Feedback control

J.B. Burl. Linear Optimal Control: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ methods. Addison-Wesley, 1998.

Outline

- State-feedback
  - Physical system
  - Controller
  - Closed-loop system
  - Stability
  - Performance
- Observer feedback
  - Physical system
  - Compensator
  - Dynamic observer
  - Closed-loop system
  - Stability
  - Performance
- Frequency domain
  - Laplace transform
  - Physical system
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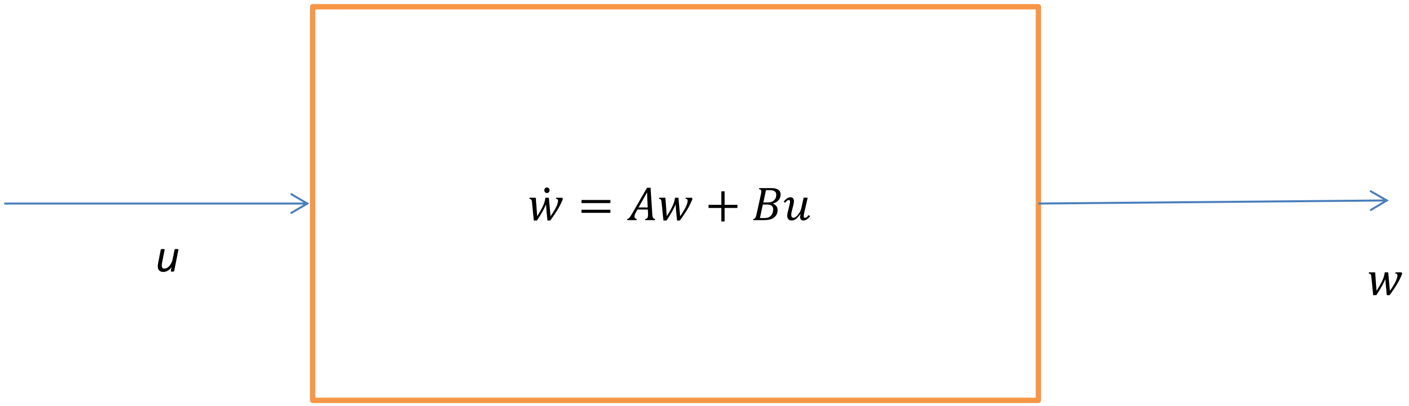
  - Closed-loop system
  - Stability
  - Performance
  - Robustness

State feedback
Physical system

Reduced-order model of dynamics:

$$\dot{w} = Aw + Bu$$

Input: u
Output: w



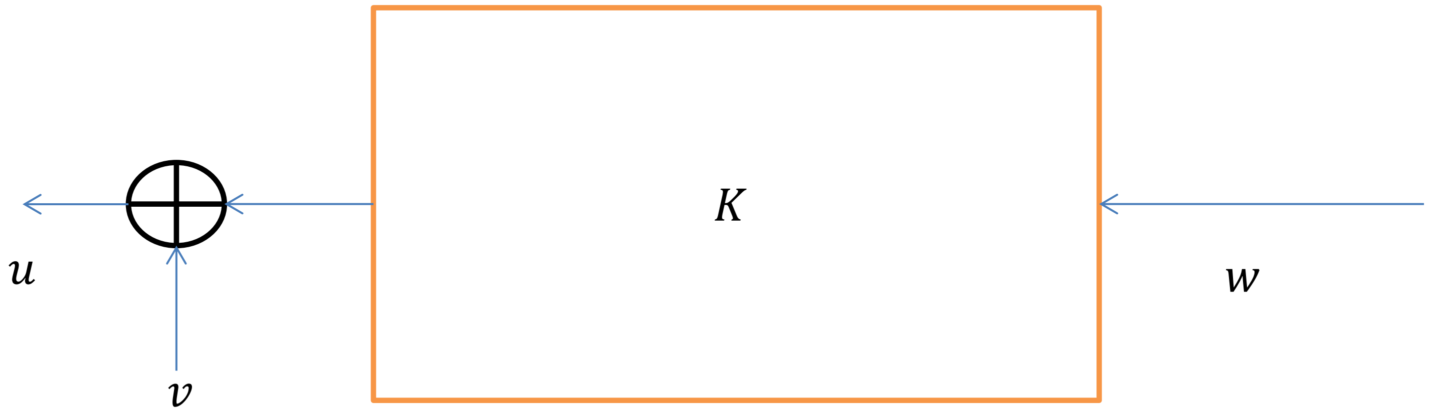
State feedback
Controller

The **controller** is a system that takes the state w and provides a control law u:

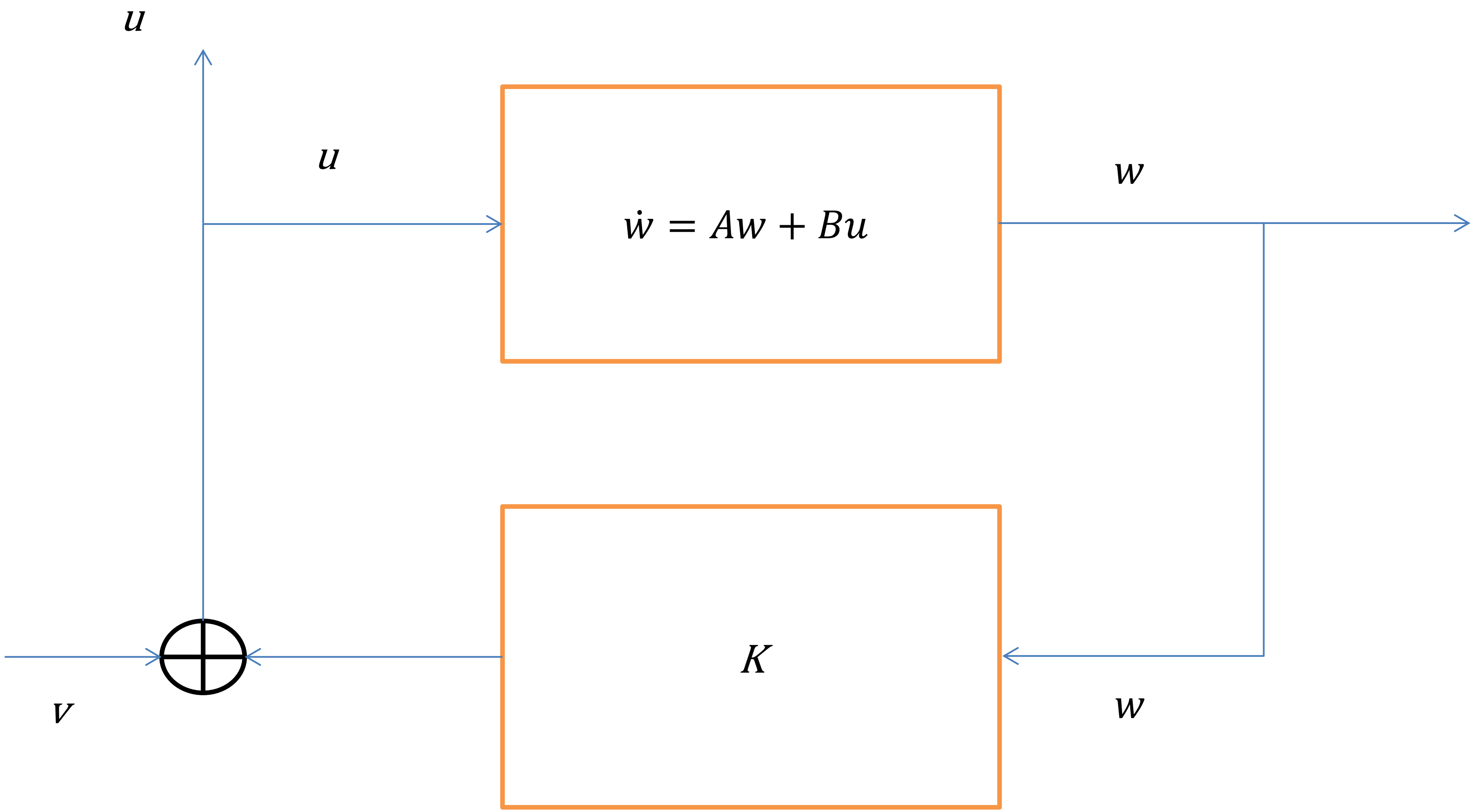
$$u = Kw + v$$

Inputs: the state w and the actuator noise v

Output: the control law u



State feedback
Closed-loop system



State feedback
Stability

Governing equation with **state feedback**:

$$\begin{cases} \dot{w} = Aw + Bu \\ u = Kw + v \text{ (Noisy actuator)} \end{cases}$$

Input: actuator noise v.

Hence:

$$\dot{w} = Aw + BKw + Bv = (A + BK)w + Bv$$

K is chosen so that $\boldsymbol{A + BK}$ **is stable**.

$$w(t) = \int_0^t e^{(A+BK)(t-\tau)} Bv(\tau) d\tau$$

State feedback
Performance

If performance is assessed by the measurement $y = Cw$, then:

$$y(t) = \int_0^t Ce^{(A+BK)(t-\tau)} Bv(\tau) d\tau.$$

In presence of white-noise v, the standard deviation of y, $\sqrt{E(y^2)}$, is proportional to the 2-norm of the closed-loop impulse function: $\|\mathrm{Z}^{\mathrm{cl}}(\mathrm{t})\|_2 = \sqrt{\int_0^\infty |Z^{cl}(t)|^2 dt}$, where $\mathrm{Z}^{\mathrm{cl}}(\mathrm{t}) = Ce^{(A+BK)t}B$.

<u>Standard deviation of an output signal.</u> Let us consider a stable system:

$$\dot{w} = A'w + B'v$$
$$y = C'w$$

If v is white-noise characterized by a PSD (Power Spectral Density) S, then the standard deviation of the output y is equal to:

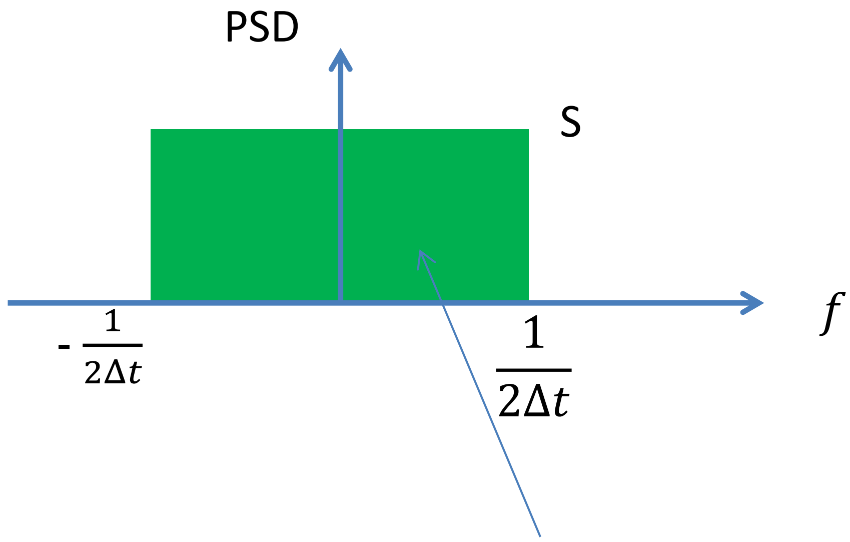
$$\sqrt{E(y^2)} = \|Z'(\mathrm{t})\|_2 \sqrt{S}$$

where $Z'(t) = C'e^{A't}B'$ is the impulse response of the system.

