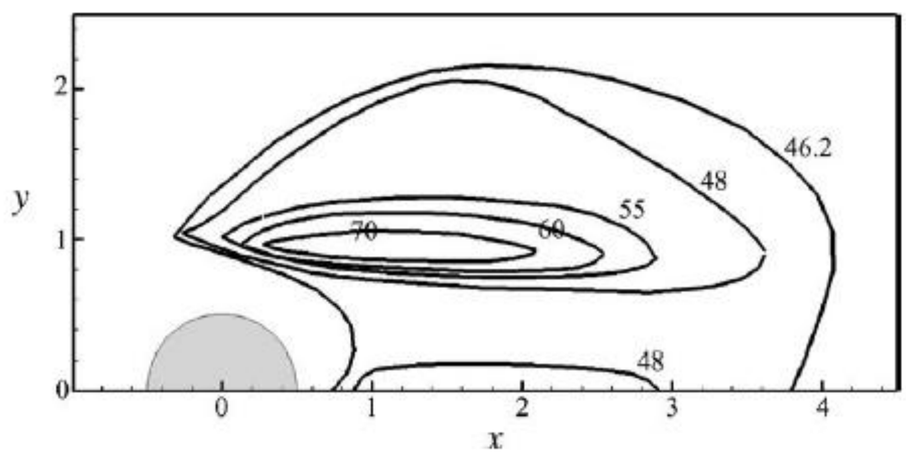
Cylinder: control maps

Theory



Experiment



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