Amplitude equations for control

Weakly-nonlinear solutions and multiple-timescale analysis

Outline

- Vortex shedding in cylinder flow at $Re = 100$
- The Van der Pol oscillator: a model problem
- The Van der oscillator: approximations
  - One time-scale approach
  - Two times-scales approach
  - With forcing term
- The Ginzburg-Landau eq.
  - Forcing with $\omega_f \neq \omega_0$
  - Forcing with ω_f close to ω_0
- The forced Navier-Stokes eq.
  - Forcing with $\omega_f \neq \omega_0$
  - Forcing with ω_f close to ω_0
  - Discussion of amplitude eq.

DNS simulation of cylinder flow at Re=100

Unforced simulation (Q1)

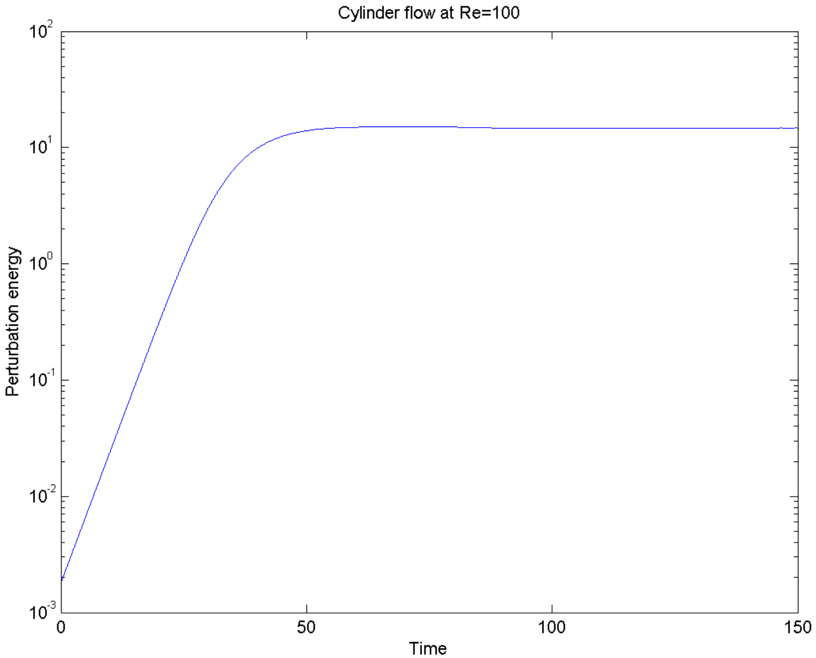
$$\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w = 0$$

How does the system respond to harmonic forcing?

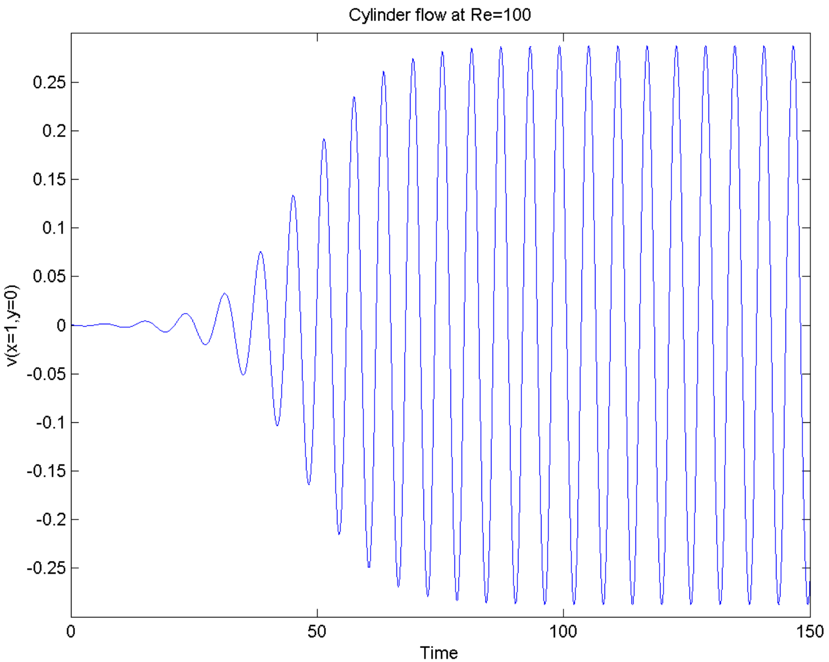
$$\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w = \tilde{E}e^{i\omega_f}f + \text{c.c}$$

Influence of ω_f, f, $\tilde{E}$?

DNS simulation of cylinder flow at Re=100



Energy vs Time



$v(x = 1, y = 0)$ vs Time

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The Van der Pol Oscillator: a model problem

ODE (similar to Navier-Stokes)

$$w'' + \omega_0^2 w = 2\underbrace{\tilde{\delta}}_{\epsilon\delta} w' - w^2 w', \epsilon \ll 1, \delta = O(1)$$

$$w(0) = w_I, w'(0) = 0$$

Fixed point: $w = 0$

Stability of fixed point: $|w| \ll 1$

$$w'' + \omega_0^2 w = 2\epsilon\delta w'$$

Global modes:

$$w = e^{\lambda t}\widehat{w} \Rightarrow \lambda^2 + \omega_0^2 = 2\epsilon\delta\lambda \Rightarrow \lambda = \epsilon\delta \pm i\sqrt{\omega_0^2 - \epsilon^2\delta^2} \approx \epsilon\delta \pm i\omega_0$$

Conclusion:

1/ Hopf bifurcation at $\epsilon = 0$

2/ If $\epsilon \ll 1$, slow time-scale on amplification rate and fast time scale on frequency

The Van der Pol Oscillator: a model problem

For $\delta > 0$, at saturation, the instability term $2\epsilon\delta w'$ is cancelled by the nonlinear term $w^2 w'$ when $w^2 \approx 2\epsilon\delta$. We expect that the saturation amplitude is about $\sqrt{\epsilon\delta}$.

We look for the solution under the form:

$$w = \epsilon^{\frac{1}{2}} y$$
$$w_I = \epsilon^{\frac{1}{2}} y_I = \epsilon^{\frac{1}{2}} y(0)$$

Hence:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2$$
$$y(0) = y_I, y'(0) = 0$$

(Q2a)

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The Van der Pol Oscillator: approximations

ODE:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2$$

Solutions under the form:

$$y = y_0 + \epsilon y_1 + \cdots$$

One time-scale approach:

$$y(t) = y_0(t) + \epsilon y_1(t) + \cdots$$

Two time-scales approach:

$$y = y_0(\mathrm{t}, \tau = \epsilon t) + \epsilon y_1(t, \tau = \epsilon t) + \cdots$$

The Van der Pol Oscillator: one time-scale approach

ODE:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2$$

We look for a solution with one time-scale:

$$y = y_0(\mathrm{t}) + \epsilon y_1(t) + o(\epsilon)$$

Then:

$$y' = y_0' + \epsilon y_1' + o(\epsilon)$$
$$y'' = y_0'' + \epsilon y_1'' + o(\epsilon)$$

Initial conditions:

$$y(0) = y_I \Rightarrow y_0(0) + \epsilon y_1(0) + o(\epsilon) = y_I \Rightarrow y_0(0) = y_I, y_1(0) = 0$$
$$y'(0) = 0 \Rightarrow y_0'(0) + \epsilon y_1'(0) + o(\epsilon) = 0 \Rightarrow y_0'(0) = 0, y_1'(0) = 0$$

The Van der Pol Oscillator: one time-scale approach

ODE:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2$$

$$\Rightarrow y_0'' + \epsilon y_1'' + \omega_0^2(y_0 + \epsilon y_1) = 2\delta\epsilon(y_0' + \epsilon y_1') - \epsilon(y_0' + \epsilon y_1')(y_0 + \epsilon y_1)^2$$

Order ϵ^0:

$$y_0'' + \omega_0^2 y_0 = 0$$

Order ϵ^1:

$$y_1'' + \omega_0^2 \mathrm{y}_1 = 2\delta y_0' - y_0^2 y_0'$$

The Van der Pol Oscillator: one time-scale approach

Solution of order ϵ^0, knowing that the solution is real,

$$y_0 = Ae^{i\omega_0 t} + \text{c.c.} = 2|A|\cos(\omega_0 t + \phi)$$

Determination of A with initial conditions:

$$y_0(0) = y_I \Rightarrow 2|A|\cos\phi = y_I$$
$$y_0'(0) = 0 \Rightarrow -2|A|\omega_0 \sin\phi = 0$$

$$\Rightarrow \phi = 0, A = \frac{y_I}{2}$$

The Van der Pol Oscillator: one time-scale approach

We recast this solition into the next order: $y_1'' + \omega_0^2 \mathrm{y}_1 = 2\delta y_0' - y_0^2 y_0'$

$$y_1'' + \omega_0^2 \mathrm{y}_1 = 2i\omega_0 \delta A e^{i\omega_0 t} + \text{c.c.} - \frac{1}{3}\frac{d\left(A^3 e^{3i\omega_0 t} + 3A^2 A^* e^{i\omega_0 t} + \text{c.c.}\right)}{dt}$$

$$= 2i\omega_0 \delta A e^{i\omega_0 t} + \text{c.c.} - \frac{1}{3}\left(3i\omega_0 A^3 e^{3i\omega_0 t} + 3i\omega_0 A^2 A^* e^{i\omega_0 t} + \text{c.c.}\right)$$

$$= -\mathrm{i}\omega_0 \mathrm{A}^3 e^{3i\omega_0 t} + i\omega_0 (2\delta \mathrm{A} - \mathrm{A}^2 \mathrm{A}^*) \mathrm{e}^{\mathrm{i}\omega_0 \mathrm{t}} + \text{c.c.}$$