State feedback
Performance

<u>Link between PSD, sampling time and variance of white noise</u> : If Δt is the sampling time, then the variance of the white-noise is:

$$E(v^2) = \frac{S}{\Delta t}$$



Variance of signal is green area (Parseval)

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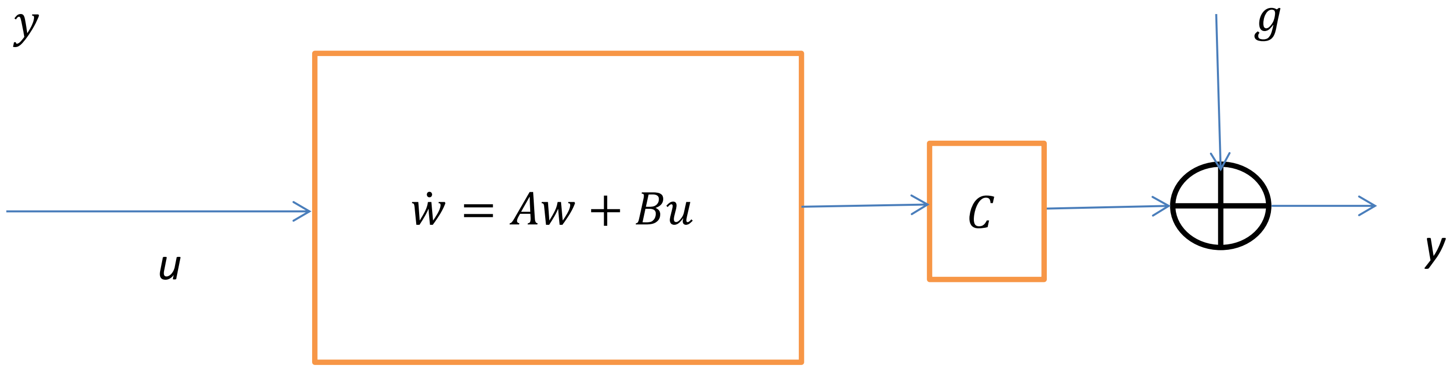
Observer feedback
Physical system

In fluid systems, state feedback is not realistic since w is unknown. Can we apply K on an estimate w_e of w (using the measurement y) ?

Reduced-order model of input-output dynamics

$$\begin{cases} \dot{w} = Aw + Bu \\ y = Cw + g \end{cases}$$

Inputs: (u, g)
Output: y



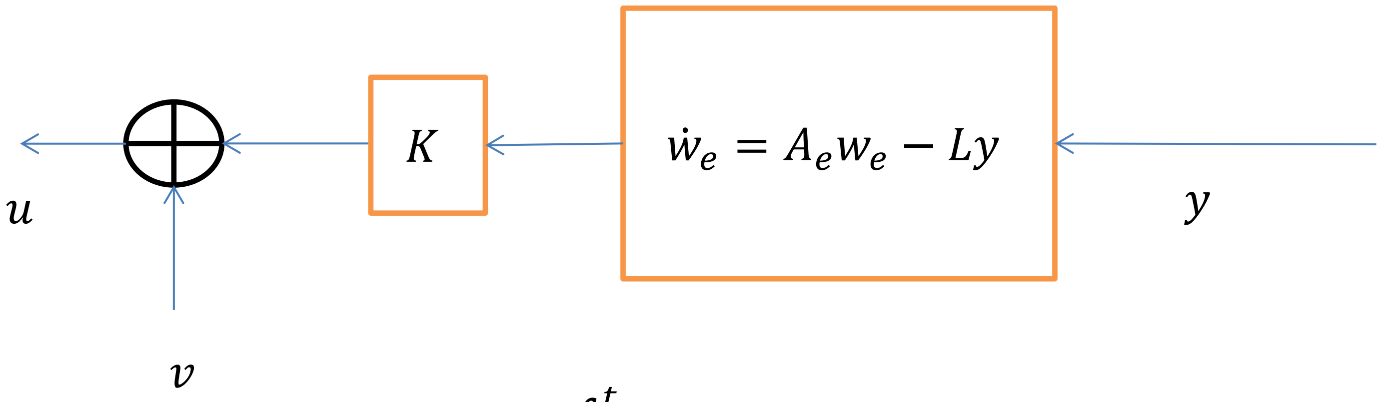
$$y(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + g(t)$$

Observer feedback Compensator

The **compensator** is a system that takes the measurement y and provides a control law u. The controller may be represented as a linear input-output system:

$$\begin{cases} \dot{w}_e = A_e w_e - Ly \\ u = K w_e + v \end{cases}$$

Inputs: the measurement y and the actuator noise v
Output: the control law u



$$u(t) = -\int_0^t K e^{A_e(t-\tau)} L y(\tau) d\tau + v(t)$$

Observer feedback
Dynamic observer

How to choose A_e and L?
Equation governing system with control input:

$$\begin{cases} \dot{w} = Aw + Bu \\ u = Kw_e + v \text{ (Noisy actuator)} \\ y = Cw + g \text{ (Noisy sensor)} \end{cases}$$