The Van der Pol Oscillator: one time-scale approach

Theorem:

$$y'' + \omega^2 y = \left(ae^{i\Omega t} + \text{c.c.}\right)$$

If $\Omega \neq \omega$, then the most general real solution is (with k as any complex constant):

$$y = \left(ke^{i\omega t} + \frac{a}{\omega^2 - \Omega^2} e^{i\Omega t} + \text{c.c.}\right)$$

If $\Omega = \omega$, then the most general real solution is (with k as any complex constant):

$$y = \left(ke^{i\omega t} - a\left(\frac{1 + 2i\omega t}{4\omega^2}\right) e^{i\omega t} + \text{c.c.}\right)$$

Proof (resonant case only):

$$y = ke^{i\omega t} - a\left(\frac{1 + 2i\omega t}{4\omega^2}\right) e^{i\omega t} + \text{c.c}$$

$$y' = i\omega ke^{i\omega t} - a\left(\frac{2i\omega}{4\omega^2}\right) e^{i\omega t} - i\omega a\left(\frac{1 + 2i\omega t}{4\omega^2}\right) e^{i\omega t} + \text{c.c.}$$

$$y'' = -\omega^2 ke^{i\omega t} + ae^{i\omega t} + a\left(\frac{1 + 2i\omega t}{4}\right) e^{i\omega t} + \text{c.c.}$$

$$y'' + \omega^2 y = ae^{i\omega t} + \text{c.c.}$$

The Van der Pol Oscillator: one time-scale approach

Solution under the form:

$$y_1 = ke^{i\omega_0 t} + \frac{iA^3}{8\omega_0^2}\omega_0 e^{3i\omega_0 t} - (2i\delta A - iA^2A^*)\omega_0\left(\frac{1+2i\omega_0 t}{4\omega_0^2}\right)e^{i\omega_0 t} + \text{c.c.}$$

Determination of k with initial conditions:

$$y_1(0) = 0 \Rightarrow k + \frac{iA^3}{8\omega_0^2}\omega_0 - (2i\delta A - iA^2A^*)\omega_0\left(\frac{1}{4\omega_0^2}\right) + \text{c.c.} \Rightarrow k_r = 0$$

$$y_1'(0) = 0 \Rightarrow ik_i i\omega_0 - \frac{iA^3}{8\omega_0^2}\omega_0 3i\omega_0 - \frac{(2i\delta A - iA^2A^*)}{4\omega_0^2}\omega_0(i\omega_0 + 2i\omega_0) + \text{c.c.} = 0$$

$$k_i = \frac{A^3}{8\omega_0^2}\omega_0 3 + \frac{(2\delta A - A^3)}{8\omega_0^2}\omega_0 6$$

$$\Rightarrow k_i = \frac{-3A^3 + 12\delta A}{8\omega_0}$$

Finally:

$$y_1 = \frac{-3A^3 + 12\delta A}{8\omega_0} ie^{i\omega_0 t} + \frac{iA^3}{8\omega_0}e^{3i\omega_0 t} - (2\delta A - A^3)\left(\frac{1+2i\omega_0 t}{4\omega_0}\right)ie^{i\omega_0 t} + \text{c.c.}$$

The Van der Pol Oscillator: one time-scale approach

Coming back to initial unknown w:

$$w = \epsilon^{\frac{1}{2}} y = \epsilon^{\frac{1}{2}} y_0(\mathrm{t}) + \epsilon^{\frac{3}{2}} y_1(t)$$

$$= \epsilon^{\frac{1}{2}}\left(Ae^{i\omega_0 t} + \mathrm{c.c.}\right)$$

$$+ \epsilon^{\frac{3}{2}}\left(\frac{-3\mathrm{A}^3 + 12\delta\mathrm{A}}{8\omega_0} ie^{i\omega_0 t} + \frac{\mathrm{i}\mathrm{A}^3}{8\omega_0} e^{3i\omega_0 t} - (2\delta\mathrm{A} - \mathrm{A}^3)\left(\frac{1 + 2i\omega_0 t}{4\omega_0}\right) ie^{i\omega_0 t} + \mathrm{c.c.}\right)$$

$$= \left(\tilde{A}e^{i\omega_0 t} + \mathrm{c.c.}\right)$$

$$+ \left(\frac{-3\tilde{A}^3 + 12\tilde{\delta}\tilde{A}}{8\omega_0} ie^{i\omega_0 t} + \frac{\mathrm{i}\tilde{A}^3}{8\omega_0} e^{3i\omega_0 t} - \left(2\tilde{\delta}\tilde{A} - \tilde{A}^3\right)\left(\frac{1 + 2i\omega_0 t}{4\omega_0}\right) ie^{i\omega_0 t} + \mathrm{c.c.}\right)$$

$$\tilde{A} = \frac{w_I}{2}$$

(Q2b)

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The Van der Pol Oscillator: two time-scales approach

PDE:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2$$

We look for a solution with two time-scales:

$$y = y_0(\mathrm{t}, \tau = \epsilon t) + \epsilon y_1(t, \tau = \epsilon t) + o(\epsilon)$$

Then:

$$y' = \frac{\partial y_0}{\partial t} + \epsilon\left(\frac{\partial y_1}{\partial t} + \frac{\partial y_0}{\partial \tau}\right) + o(\epsilon)$$

$$y'' = \frac{\partial^2 y_0}{\partial t^2} + \epsilon\left(\frac{\partial^2 y_1}{\partial t^2} + 2\frac{\partial^2 y_0}{\partial t \partial \tau}\right) + o(\epsilon)$$

The Van der Pol Oscillator: two time-scales approach

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2$$

$$\frac{\partial^2 y_0}{\partial t^2} + \epsilon\left(2\frac{\partial^2 y_0}{\partial t \partial \tau} + \frac{\partial^2 y_1}{\partial t^2}\right) + \omega_0^2(y_0 + \epsilon y_1)$$
$$= 2\delta\epsilon\left(\frac{\partial y_0}{\partial t} + \epsilon\left(\frac{\partial y_0}{\partial \tau} + \frac{\partial y_1}{\partial t}\right)\right) - \epsilon\left(\frac{\partial y_0}{\partial t} + \epsilon\left(\frac{\partial y_0}{\partial \tau} + \frac{\partial y_1}{\partial t}\right)\right)(y_0 + \epsilon y_1)^2$$

Order ϵ^0:

$$\frac{\partial^2 y_0}{\partial t^2} + \omega_0^2 y_0 = 0$$

Order ϵ^1:

$$\frac{\partial^2 y_1}{\partial t^2} + \omega_0^2 \mathrm{y}_1 = 2\delta\frac{\partial y_0}{\partial t} - y_0^2\frac{\partial y_0}{\partial t} - 2\frac{\partial^2 y_0}{\partial t \partial \tau}$$

The Van der Pol Oscillator: two time-scales approach

Solution at order ϵ^0, knowing that the solution is real,

$$y_0 = A(\tau)e^{i\omega_0 t} + \text{c.c.}$$

We recast this solution into the next order:

$$\frac{\partial^2 y_1}{\partial t^2} + \omega_0^2 \mathrm{y}_1$$

$$= 2i\omega_0\delta A e^{i\omega_0 t} + \text{c.c.} - \frac{1}{3}\frac{\partial\left(A^3 e^{3i\omega_0 t} + 3A^2A^* e^{i\omega_0 t} + \text{c.c.}\right)}{\partial t} - 2i\omega_0\frac{dA}{d\tau}e^{i\omega_0 t} + \text{c.c.}$$

$$= 2i\omega_0\delta A e^{i\omega_0 t} + \text{c.c.} - \frac{1}{3}\left(3i\omega_0 A^3 e^{3i\omega_0 t} + 3i\omega_0 A^2A^* e^{i\omega_0 t} + \text{c.c.}\right) - 2i\omega_0\frac{dA}{d\tau}e^{i\omega_0 t} + \text{c.c.}$$

$$= -i\omega_0 A^3 e^{3i\omega_0 t} + i\omega_0\left(2\delta\mathrm{A} - \mathrm{A}^2\mathrm{A}^* - 2\frac{dA}{d\tau}\right)\mathrm{e}^{i\omega_0 \mathrm{t}} + \text{c.c.}$$

The Van der Pol Oscillator: two time-scales approach

Theorem:

$$y'' + \omega^2 y = \left(a e^{i\Omega t} + \text{c.c.}\right)$$

If $\Omega \neq \omega$, then the most general real solution is (with k as any complex constant):

$$y = \left(k e^{i\omega t} + \frac{a}{\omega^2 - \Omega^2} e^{i\Omega t} + \text{c.c.}\right)$$

If $\Omega = \omega$, then the most general real solution is (with k as any complex constant):

$$y = \left(k e^{i\omega t} - a\left(\frac{1 + 2i\omega t}{4\omega^2}\right) e^{i\omega t} + \text{c.c.}\right)$$

Proof (resonant case):

$$y = k e^{i\omega t} - a\left(\frac{1 + 2i\omega t}{4\omega^2}\right) e^{i\omega t} + \text{c.c}$$

$$y' = i\omega k e^{i\omega t} - a\left(\frac{2i\omega}{4\omega^2}\right) e^{i\omega t} - i\omega a\left(\frac{1 + 2i\omega t}{4\omega^2}\right) e^{i\omega t} + \text{c.c.}$$

$$y'' = -\omega^2 k e^{i\omega t} + a e^{i\omega t} + a\left(\frac{1 + 2i\omega t}{4}\right) e^{i\omega t} + \text{c.c.}$$

$$y'' + \omega^2 y = a e^{i\omega t} + \text{c.c}$$

The Van der Pol Oscillator: two time-scales approach

For the solution to be valid uniformly in time, we kill the secular term:

$$\frac{dA}{d\tau} = \delta \mathrm{A} - \frac{1}{2}\mathrm{A}^2\mathrm{A}^*$$

Final first order solution:

$$w(t) = \epsilon^{\frac{1}{2}}\left(Ae^{i\omega_0 t} + \mathrm{c.c}\right) = \left(\tilde{A}e^{i\omega_0 t} + \mathrm{c.c}\right)$$

with:

$$\frac{d\tilde{A}}{dt} = \tilde{\delta}\tilde{A} - \frac{1}{2}\tilde{A}^3$$

$$\tilde{A}(0) = \frac{w_I}{2}$$

(Q2c)

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The Van der Pol Oscillator with forcing term

PDE:

$$w'' + \omega_0^2 w = 2\underbrace{\tilde{\delta}}_{\epsilon\delta} w' - w^2 w' + \underbrace{\tilde{E}}_{\epsilon^{\frac{1}{2}}E} \cos\omega_f t\,, \epsilon \ll 1, \delta = O(1)$$

We look for the solution under the form:

$$w(t) = \epsilon^{\frac{1}{2}} y(t)$$

Hence:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2 + E\cos\omega_f t$$

The Van der Pol oscillator with forcing term

PDE:

$$y'' + \omega_0^2 y = 2\delta\epsilon y' - \epsilon y' y^2 + E\cos\omega_f t$$

Expansion:

$$y = y_0(t,\tau) + \epsilon y_1(t,\tau) + o(\epsilon)$$

Order ϵ^0:

$$\frac{\partial^2 y_0}{\partial t^2} + \omega_0^2 y_0 = E\cos\omega_f t \Rightarrow \mathrm{y}_0 = A(\tau)e^{i\omega_0 t} + \text{c.c.} + 2\underbrace{\frac{1}{2}\frac{E}{\omega_0^2-\omega_f^2}}_{\eta}\cos\omega_f t$$

Order ϵ:

$$\frac{\partial^2 y_1}{\partial t^2} + \omega_0^2 y_1 = 2\delta\frac{\partial y_0}{\partial t} - \frac{1}{3}\frac{\partial y_0^3}{\partial t} - 2\frac{\partial^2 y_0}{\partial t\partial\tau} = i\omega_0 e^{i\omega_0 t}\left(2\delta A - A|A|^2 - 2A\eta^2 - 2\frac{\partial A}{\partial\tau}\right) + \cdots$$

$$y_0^3 = \left(Ae^{i\omega_0 t} + A^* e^{-i\omega_0 t} + \eta e^{i\omega_f t} + \eta e^{-i\omega_f t}\right)^2\left(Ae^{i\omega_0 t} + A^* e^{-i\omega_0 t} + \eta e^{i\omega_f t} + \eta e^{-i\omega_f t}\right)$$

$$= \Big(A^2 e^{2i\omega_0 t} + A^{*2}e^{-2i\omega_0 t} + \eta^2 e^{2i\omega_f t} + \eta^2\ e^{-2i\omega_f t} + 2|A|^2 + 2\eta^2 + 2A\eta e^{i(\omega_0+\omega_f)t}$$

$$+ 2A^*\eta e^{-i(\omega_0+\omega_f)t} + 2A\eta e^{i(\omega_0-\omega_f)t} + 2A^*\eta e^{-i(\omega_0-\omega_f)t}\Big)^2\Big(Ae^{i\omega_0 t} + A^* e^{-i\omega_0 t}$$

$$+ \eta e^{i\omega_f t} + \eta e^{-i\omega_f t}\Big) = (3A|A|^2 + 6\eta^2 A)e^{i\omega_0 t} + \cdots \qquad \omega_f \neq \left(\frac{1}{3}\omega_0, \omega_0, 3\omega_0\right)$$

The Van der Pol oscillator with forcing term

Kill resonant term to remove secular terms:

$$\frac{dA}{d\tau} = \delta A - \frac{1}{2} A|A|^2 - \frac{1}{4}\left(\frac{1}{\omega_0^2 - \omega_f^2}\right)^2 E^2 A$$

Final solution:

$$w(t) = \epsilon^{\frac{1}{2}}\left(Ae^{i\omega_0 t} + \text{c.c.} + \frac{E}{\omega_0^2 - \omega_f^2}\cos\omega_f t\right) = 2|\tilde{A}|\cos(\omega_0 t + \phi) + \frac{\tilde{E}}{\omega_0^2 - \omega_f^2}\cos\omega_f t$$

with:

$$\frac{d\tilde{A}}{dt} = \tilde{\delta}\tilde{A} - \frac{1}{2}\tilde{A}^3 - \frac{1}{4}\left(\frac{1}{\omega_0^2 - \omega_f^2}\right)^2 \tilde{E}\tilde{A}$$

Or:

$$\frac{d\tilde{A}}{dt} = \left[\tilde{\delta} - \frac{1}{4}\left(\frac{\tilde{E}}{\omega_0^2 - \omega_f^2}\right)^2\right]\tilde{A} - \frac{1}{2}\tilde{A}^3$$

(Q3)

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The Ginzburg-Landau eq. (cont'd)

8/ The forced non-linear Ginzburg-Landau equations read:

$$\partial_t w + \mathcal{L}w = cw|w|^2 + f(x,t)$$

where

$$\mathcal{L} = U\partial_x - \mu(x) - \gamma\partial_{xx}, \qquad \mu(x) = i\omega_0 + \mu_0 - \mu_2\frac{x^2}{2}$$

and c is a positive real constant, $f(x,t)$ a forcing. The least-damped global mode is $\widehat{w}(x) = \zeta e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$ with eigenvalue $\lambda = \mathrm{i}\omega_0 + \mu_0 - \mu_c$ where $\mu_c = \frac{U^2}{4\gamma} + \sqrt{\frac{\gamma\mu_2}{2}}$. The corresponding adjoint global mode is $\widetilde{w}(x) = \xi e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$. They are normalized such that: $\langle\widehat{w},\widehat{w}\rangle = \langle\widetilde{w},\widehat{w}\rangle = 1$. We choose μ_0 in the vicinity of μ_c such that:

$$\mu_0 = \mu_c + \delta',$$

where $\delta' = \epsilon\delta$ with $0 < \epsilon \ll 1$, $\delta = O(1)$. The operator $\mathcal{L}$ may therefore be written as $\mathcal{L} = \mathcal{L}_c - \epsilon\delta$, where $\mathcal{L}_c$ is the operator obtained for $\delta' = 0$, that is $\mu_0 = \mu_c$.
Show that $(i\omega_0 I + \mathcal{L}_c)\widehat{w}_c = 0$ where $\widehat{w}_c$ is the global mode for $\mu_0 = \mu_c$.

8/

$$\mu(x) = i\omega_0 + \mu_c + \epsilon\delta - \mu_2 \frac{x^2}{2}$$

$$\mathcal{L} = \underbrace{U\partial_x - i\omega_0 - \mu_c + \mu_2 \frac{x^2}{2} - \gamma\partial_{xx}}_{\mathcal{L}_c} - \epsilon\delta$$

$$i\omega_0 \hat{w}_c + \mathcal{L}_c \hat{w}_c = 0$$

The Ginzburg-Landau eq. (cont'd)

9a/ We choose a forcing such that:

$$f(x,t) = E'\delta(x - x_f)e^{i\omega_f t}$$

where $E' = \epsilon^{\frac{1}{2}}E$, $E = O(1)$ is the forcing amplitude (positive real) and $\delta(x - x_f)$ is the Dirac function at $x = x_f$ (we remind the reader that $\int_{-\infty}^{+\infty} \delta(x - x_f)w(x)dx = w(x_f)$ for any function w). The forcing frequency ω_f is chosen such that $\omega_f \neq \omega_0$.

The solution is sought under the form:

$$w = \epsilon^{\frac{1}{2}}y$$

$$y = y_0(t,\tau) + \epsilon y_1(t,\tau) + \cdots$$

where $\tau = \epsilon t$ is a slow time-scale.

What is the equation governing y_0? What is the equation governing y_1?

9b/ Show that $y_0(t,\tau) = A(\tau)e^{i\omega_0 t}\hat{w}_c + Ee^{i\omega_f t}\hat{w}_p$ is an acceptable solution for y_0, with $\hat{w}_p$ a spatial structure to be defined depending on x_f and ω_f.