Hence:

$$\dot{w} = Aw + BKw_e + Bv$$

Governing equation of **dynamic observer**: We replace unknown term Bv by a forcing term $-L(y - y_e)$ proportional to the measurement error:

$$\dot{w}_e = Aw_e + BKw_e - L(y - y_e)$$
$$y_e = Cw_e$$

Hence:

$$\dot{w}_e = (A + BK + LC)w_e - Ly$$

Observer feedback
Dynamic observer

The error $e = w - w_e$ in the state reconstruction is governed by:

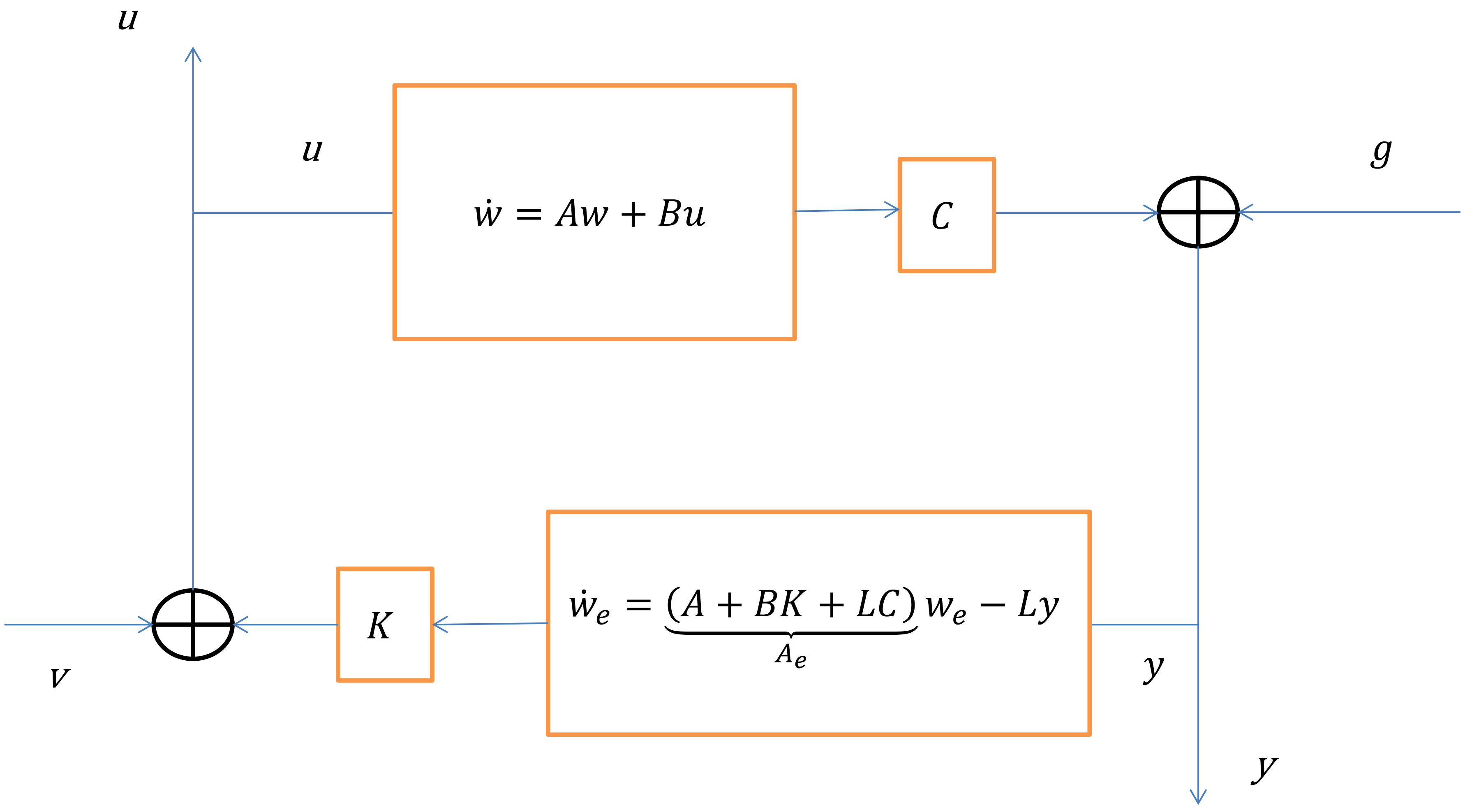
$$\dot{e} = \dot{w} - \dot{w}_e = Aw + BKw_e + Bv - Aw_e - BKw_e + L(Cw + g - Cw_e)$$
$$= (A + LC)e + (B \quad L)\begin{pmatrix} v \\ g \end{pmatrix}$$

We choose L so that $\boldsymbol{A + LC}$ **is stable**.

$$e(t) = \int_0^t e^{(A+LC)(t-\tau)}[Bv(\tau) + Lg(\tau)]d\tau$$

Hence, error e is weak in presence of noises v and g.

Observer feedback
Closed-loop system



Observer feedback
Closed-loop system

The full coupled system is governed by:

$$\begin{cases} \dot{w} = Aw + Bu \text{ (System dynamics)} \\ \dot{w}_e = (A + BK + LC)w_e - Ly \text{ (Estimator dynamics)} \\ u = Kw_e + v \text{ (Noisy actuator)} \\ y = Cw + g \text{ (Noisy measurement)} \end{cases}$$

Inputs: actuator noise v, measurement noise g
Outputs: measurement y, actuator signal u

In matrix form:

$$\begin{pmatrix} \dot{w} \\ w_e \end{pmatrix} = \overbrace{\begin{pmatrix} A & BK \\ -LC & A + BK + LC \end{pmatrix}}^{A_{cl}} \begin{pmatrix} w \\ w_e \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & -L \end{pmatrix} \begin{pmatrix} v \\ g \end{pmatrix}$$