Show that the solution $y_1(t,\tau)$ is bounded only if:

$$\frac{dA}{d\tau} = \left(\delta + 2c|E|^2 \left\langle \tilde{w}_c, \hat{w}_c \left|\hat{w}_p\right|^2 \right\rangle\right) A + cA|A|^2 \langle \tilde{w}_c, \hat{w}_c |\hat{w}_c|^2 \rangle$$

The Ginzburg-Landau eq. (cont'd)

9c/ Show that the leading−order solution of the flowfield may be given by:

$$w(x,t) = B'(t)\widehat{w}_c(x) + E' e^{i\omega_f t}\widehat{w}_p(x)$$

where:

$$\frac{dB'}{dt} = \left(i\omega_0 + \delta' + 2c|E'|^2 \left\langle \widetilde{w}_c, \widehat{w}_c \middle| \widehat{w}_p \right|^2 \right\rangle\right) B' + cB'|B'|^2 \langle \widetilde{w}_c, \widehat{w}_c |\widehat{w}_c|^2 \rangle$$

Hint: note that B' verifies $B' = \epsilon^{\frac{1}{2}} A(\tau) e^{i\omega_0 t}$

9d/ Comment in terms of open-loop control

9a/

$$\partial_t y + \mathcal{L}_c y = \epsilon\delta y + c\epsilon y|y|^2 + E\delta(x - x_f)e^{i\omega_f t}$$

$$\partial_t y = \partial_t y_0 + \epsilon(\partial_t y_1 + \partial_\tau y_0)$$

Throwing everything inside:

$$\partial_t y_0 + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \mathcal{L}_c(y_0 + \epsilon y_1) = \epsilon\delta y_0 + c\epsilon y_0|y_0|^2 + E\delta(x - x_f)e^{i\omega_f t}$$

Order 1:

$$\partial_t y_0 + \mathcal{L}_c y_0 = E\delta(x - x_f)e^{i\omega_f t}$$

$$y_0 = A(\tau)e^{i\omega_0 t}\hat{w}_c + Ee^{i\omega_f t}\hat{w}_p$$

$$i\omega_f\hat{w}_p + \mathcal{L}_c\hat{w}_p = \delta(x - x_f)$$

Orderϵ:

$$\partial_t y_1 + \partial_\tau y_0 + \mathcal{L}_c y_1 = \delta y_0 + c y_0|y_0|^2$$

9b/

$$\partial_t y_1 + \mathcal{L}_c y_1 = -\frac{dA}{d\tau} e^{i\omega_0 t}\widehat{w}_c + \delta A e^{i\omega_0 t}\widehat{w}_c + cA|A|^2 e^{i\omega_0 t}\widehat{w}_c|\widehat{w}_c|^2 + 2cA|E|^2 e^{i\omega_0 t}\widehat{w}_c\left|\widehat{w}_p\right|^2 + \cdots$$

$$y_0|y_0|^2 = \left(Ae^{i\omega_0 t}\widehat{w}_c + Ee^{i\omega_f t}\widehat{\mathrm{w}}_p\right)\left(Ae^{i\omega_0 t}\widehat{w}_c + Ee^{i\omega_f t}\widehat{\mathrm{w}}_p\right)\left(A^* e^{-i\omega_0 t}\widehat{w}_c^* + E^* e^{-i\omega_f t}\widehat{\mathrm{w}}_p^*\right)$$
$$= A|A|^2 e^{i\omega_0 t}\widehat{w}_c|\widehat{w}_c|^2 + 2A|E|^2 e^{i\omega_0 t}\widehat{w}_c\left|\widehat{w}_p\right|^2 + \cdots$$

Kill resonant terms:

$$\frac{dA}{d\tau} = \delta A + cA|A|^2\langle\widetilde{w}_c, \widehat{w}_c|\widehat{w}_c|^2\rangle + 2cA|E|^2\left\langle\widetilde{w}_c, \widehat{w}_c\left|\widehat{w}_p\right|^2\right\rangle$$

9c/

$$\frac{dB'}{dt} = \epsilon^{\frac{1}{2}}\frac{\mathrm{dA}}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{dt}e^{i\omega_0 t} + \epsilon^{\frac{1}{2}}Ai\omega_0 e^{i\omega_0 t}$$

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The Ginzburg-Landau eq. (cont'd)

10a/ The forcing frequency ω_f is chosen in the vicinity of the natural frequency ω_0 of the flow:

$$\omega_f = \omega_0 + \Omega'$$

where $\Omega' = \epsilon\Omega, \epsilon \ll 1, \Omega = O(1)$.
The forcing amplitude is:

$$E' = \epsilon^{\frac{3}{2}} E, E = O(1)$$

The solution is sought under the form:

$$w = \epsilon^{\frac{1}{2}} y$$

$$y = y_0(t,\tau) + \epsilon y_1(t,\tau) + \cdots$$

where $\tau = \epsilon t$ is a slow time-scale.
What is the equation governing y_0? What is the equation governing y_1?
Show that $y_0(t,\tau) = A(\tau) e^{i\omega_0 t} \widehat{w}_c(x)$ is an acceptable solution for y_0.
Show that the solution $y_1(t,\tau)$ is bounded only if:

$$\frac{dA}{d\tau} = \delta A + c < \widetilde{w}_c, \widehat{w}_c |\widehat{w}_c|^2 > A|A|^2 + \overline{\widetilde{w}_c(x_f)} E e^{\mathrm{i}\Omega\tau}.$$

The Ginzburg-Landau eq. (cont'd)

10b/ Show that the leading−order solution of the flowfield may be given by:

$$w(x,t) = C'(t)e^{i\omega_f t}\hat{w}_c(x)$$

where:

$$\frac{dC'}{dt} = (-i\Omega' + \delta')C' + c < \tilde{w}_c, \hat{w}_c|\hat{w}_c|^2 > C'|C'|^2 + \overline{\tilde{w}_c(x_f)}E'$$

Hint: note that C' verifies $C' = \epsilon^{\frac{1}{2}}A(\tau)e^{-i\Omega\tau}$

10c/ Numerical simulations of the equation governing C' show that there exists a threshold amplitude E'_c, such that: If $E' > E'_c$ then $C' \to C'_0$ as $t \to \infty$, where C'_0 is a complex constant. What is the frequency of the flowfield in this case? Can you comment this result in terms of open-loop control? How should the forcing location x_f be chosen to minimize the threshold amplitude E'_c?

10a/

$$\partial_t y + \mathcal{L}_c y = \epsilon\delta y + c\epsilon y|y|^2 + \epsilon E\delta(x - x_f)e^{i\omega_0 t}e^{\mathrm{i}\Omega\tau}$$

$$\partial_t y = \partial_t y_0 + \epsilon(\partial_t y_1 + \partial_\tau y_0)$$

Throwing everything inside:

$$\partial_t y_0 + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \mathcal{L}_c(y_0 + \epsilon y_1) = \epsilon\delta y_0 + c\epsilon y_0|y_0|^2 + \epsilon E\delta(x - x_f)e^{i\omega_0 t}e^{\mathrm{i}\Omega\tau}$$

Order 1:

$$\partial_t y_0 + \mathcal{L}_c y_0 = 0$$

$$y_0 = A(\tau)e^{i\omega_0 t}\widehat{w}_c$$

Order ϵ:

$$\partial_t y_1 + \partial_\tau y_0 + \mathcal{L}_c y_1 = \delta y_0 + c y_0|y_0|^2 + E\delta(x - x_f)e^{i\omega_0 t}e^{\mathrm{i}\Omega\tau}$$

$$\partial_t y_1 + \mathcal{L}_c y_1 = -\frac{dA}{d\tau} e^{i\omega_0 t}\hat{w}_c + \delta A e^{i\omega_0 t}\hat{w}_c + cA|A|^2 e^{i\omega_0 t}\hat{w}_c|\hat{w}_c|^2 + E\delta(x - x_f)e^{i\omega_0 t}e^{\mathrm{i}\Omega\tau}$$

Compatibility condition:

$$< \tilde{w}_c, -\frac{dA}{d\tau} e^{i\omega_0 t}\hat{w}_c + \delta A e^{i\omega_0 t}\hat{w}_c + cA|A|^2 e^{i\omega_0 t}\hat{w}_c|\hat{w}_c|^2 + E\delta(x - x_f)e^{i\omega_0 t}e^{\mathrm{i}\Omega\tau} >= 0$$

$$< \tilde{w}_c, -\frac{dA}{d\tau}\phi + \delta A\hat{w}_c + cA|A|^2\hat{w}_c|\hat{w}_c|^2 + E\delta(x - x_f)e^{\mathrm{i}\Omega\tau} >= 0$$

$$-\frac{dA}{d\tau} < \tilde{w}_c, \hat{w}_c > +\delta A < \tilde{w}_c, \hat{w}_c > +cA|A|^2 < \tilde{w}_c, \hat{w}_c|\hat{w}_c|^2 > +Ee^{\mathrm{i}\Omega\tau} < \tilde{w}_c, \delta(x - x_f) >= 0$$

$$\frac{dA}{d\tau} = \delta A + c < \tilde{w}_c, \hat{w}_c|\hat{w}_c|^2 > A|A|^2 + \overline{\tilde{w}_c(x_f)}Ee^{\mathrm{i}\Omega\tau}$$

10b/

$$\frac{dC'}{dt} = \epsilon^{\frac{1}{2}}\left(-i\Omega + \frac{1}{A}\frac{dA}{d\tau}\right)Ae^{-i\Omega\tau}\epsilon$$

$$= \epsilon^{\frac{1}{2}}\left(-i\Omega + \delta + c < \widetilde{w}_c, \widehat{w}_c|\widehat{w}_c|^2 > |A|^2 + \overline{\widetilde{w}_c(x_f)}\frac{E}{A}e^{\mathrm{i}\Omega\tau}\right)Ae^{-i\Omega\tau}\epsilon$$

$$= \left(-i\Omega' + \delta' + c < \widetilde{w}_c, \widehat{w}_c|\widehat{w}_c|^2 > |C'|^2 + \epsilon\overline{\widetilde{w}_c(x_f)}\frac{E}{C'\epsilon^{-\frac{1}{2}}e^{\mathrm{i}\Omega\tau}}e^{\mathrm{i}\Omega\tau}\right)C'$$

$$\frac{dC'}{dt} = -i\Omega'C' + \delta'C' + c < \widetilde{w}_c, \widehat{w}_c|\widehat{w}_c|^2 > C'|C'|^2 + \overline{\widetilde{w}_c(x_f)}E'$$

$$w = C'(t)e^{i\omega_f t}\widehat{w}_c$$

Theorem: Let w be the solution of the following equation

$$\partial_t w + \mathcal{L}\mathrm{w} = e^{i\omega t} f$$

1/ If $i\omega$ does not belong to the eigenvalues of the Jacobian, then the most general real solution to this equation is

$$w = \sum k_j e^{\lambda_j t} \widehat{w}_j + e^{i\omega t} \widehat{w}_p$$

With the particular solution determined from:

$$i\omega \widehat{w}_p + \mathcal{L}\widehat{w}_p = f$$

And the homogeneous one from the global modes:

$$\lambda_j \widehat{w}_j + \mathcal{L}\widehat{w}_j = 0$$

2/ If $(\mathrm{i}\omega, \widehat{w})$ corresponds to one of the eigenvalues of the Jacobian:

$$\mathrm{i}\omega \widehat{w} + \mathcal{L}\widehat{w} = 0$$

Then, the most general real solution is

$$w(t) = \sum k_j e^{\lambda_j t} \widehat{w}_j + e^{i\omega t} < \widetilde{w}, f > t\widehat{w} + \sum_{\omega_j \neq \omega} \frac{< \widetilde{\mathrm{w}}_j, f >}{i\omega - \lambda_j} e^{i\omega t} \widehat{w}_j$$

Where $\widetilde{w}_j$ (including $\widetilde{w}$) is the bi-orthogonal basis corresponding to $\widehat{w}_k$ (including $\widehat{w}$)

$$< \widetilde{w}_j, \widehat{w}_k > = \delta_{jk}$$

Proof: particular solution with method of variation of constant

$$w(t) = \alpha(t)e^{i\omega t}\widehat{w} + \sum_{\omega_j \neq \omega} \alpha_j e^{i\omega t}\widehat{w}_j$$

Inserting this solution in governing equation:

$$\frac{d\alpha}{dt}\widehat{w} + \alpha(\mathrm{i}\omega\widehat{w} + \mathcal{L}\widehat{w}) + \sum_{\omega_j \neq \omega} \alpha_j\left(i\omega\widehat{w}_j + \mathcal{L}\widehat{w}_j\right) = f$$

$$\frac{d\alpha}{dt}\widehat{w} + \sum_{\omega_j \neq \omega} \alpha_j\left(i\omega - \lambda_j\right)\widehat{w}_j = f$$

Scalar product with $\widetilde{w}$ and $\widetilde{w}_j \neq \widetilde{w}$:

$$\frac{d\alpha}{dt} = < \widetilde{w}, f >$$

$$\alpha_j = \frac{\left\langle \widetilde{w}_j, f \right\rangle}{i\omega - \lambda_j}$$

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The forced Navier-Stokes equations

Forced Navier-Stokes equations with viscosity $\nu = \nu_c - \tilde{\delta}$:

$$\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w = \tilde{\delta}\mathcal{M}w + \left(\tilde{E}e^{i\omega_f t} f + \text{c.c}\right)$$

$$w = \begin{pmatrix} u \\ v \\ p \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{N}(w_1, w_2) = \begin{pmatrix} u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\ 0 \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} -\nu_c \Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}, \mathcal{M} = \begin{pmatrix} -\Delta & 0 \\ 0 & 0 \end{pmatrix}$$

Scalings for $\omega_f \neq \omega_0$:

$$w = w_0 + \epsilon^{1/2} y, \qquad \epsilon \ll 1,$$

$$\tilde{\delta} = \epsilon\delta, \qquad \delta = O(1)$$

$$\tilde{E} = \epsilon^{\frac{1}{2}} E, \qquad E = O(1)$$

Base-flow:

$$\frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = 0$$

Perturbation dynamics:

$$\mathcal{B}\partial_t y + \mathcal{N}_{w_0} y + \mathcal{L}y = \epsilon^{\frac{1}{2}}\delta\mathcal{M}w_0 + \epsilon\delta\mathcal{M}y - \frac{1}{2}\epsilon^{\frac{1}{2}}\mathcal{N}(y,y) + \left(Ee^{i\omega_f t} f + \text{c.c}\right)$$

The forced Navier-Stokes equations

Expansion:

$$y = y_0(t, \tau = \epsilon t) + \epsilon^{1/2} y_{1/2}(t, \tau = \epsilon t) + \epsilon^1 y_1(t, \tau = \epsilon t) + \cdots$$

In particular:

$$\partial_t y = \partial_t y_0 + \epsilon^{1/2} \partial_t y_{1/2} + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \cdots$$

The forced Navier-Stokes equations

$$\mathcal{B}\left(\partial_t y_0 + \epsilon^{1/2}\partial_t y_{1/2} + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \cdots\right) + (\mathcal{N}_{w_0} + \mathcal{L})(y_0 + \epsilon^{1/2} y_{1/2} + \epsilon^1 y_1)$$

$$= \epsilon^{\frac{1}{2}}\delta\mathcal{M}w_0 + \epsilon\delta\mathcal{M}y_0 - \frac{1}{2}\epsilon^{\frac{1}{2}}\mathcal{N}(y_0, y_0) - \epsilon\mathcal{N}\left(y_0, y_{1/2}\right) + \left(Ee^{i\omega_f t}f + \text{c.c}\right)$$

$$\Rightarrow$$

$$\mathcal{B}\partial_t y_0 + \mathcal{N}_{w_0} y_0 + \mathcal{L}y_0 = \left(Ee^{i\omega_f t}f + \text{c.c}\right)$$

$$\mathcal{B}\partial_t y_{1/2} + \mathcal{N}_{w_0} y_{1/2} + \mathcal{L}y_{1/2} = \delta\mathcal{M}w_0 - \frac{1}{2}\mathcal{N}(y_0, y_0)$$

$$\mathcal{B}\partial_t y_1 + \mathcal{N}_{w_0} y_1 + \mathcal{L}y_1 = -\mathcal{B}\partial_\tau y_0 + \delta\mathcal{M}y_0 - \mathcal{N}\left(y_0, y_{1/2}\right)$$

Order ϵ^0

PDE:

$$\mathcal{B}\partial_t y_0 + \mathcal{N}_{w_0} y_0 + \mathcal{L} y_0 = \left(E e^{i\omega_f t} f + \text{c.c}\right)$$

We choose:

$$y_0 = \left(A(\tau) e^{i\omega_c t} y_A + \text{c.c}\right) + \left(E e^{i\omega_f t} y_E + \text{c.c}\right)$$

Hence:

$$\text{A}e^{i\omega_c t} \underbrace{\left(i\omega_c \mathcal{B} y_A + \mathcal{N}_{w_0} y_A + \mathcal{L} y_A\right)}_{0} + E e^{i\omega_f t}\left(i\omega_f \mathcal{B} y_E + \mathcal{N}_{w_0} y_E + \mathcal{L} y_E\right) + \text{c.c.} = E e^{i\omega_f t} f + \text{c.c}$$

which leads to:

$$i\omega_f \mathcal{B} y_E + \mathcal{N}_{w_0} y_E + \mathcal{L} y_E = f$$

This linear problem can be solved because ω_f is not an eigenvalue of the Jacobian.