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} 0 & K \\ C & 0 \end{pmatrix} \begin{pmatrix} w \\ w_e \end{pmatrix} + \begin{pmatrix} v \\ g \end{pmatrix}$$

Observer feedback
Stability

The dynamics of the coupled system can be analysed by introducing $e = w - w_e$:

$$\begin{cases} \dot{w} = Aw + BKw_e + Bv = Aw + BK(w - e) + Bv = (A + BK)w - BKe + Bv \\ \dot{e} = (A + LC)e + Bv + Lg \end{cases}$$

Hence:

$$\begin{pmatrix} \dot{w} \\ e \end{pmatrix} = \begin{pmatrix} A + BK & -BK \\ 0 & A + LC \end{pmatrix} \begin{pmatrix} w \\ e \end{pmatrix} + \begin{pmatrix} B & 0 \\ B & L \end{pmatrix} \begin{pmatrix} v \\ g \end{pmatrix}$$

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} K & -K \\ C & 0 \end{pmatrix} \begin{pmatrix} w \\ e \end{pmatrix} + \begin{pmatrix} v \\ g \end{pmatrix}$$

Eigenvalues of coupled system are those of $A + BK$ and $A + LC$, which by design of K and L, exhibit negative real parts.

Observer feedback
Stability

The compensator is given by:

$$\dot{w}_e = (A + BK + LC)w_e - Ly$$
$$u = Kw_e + v$$

Note that $A + BK + LC$ is not necessarily stable. Only $A + BK$ and $A + LC$ are stable.

Observer feedback
Performance

The performance of the compensator is best when the 2-norm of the closed-loop impulse response is weak. For example, from v to y, this impulse response is:

$$Z^{cl}(t) = \begin{pmatrix} C & 0 \end{pmatrix} \exp\left[\begin{pmatrix} A & BK \\ -LC & A + BK + LC \end{pmatrix} t\right] \begin{pmatrix} B \\ 0 \end{pmatrix}$$

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Laplace transform

Laplace transform:

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t) dt$$

with $s = \sigma + i\omega$. $\hat{f}(s)$ can be evaluated only if $\Re(s)$ is sufficiently large.

Laplace transform is analogous to Fourier transform, but also holds for unbounded functions.

Inverse Laplace –transform:

$$f(t) = \frac{1}{2\pi i} \lim_{\Omega \to \infty} \int_{\gamma - i\Omega}^{\gamma + i\Omega} e^{st} \hat{f}(s) ds$$

where γ is chosen to the right of all poles of $\hat{f}(s)$ (causality condition).

Laplace transform

Some useful properties:

1/ $\widehat{af+bg} = a\hat{f} + b\hat{g}$

2/ $\widehat{f'}(s) = \int_0^{+\infty} e^{-st} f'(t)dt = [e^{-st}f(t)]_0^{\infty} - \int_0^{\infty} -se^{-st} f(t)dt = \hat{f}(s) - f(0)$

3/ $\widehat{H(t)e^{at}}(s) = \int_0^{\infty} e^{-st}e^{at}dt = \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^{\infty} = \frac{1}{s-a}$ for $s_r > a_r$

$H(t)$ is the Heaviside step function

4/ $\widehat{f*g}(s) = \hat{f}(s)\hat{g}(s),\ (f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$

5/ If $g(t) = 0$ for $0 \le t < \tau$ and $g(t) = f(t-\tau)$ for $t \ge \tau$:

$$\hat{g}(s) = \int_0^{\infty} e^{-st} g(t)dt = \int_{\tau}^{\infty} e^{-st} g(t)dt = \int_{\tau}^{\infty} e^{-st} f(t-\tau)dt$$

$$= e^{-s\tau}\int_0^{\infty} e^{-st'} f(t')dt' = e^{-s\tau}\hat{f}(s) \Rightarrow \arg\hat{g}\,(i\omega) = \arg\hat{f} - \tau\omega$$

Frequency space
Physical system

Performing a **Laplace-transform** of the governing equations:

$$\begin{cases} \dot{w} = Aw + Bu \\ y = Cw + g \text{ (Noisy sensor} \end{cases}$$

Yields:

$$\begin{cases} s\hat{w} - w(0) = A\hat{w} + B\hat{u} \\ \hat{y} = C\hat{w} + \hat{g} \end{cases}$$

Hence:

$$(sI - A)\hat{w} = w(0) + B\hat{u}$$

$$\Rightarrow \hat{y} = C(sI - A)^{-1}w(0) + C(sI - A)^{-1}B\hat{u} + \hat{g}$$

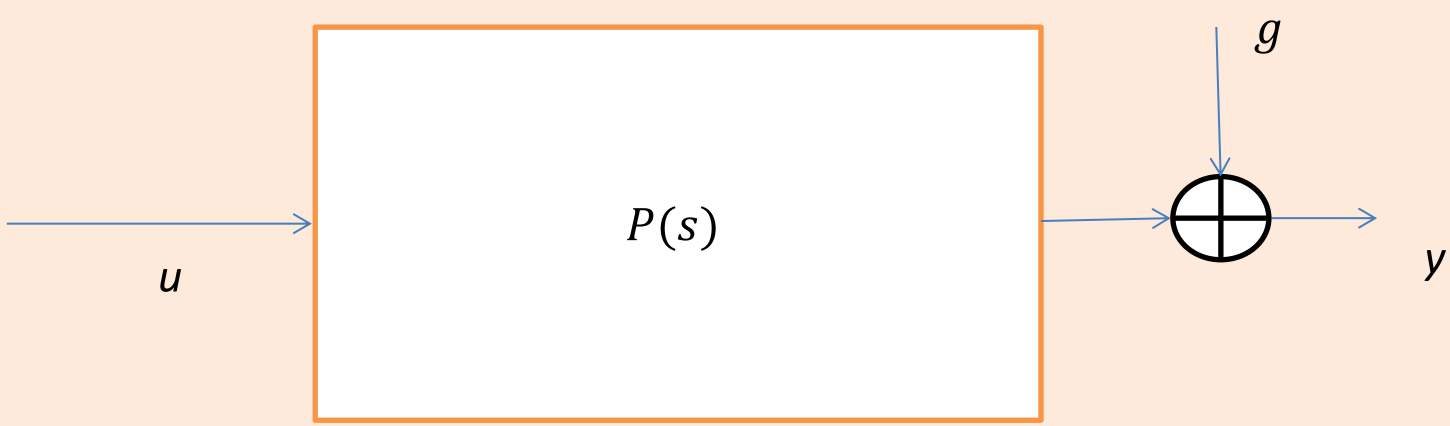
With $w(0) = 0$:

$$\hat{y} = P(s)\hat{u} + \hat{g}$$

where:

$$P(s) = C(sI - A)^{-1}B$$

Frequency space
Physical system



$$P(s) = C(sI - A)^{-1}B$$

Frequency space
Physical system

The **bode plot** of $P(s)$ presents $|P(i\omega)|$ and $\arg P(i\omega)$ as a function of frequency ω.

Introducing the adjugate adj(), the transfer function can be rewritten as:

$$P(s) = \frac{C\mathrm{adj}(sI - A)B}{\det(sI - A)} = \frac{\mathrm{num(s)}}{\mathrm{den(s)}}$$

The **poles** of $P(s)$ are defined as the zeros of $\mathrm{den(s)}$ and correspond to the eigenvalues of A: $A\hat{w} = s\hat{w} \Rightarrow \det(sI - A) = 0$.

The **zeros** of $P(s)$ are the zeros of $\mathrm{num}(s)$.

Adjugate

Theorem:

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}(A)$$

where $\operatorname{adj}(A)$ is the transpose of the matrix of co-factors. For a matrix of order n, the cofactor $A_{i,j}$ is defined as the determinant of the square matrix of order (n-1) obtained from A by removing the row number i and the column number j multiplied by $(-1)^{i+j}$.

Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^{*} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Frequency space
Compensator

Taking the Laplace tranform of the equations governing the compenstator:

$$\dot{w}_e = (A + BK + LC)w_e - Ly$$
$$u = Kw_e + v$$

we obtain:

$$s\widehat{w}_e - w_e(0) = (A + BK + LC)\widehat{w}_e - L\hat{y}$$
$$\hat{u} = K\widehat{w}_e + \hat{v}$$

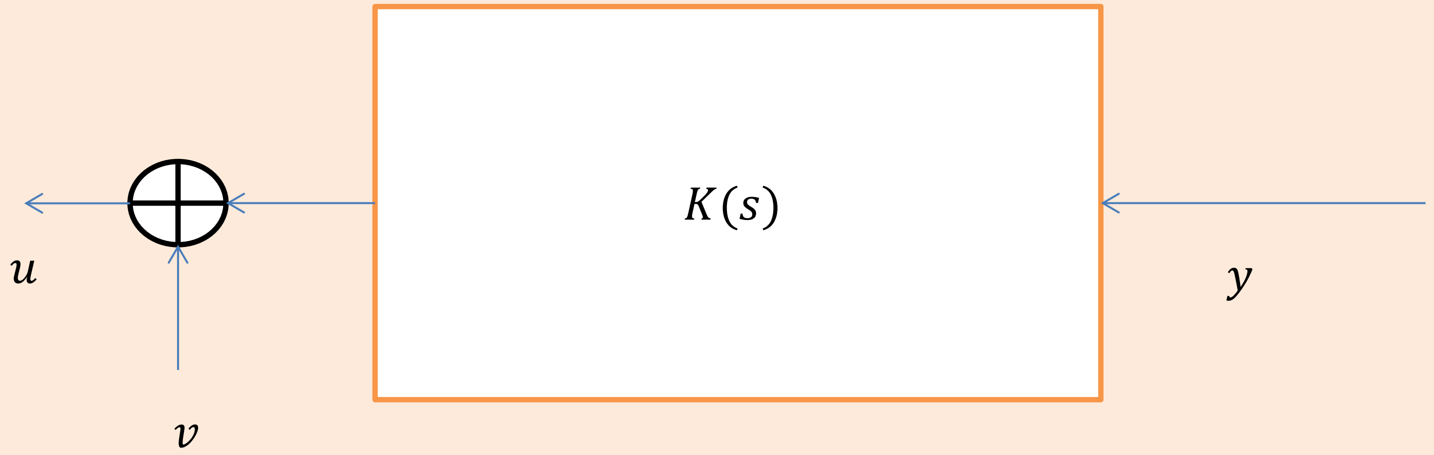
Hence, eliminating $\widehat{w}_e$ and setting $w_e(0) = 0$:

$$\hat{u} = K(s)\hat{y} + \hat{v}$$

where:

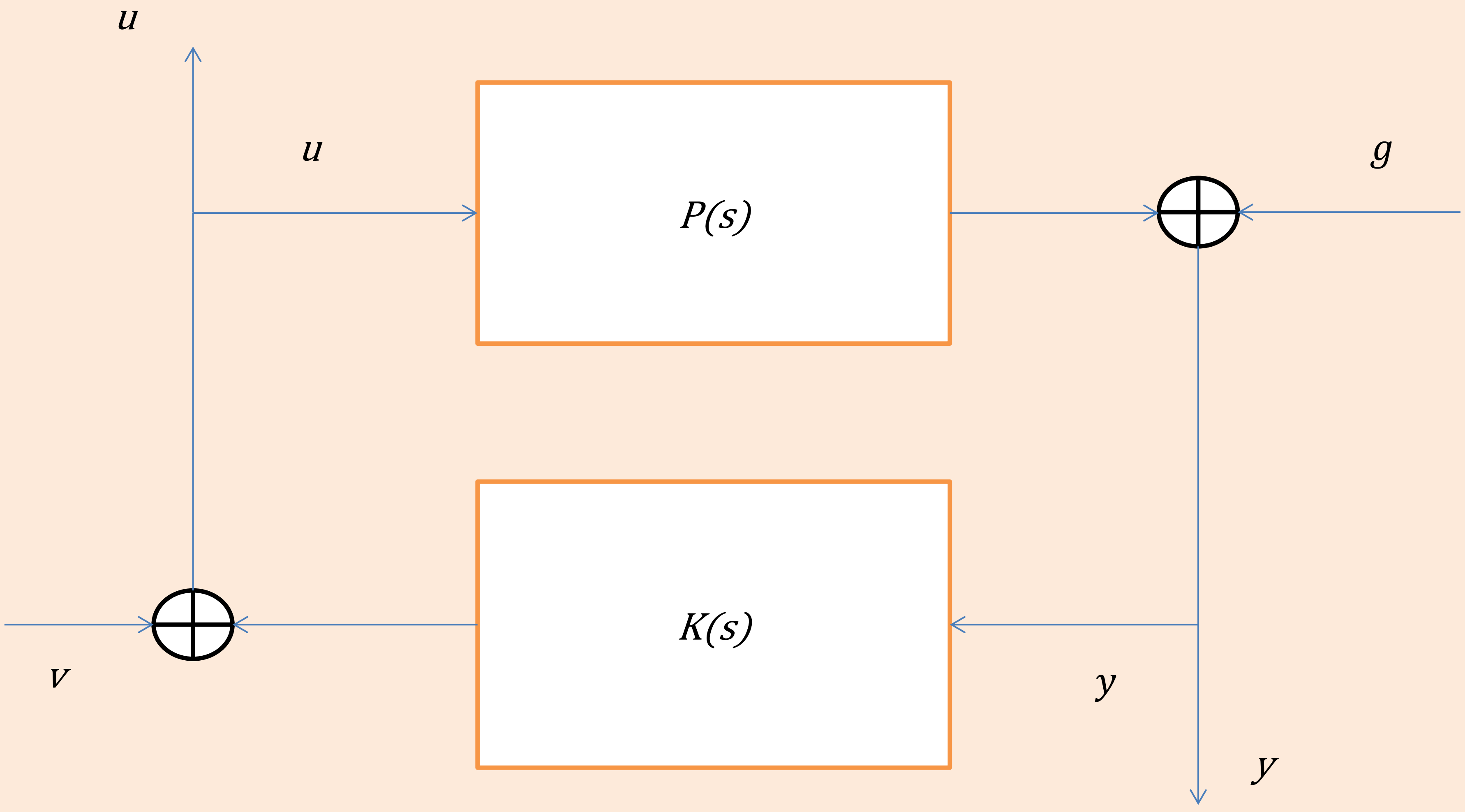
$$K(s) = -K\big(sI - (A + BK + LC)\big)^{-1}L$$

Frequency space Compensator



$$K(s) = -K\left(sI - (A + BK + LC)\right)^{-1} L$$

Frequency space
Closed-loop system



Frequency space
Stability

Closed-loop system is governed by (considering zero initial conditions):

$$\hat{y} = P(s)\hat{u} + \hat{g}$$
$$\hat{u} = K(s)\hat{y} + \hat{v}$$

If we eliminate $\hat{u}$, we have:

$$\hat{y} = P(s)K(s)\hat{y} + P(s)\hat{v} + \hat{g}$$

The **closed-loop transfer-functions** from $(g, v\ )$ to (y, u) may be obtained from:

$$\hat{y} = \overbrace{\frac{P(s)}{1 - P(s)K(s)}}^{T^{cl}_{yv}(s)} \hat{v} + \overbrace{\frac{1}{1 - P(s)K(s)}}^{T^{cl}_{yg}(s)} \hat{g}$$
$$\hat{u} = \underbrace{\frac{1}{1 - P(s)K(s)}}_{T^{cl}_{uv}(s)} \hat{v} + \underbrace{\frac{K(s)}{1 - P(s)K(s)}}_{T^{cl}_{ug}(s)} \hat{g}$$

The **stability of the closed-loop system** is assessed by scrutinizing the poles of the closed-loop transfer-functions. The compensator $K(s)$ is designed to stabilize all of them!

Frequency space
Stability

<u>Theorem:</u>
The poles of all closed-loop transfer-functions correspond to the zeros of $1 - P(s)K(s)$.

<u>Proof:</u>
Poles of $T_{yv}^{cl}(s) = \frac{P(s)}{1-P(s)K(s)}$: the poles of $P(s)$ in the numerator are cancelled by the poles of $P(s)$ in the denominator. Note also, that the poles of $K(s)$ in the denominator become zeros of the Transfer-function.

Poles of $T_{yg}^{cl}(s) = T_{uv}^{cl}(s) = \frac{1}{1-P(s)K(s)}$. The poles of $P(s)$ and $K(s)$ become zeros of the transfer-function.

Poles of $T_{ug}^{cl}(s) = \frac{K(s)}{1-P(s)K(s)}$: the poles of $K(s)$ in the numerator are cancelled by the poles of $K(s)$ in the denominator. Note also, that the poles of $P(s)$ in the denominator become zeros of the Transfer-function.