Order $\epsilon^{\frac{1}{2}}$

PDE:

$$\mathcal{B}\partial_t y_{1/2} + \mathcal{N}_{w_0} y_{1/2} + \mathcal{L} y_{1/2} = \delta \mathcal{M} w_0 - \frac{1}{2}\mathcal{N}(y_0, y_0)$$

$$= \delta \mathcal{M} w_0 + \left(-\frac{1}{2}A^2 e^{2i\omega_c t}\mathcal{N}(y_A, y_A) + \text{c.c.}\right) - |A|^2 \mathcal{N}(y_A, \bar{y}_A) - |E|^2 \mathcal{N}(y_E, \bar{y}_E)$$

$$+ \left(-\frac{1}{2}E^2 e^{2i\omega_f t}\mathcal{N}(y_E, y_E) + \text{c.c.}\right) + \left(-AE e^{i(\omega_c+\omega_f)t}\mathcal{N}(y_A, y_E) + \text{c.c.}\right)$$

$$+ \left(-A\bar{E} e^{i(\omega_c-\omega_f)t}\mathcal{N}(y_A, \bar{y}_E) + \text{c.c.}\right)$$

We choose:

$$y_{1/2} = \delta w_\delta + \left(A^2 e^{2i\omega_c t} y_{AA} + \text{c.c.}\right) + |A|^2 y_{A\bar{A}} + \left(E^2 e^{2i\omega_f t} y_{EE} + \text{c.c.}\right) + |E|^2 y_{E\bar{E}}$$

$$+ \left(AE e^{i(\omega_c+\omega_f)t} y_{AE} + \text{c.c.}\right) + \left(A\bar{E} e^{i(\omega_c-\omega_f)t} y_{A\bar{E}} + \text{c.c.}\right)$$

Order $\epsilon^{\frac{1}{2}}$

We have:

$$\mathcal{N}_{w_0} y_\delta + \mathcal{L} y_\delta = \mathcal{M} w_0$$

$$2i\omega_c \mathcal{B} y_{AA} + \mathcal{N}_{w_0} y_{AA} + \mathcal{L} y_{AA} = -\frac{1}{2}\mathcal{N}(y_A, y_A)$$

$$\mathcal{N}_{w_0} y_{A\bar{A}} + \mathcal{L} y_{A\bar{A}} = -\mathcal{N}(y_A, \bar{y}_A)$$

$$2i\omega_f \mathcal{B} y_{EE} + \mathcal{N}_{w_0} y_{EE} + \mathcal{L} y_{EE} = -\frac{1}{2}\mathcal{N}(y_E, y_E)$$

$$\mathcal{N}_{w_0} y_{E\bar{E}} + \mathcal{L} y_{E\bar{E}} = -\mathcal{N}(y_E, \bar{y}_E)$$

$$2i(\omega_c + \omega_f)\mathcal{B} y_{AE} + \mathcal{N}_{w_0} y_{AE} + \mathcal{L} y_{AE} = -\mathcal{N}(y_A, y_E)$$

$$2i(\omega_c - \omega_f)\mathcal{B} y_{A\bar{E}} + \mathcal{N}_{w_0} y_{A\bar{E}} + \mathcal{L} y_{A\bar{E}} = -\mathcal{N}(y_A, \bar{y}_E)$$

Order ϵ^1

PDE:

$$\mathcal{B}\partial_t y_1 + \mathcal{N}_{w_0} y_1 + \mathcal{L} y_1 = -\mathcal{B}\partial_\tau y_0 + \delta\mathcal{M} y_0 - \mathcal{N}(y_0, y_{1/2})$$

$$= e^{i\omega_c t}\Big[-\frac{dA}{d\tau}\mathcal{B} y_A + \delta A \mathcal{M} y_A - \delta A \mathcal{N}(y_A, y_\delta) - A|A|^2\mathcal{N}(y_A, y_{A\bar{A}})$$

$$- A|A|^2\mathcal{N}(\bar{y}_A, y_{AA}) - A|E|^2\mathcal{N}(y_A, y_{E\bar{E}}) - A|E|^2\mathcal{N}(y_{\bar{E}}, y_{AE})$$

$$- A|E|^2\mathcal{N}(y_E, y_{A\bar{E}})\Big] + \text{c.c} + \cdots$$

Order ϵ^1

We kill the resonant terms to remove secular terms:

$$-\frac{dA}{d\tau} < \tilde{y}_A, \mathcal{B} y_A > + < \tilde{y}_A, \mathcal{M} y_A - \mathcal{N}(y_A, y_\delta) > \delta A$$
$$- < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) > A|A|^2 -$$
$$< \tilde{y}_A, \mathcal{N}(y_A, y_{E\bar{E}}) + \mathcal{N}(y_{\bar{E}}, y_{AE}) + \mathcal{N}(y_E, y_{A\bar{E}}) > A|E|^2 = 0$$

Where the adjoint global mode is:

$$-i\omega_c \mathcal{B} \tilde{y}_A + \widetilde{\mathcal{N}}_{w_0} \tilde{y}_A + \tilde{\mathcal{L}} \tilde{y}_A = 0$$

And has been scaled following:

$$< \tilde{y}_A, \mathcal{B} y_A > = 1$$

Hence:

$$\frac{dA}{d\tau} = \lambda \delta A - \mu A|A|^2 - \pi A|E|^2$$

With:

$$\lambda = < \tilde{y}_A, \mathcal{M} y_A > - < \tilde{y}_A, \mathcal{N}(y_A, y_\delta)$$
$$\mu = < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) >$$
$$\pi = < \tilde{y}_A, \mathcal{N}(y_A, y_{E\bar{E}}) + \mathcal{N}(y_{\bar{E}}, y_{AE}) + \mathcal{N}(y_E, y_{A\bar{E}}) >$$

The forced Navier-Stokes equations

Final solution:

$$w=\epsilon^{\frac{1}{2}}\Bigg(\left(A(\tau)e^{i\omega_c t}y_A+\text{c.c}\right)+\left(Ee^{i\omega_f t}y_E+\text{c.c}\right)$$
$$+\epsilon^{\frac{1}{2}}\Big(\delta w_\delta+\left(A^2e^{2i\omega_c t}y_{AA}+\text{c.c.}\right)+|A|^2y_{A\bar{A}}+\left(E^2e^{2i\omega_f t}y_{EE}+\text{c.c.}\right)+|E|^2y_{E\bar{E}}$$
$$+\left(AEe^{i(\omega_c+\omega_f)t}y_{AE}+\text{c.c.}\right)+\left(A\bar{E}e^{i(\omega_c-\omega_f)t}y_{A\bar{E}}+\text{c.c.}\right)\Big)+\cdots\Bigg)$$

Or:

$$w=\left(\tilde{A}e^{i\omega_c t}y_A+\text{c.c}\right)+\left(\tilde{E}e^{i\omega_f t}y_E+\text{c.c}\right)+\tilde{\delta}w_\delta+\left(\tilde{A}^2e^{2i\omega_c t}y_{AA}+\text{c.c.}\right)+\left|\tilde{A}\right|^2y_{A\bar{A}}$$
$$+\left(\tilde{E}^2e^{2i\omega_t t}y_{EE}+\text{c.c.}\right)+\left|\tilde{E}\right|^2y_{E\bar{E}}+\left(\tilde{A}\tilde{E}e^{i(\omega_c+\omega_f)t}y_{AE}+\text{c.c.}\right)$$
$$+\left(\tilde{A}\bar{\tilde{E}}e^{i(\omega_c-\omega_f)t}y_{A\bar{E}}+\text{c.c.}\right)+\cdots$$

With:

$$\frac{d\tilde{A}}{dt}=\lambda\tilde{\delta}\tilde{A}-\mu\tilde{A}\left|\tilde{A}\right|^2-\pi\tilde{A}\left|\tilde{E}\right|^2 \qquad \text{(Q4)}$$

- It corresponds to an exact solution of the non-linear Navier-Stokes Eq.
- But no guarantee that it can be observed (stability? Unique-ness?)
- Valid near bifurcation threshold: $\left|\tilde{\delta}\right|\ll 1$

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The forced Navier-Stokes equations

Forced Navier-Stokes equations with viscosity $\nu = \nu_c - \tilde{\delta}$:

$$\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w = \tilde{\delta}\mathcal{M}w + \left(\tilde{E}e^{i\omega_f t}f + \text{c.c}\right)$$

$$w = \begin{pmatrix} u \\ v \\ p \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{N}(w_1, w_2) = \begin{pmatrix} u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\ 0 \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} -\nu_c \Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}, \mathcal{M} = \begin{pmatrix} -\Delta & 0 \\ 0 & 0 \end{pmatrix}$$

Scalings for $\omega_f = \omega_0 + \widetilde{\Omega}$:

$$w = w_0 + \epsilon^{1/2} y, \qquad \epsilon \ll 1,$$

$$\tilde{\delta} = \epsilon\delta, \qquad \delta = O(1)$$

$$\tilde{E} = \epsilon^{\frac{3}{2}} E, \qquad E = O(1)$$

$$\widetilde{\Omega} = \epsilon\Omega, \qquad \Omega = O(1)$$

Base-flow:

$$\frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = 0$$

Perturbation dynamics:

$$\mathcal{B}\partial_t y + \mathcal{N}_{w_0} y + \mathcal{L}y = \epsilon^{\frac{1}{2}}\delta\mathcal{M}w_0 + \epsilon\delta\mathcal{M}y - \frac{1}{2}\epsilon^{\frac{1}{2}}\mathcal{N}(y,y) + \epsilon\left(Ee^{i\omega_0 t}e^{i\epsilon\Omega t}f + \text{c.c}\right)$$

The forced Navier-Stokes equations

Expansion:

$$y = y_0(t, \tau = \epsilon t) + \epsilon^{1/2} y_{1/2}(t, \tau = \epsilon t) + \epsilon^1 y_1(t, \tau = \epsilon t) + \cdots$$

In particular:

$$\partial_t y = \partial_t y_0 + \epsilon^{1/2} \partial_t y_{1/2} + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \cdots$$

The forced Navier-Stokes equations

$$\mathcal{B}\left(\partial_t y_0 + \epsilon^{1/2}\partial_t y_{1/2} + \epsilon(\partial_t y_1 + \partial_\tau y_0) + \cdots\right) + (\mathcal{N}_{w_0} + \mathcal{L})(y_0 + \epsilon^{1/2} y_{1/2} + \epsilon^1 y_1)$$
$$= \epsilon^{\frac{1}{2}}\delta\mathcal{M}w_0 + \epsilon\delta\mathcal{M}y_0 - \frac{1}{2}\epsilon^{\frac{1}{2}}\mathcal{N}(y_0, y_0) - \epsilon\mathcal{N}\left(y_0, y_{1/2}\right)$$
$$+ \epsilon\left(Ee^{i\omega_0 t}e^{i\Omega\tau}f + \text{c.c}\right)$$

$$\Rightarrow$$
$$\mathcal{B}\partial_t y_0 + \mathcal{N}_{w_0} y_0 + \mathcal{L}y_0 = 0$$
$$\mathcal{B}\partial_t y_{1/2} + \mathcal{N}_{w_0} y_{1/2} + \mathcal{L}y_{1/2} = \delta\mathcal{M}w_0 - \frac{1}{2}\mathcal{N}(y_0, y_0)$$
$$\mathcal{B}\partial_t y_1 + \mathcal{N}_{w_0} y_1 + \mathcal{L}y_1 = -\mathcal{B}\partial_\tau y_0 + \delta\mathcal{M}y_0 - \mathcal{N}_{\left(y_0, y_{\frac{1}{2}}\right)} + \left(Ee^{i\omega_c t}e^{i\Omega\tau}f + \text{c.c}\right)$$

Order ϵ^0

PDE:

$$\mathcal{B}\partial_t y_0 + \mathcal{N}_{w_0} y_0 + \mathcal{L} y_0 = 0$$

We choose:

$$y_0 = \left(A(\tau) e^{i\omega_c t} y_A + \text{c.c}\right)$$

Order $\epsilon^{\frac{1}{2}}$

PDE:

$$\mathcal{B}\partial_t y_{1/2} + \mathcal{N}_{w_0} y_{1/2} + \mathcal{L} y_{1/2} = \delta \mathcal{M} w_0 - \frac{1}{2}\mathcal{N}(y_0, y_0)$$

$$= \delta \mathcal{M} w_0 + \left(-\frac{1}{2} A^2 e^{2i\omega_c t} \mathcal{N}(y_A, y_A) + \text{c.c.}\right) - |A|^2 \mathcal{N}(y_A, \bar{y}_A)$$

We choose:

$$y_{1/2} = \delta w_\delta + \left(A^2 e^{2i\omega_c t} y_{AA} + \text{c.c.}\right) + |A|^2 y_{A\bar{A}}$$

We have:

$$\mathcal{N}_{w_0} y_\delta \quad + \mathcal{L} y_\delta = \mathcal{M} w_0$$

$$2i\omega_c \mathcal{B} y_{AA} + \mathcal{N}_{w_0} y_{AA} + \mathcal{L} y_{AA} = -\frac{1}{2}\mathcal{N}(y_A, y_A)$$

$$\mathcal{N}_{w_0} y_{A\bar{A}} + \mathcal{L} y_{A\bar{A}} = -\mathcal{N}(y_A, \bar{y}_A)$$

Order ϵ^1

PDE:

$$\mathcal{B}\partial_t y_1 + \mathcal{N}_{w_0} y_1 + \mathcal{L} y_1 = -\mathcal{B}\partial_\tau y_0 + \delta\mathcal{M} y_0 - \mathcal{N}\left(y_0, y_{1/2}\right)$$

$$= e^{i\omega_c t}\left[-\frac{dA}{d\tau}\mathcal{B} y_A + \delta A \mathcal{M} y_A - \delta A \mathcal{N}(y_A, y_\delta) - A|A|^2 \mathcal{N}(y_A, y_{A\bar{A}}) - A|A|^2 \mathcal{N}(\bar{y}_A, y_{AA}) + E e^{i\Omega\tau} f\right] + \text{c.c} + \cdots$$

Order ϵ^1

We kill the resonant terms to remove secular terms:

$$-\frac{dA}{d\tau} < \tilde{y}_A, \mathcal{B} y_A > + < \tilde{y}_A, \mathcal{M} y_A - \mathcal{N}(y_A, y_\delta) > \delta A$$
$$- < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) > A|A|^2 + < \tilde{y}_A, f > E e^{i\Omega\tau} = 0$$

Where the adjoint global mode is:

$$-i\omega_c \mathcal{B} \tilde{y}_A + \widetilde{\mathcal{N}}_{w_0} \tilde{y}_A + \tilde{\mathcal{L}} \tilde{y}_A = 0$$

And has been scaled following:

$$< \tilde{y}_A, \mathcal{B} y_A > = 1$$

Hence:

$$\frac{dA}{d\tau} = \lambda \delta A - \mu A|A|^2 + \pi E e^{i\Omega\tau}$$

With:

$$\lambda = < \tilde{y}_A, \mathcal{M} y_A > - < \tilde{y}_A, \mathcal{N}(y_A, y_\delta)$$
$$\mu = < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) >$$
$$\pi = < \tilde{y}_A, f >$$

The forced Navier-Stokes equations

Final solution:

$$w = \epsilon^{\frac{1}{2}}\left(\left(A(\tau)e^{i\omega_c t}y_A + \text{c.c}\right) + \epsilon^{\frac{1}{2}}\left(\delta w_\delta + \left(A^2 e^{2i\omega_c t}y_{AA} + \text{c.c.}\right) + |A|^2 y_{A\bar{A}}\right) + \cdots\right)$$

Or:

$$w = \left(\tilde{A}e^{i\omega_c t}y_A + \text{c.c}\right) + \tilde{\delta}w_\delta + \left(\tilde{A}^2 e^{2i\omega_c t}y_{AA} + \text{c.c.}\right) + \left|\tilde{A}\right|^2 y_{A\bar{A}} + \cdots$$

With:

$$\frac{d\tilde{A}}{dt} = \lambda\tilde{\delta}\tilde{A} - \mu\tilde{A}\left|\tilde{A}\right|^2 + \pi\tilde{E}e^{i\tilde{\Omega}t}$$

- It corresponds to an exact solution of the non-linear Navier-Stokes Eq.
- But no guarantee that it can be observed (stability? Unique-ness?)
- Valid near bifurcation threshold: $\left|\tilde{\delta}\right| \ll 1$

(Q4)

Outline

- Vortex shedding in cylinder flow at $Re = 100$
- The Van der Pol oscillator: a model problem
- The Van der oscillator: approximations
  - One time-scale approach
  - Two times-scales approach
  - With forcing term
- The Ginzburg-Landau eq.
  - Forcing with $\omega_f \neq \omega_0$
  - Forcing with ω_f close to ω_0
- The forced Navier-Stokes eq.
  - Forcing with $\omega_f \neq \omega_0$
  - Forcing with ω_f close to ω_0
  - Discussion of amplitude eq.

Discussion

Amplitude equation with $\omega_f \neq \omega_c$:

$$\frac{d\tilde{A}}{dt} = \lambda\tilde{\delta}\tilde{A} - \mu\tilde{A}\left|\tilde{A}\right|^2 - \pi\tilde{A}\left|\tilde{E}\right|^2$$

Polar coordinates:

$$\tilde{A} = re^{i\phi}$$

Inserting this expression into the amplitude equation:

$$\frac{dr}{dt}e^{i\phi} + ri\frac{d\phi}{dt}e^{i\phi} = (\lambda_r + i\lambda_i)\tilde{\delta}re^{i\phi} - (\mu_r + i\mu_i)r^3e^{i\phi} - (\pi_r + i\pi_i)re^{i\phi}\left|\tilde{E}\right|^2$$

Hence, removing $e^{i\phi}$ and taking the real and imaginary parts:

$$\frac{d}{dt}(\ln r) = \lambda_r\tilde{\delta} - \mu_r r^2 - \pi_r\left|\tilde{E}\right|^2$$

$$\frac{d\phi}{dt} = \lambda_i\tilde{\delta} - \mu_i r^2 - \pi_i\left|\tilde{E}\right|^2$$

Discussion: $|\tilde{A}| \ll 1$ and $\tilde{E} = 0$

Amplitude equation:

$$\frac{d}{dt}(\ln r) = \lambda_r \tilde{\delta}$$
$$\frac{d\phi}{dt} = \lambda_i \tilde{\delta}$$

Hence:

$$A = e^{\lambda_r \tilde{\delta} t} e^{(i\lambda_i \tilde{\delta})}$$

Amplification rate:

$$\sigma = \lambda_r \tilde{\delta}$$
$$=> \lambda_r > 0 \text{ if } \mathrm{Re} > \mathrm{Re}_c$$

Frequency shift:

$$\frac{d\phi}{dt} = \lambda_i \tilde{\delta}$$

Discussion: $|\tilde{A}| = O(1)$ and $\tilde{E} = 0$

Amplitude equations:

$$\frac{d}{dt}(\ln r) = \lambda_r \tilde{\delta} - \mu_r r^2$$
$$\frac{d\phi}{dt} = \lambda_i \tilde{\delta} - \mu_i r^2$$

Fixed point for r if $\mu_r > 0$:

$$\frac{d}{dt}(\ln r) = 0 \Rightarrow r = \sqrt{\frac{\lambda_r}{\mu_r}\tilde{\delta}}$$

Frequency shift on limit-cycle:

Square root of $\tilde{\delta}$!

$$\frac{d\phi}{dt} = \left(\lambda_i - \mu_i \frac{\lambda_r}{\mu_r}\right)\tilde{\delta}$$

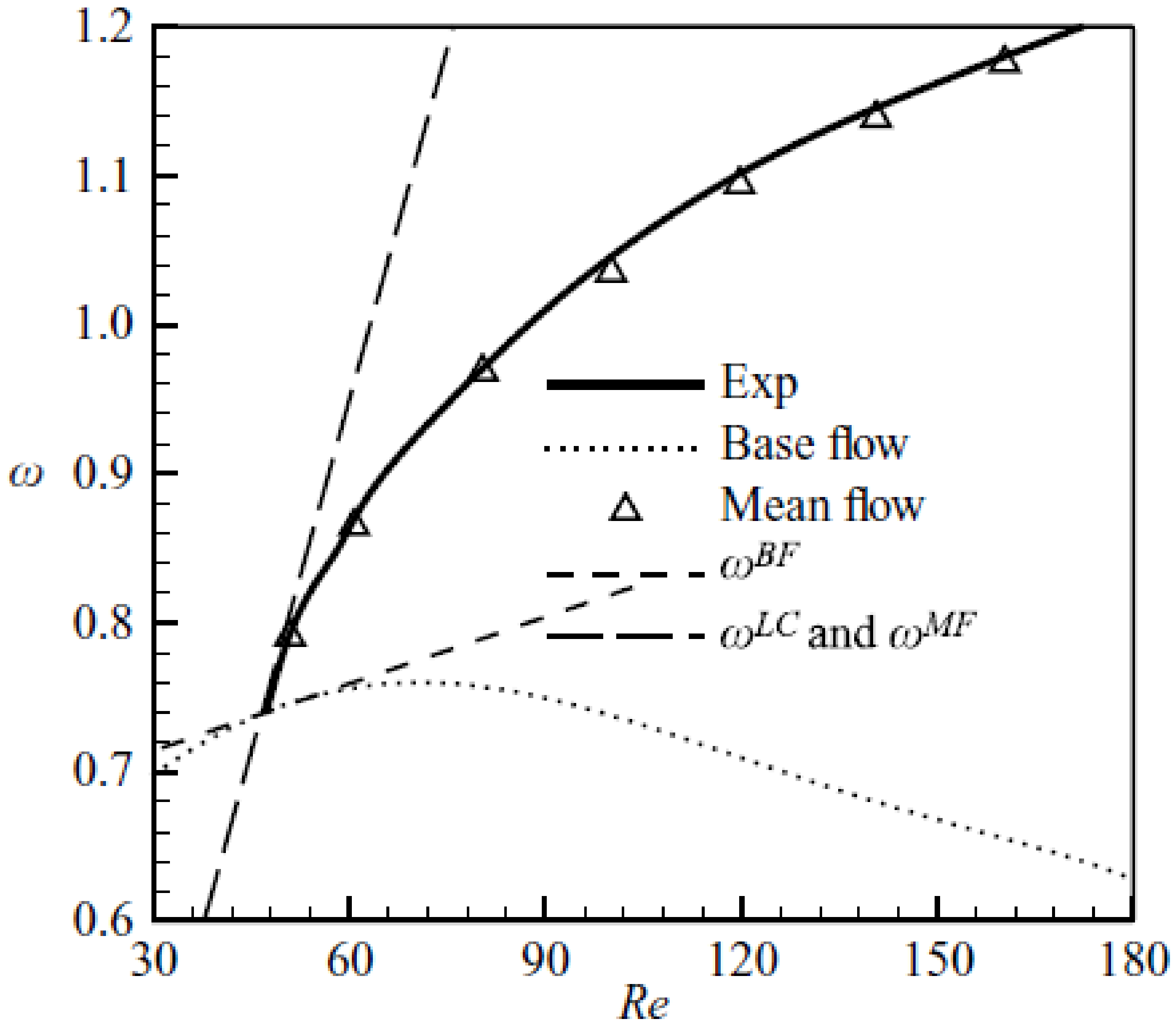
Solution on limit-cycle:

$$w = w_0 + \sqrt{\frac{\lambda_r}{\mu_r}\tilde{\delta}}\left[e^{i\left(\omega_c + \left(\lambda_i - \mu_i \frac{\lambda_r}{\mu_r}\right)\tilde{\delta}\right)t} y_A + \text{c.c}\right] + \cdots$$

Actual frequency on limit-cycle:

$$\omega = \omega_c + \underbrace{\lambda_i \tilde{\delta}}_{\text{Linear}} \quad \underbrace{-\mu_i \frac{\lambda_r}{\mu_r}\tilde{\delta}}_{\text{Non-linear interaction}}$$

Discussion: $|\tilde{A}| = O(1)$ and $\tilde{E} = 0$



Sipp & Lebedev JFM 2007

Discussion: $\tilde{E} = 0$

Where does the instability come from? $\lambda_r > 0$

$$\lambda = < \tilde{y}_A, \mathcal{M} y_A > - < \tilde{y}_A, \mathcal{N}(y_A, y_\delta)$$

Where does saturation come from? $\mu_r > 0$

$$\mu = < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) >$$

Exact solution (with w_0):

$$w = w_0 + \left(\tilde{A} e^{i\omega_c t} y_A + \text{c.c}\right) + \tilde{\delta} w_\delta + \left(\tilde{A}^2 e^{2i\omega_c t} y_{AA} + \text{c.c.}\right) + \left|\tilde{A}\right|^2 y_{A\bar{A}} + \cdots$$

Mean-flow: $\overline{w} = \lim_{T\to\infty} \frac{1}{T} \int_0^\infty w(t) dt$

$$\overline{w} = w_0 + \tilde{\delta} w_\delta + \left|\tilde{A}\right|^2 y_{A\bar{A}} + \cdots$$

Saturation does not only stem from mean-flow harmonic $\left|\tilde{A}\right|^2 y_{A\bar{A}}$ but also from second harmonic $\tilde{A}^2 e^{2i\omega_c t} y_{AA}$.

Discussion: $|\tilde{A}| \ll 1$ and $\tilde{E} > 0$

Amplitude equation:

$$\frac{d}{dt}(\ln r) = \lambda_r \tilde{\delta} - \pi_r \left|\tilde{E}\right|^2$$

$$\frac{d\phi}{dt} = \lambda_i \tilde{\delta} - \pi_i \left|\tilde{E}\right|^2$$

Amplification rate:

$$\sigma = \lambda_r \tilde{\delta} - \pi_r \left|\tilde{E}\right|^2$$

Control has stabilizing effect if $\pi_r > 0$ and destabilizing effect if $\pi_r < 0$.

Flow linearly stable if:

$$\lambda_r \tilde{\delta} - \pi_r \left|\tilde{E}\right|^2 < 0 \Rightarrow \left|\tilde{E}\right| > \sqrt{\frac{\lambda_r}{\pi_r}\tilde{\delta}}$$

Discussion: $|\tilde{A}| \ll 1$ and $\tilde{E} > 0$

Where does stabilization come from? $\pi_r > 0$

$$\pi =< \tilde{y}_A, \mathcal{N}(y_A, y_{E\bar{E}}) + \mathcal{N}(y_{\bar{E}}, y_{AE}) + \mathcal{N}(y_E, y_{A\bar{E}}) >$$

Exact solution (with w_0):

$$w = w_0 + \left(\tilde{A}e^{i\omega_c t}y_A + \text{c.c}\right) + (\tilde{E}e^{i\omega_f t}y_E + \text{c.c}) + \tilde{\delta}w_\delta + \left|\tilde{E}\right|^2 y_{E\bar{E}}$$
$$+ \left(\tilde{E}^2 e^{2i\omega_t t}y_{EE} + \text{c.c.}\right) + \left(\tilde{A}\tilde{E}e^{i(\omega_c+\omega_f)t}y_{AE} + \text{c.c.}\right)$$
$$+ \left(\tilde{A}\bar{\tilde{E}}e^{i(\omega_c-\omega_f)t}y_{A\bar{E}} + \text{c.c.}\right) + \cdots$$

Mean-flow:

$$w = w_0 + \tilde{\delta}w_\delta + \left|\tilde{E}\right|^2 y_{E\bar{E}} + \cdots$$

Stabilization is not only due to mean-flow harmonic $\left|\tilde{E}\right|^2 y_{E\bar{E}}$ but also to other harmonics $\tilde{A}\tilde{E}e^{i(\omega_c+\omega_f)t}y_{AE}$ and $\tilde{A}\bar{\tilde{E}}e^{i(\omega_c-\omega_f)t}y_{A\bar{E}}$.

(Q5)