Frequency space
Performance

If the closed-loop system is stable, then the **performance** achieved by the compensator is given by the closed-loop transfer functions from v and g to y:

$$T_{yv}^{cl} = \frac{P(s)}{1 - P(s)K(s)}$$

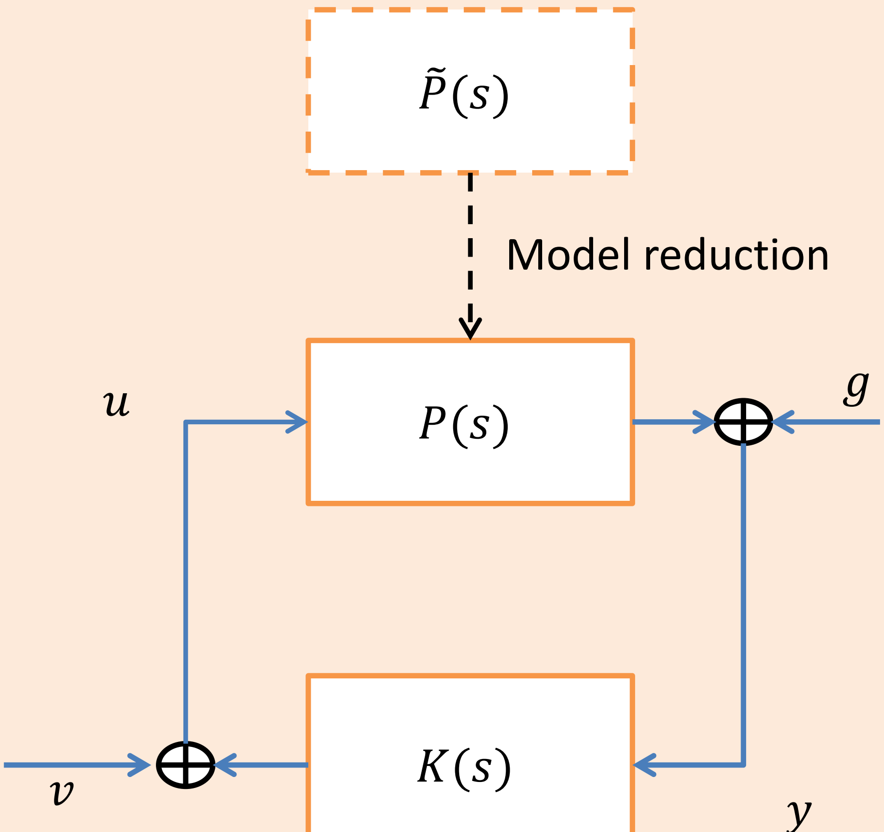
$$T_{yg}^{cl} = \frac{1}{1 - P(s)K(s)}$$

The **control-cost** and the **operating conditions** of the compensator are given by the closed-loop transfer functions from v and g to $\hat{u} = K(s)\hat{y} + \hat{v}$:

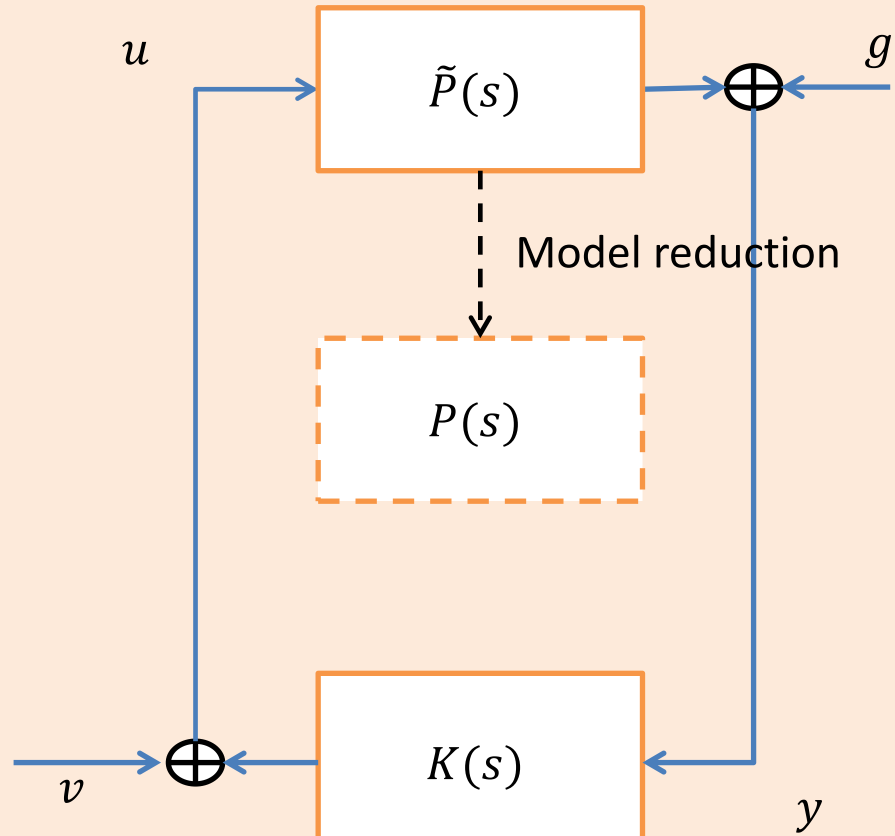
$$T_{uv}^{cl}(s) = \frac{1}{1 - P(s)K(s)}$$

$$T_{ug}^{cl}(s) = \frac{K(s)}{1 - P(s)K(s)}$$

Robustness



Stable by design
Nominal performances



Stable ?
Performance ?

Robustness

The compensator $K(s)$ has been designed based on an estimate of the transfer function: $P(s)$. Yet, the real system may exhibit a slightly different transfer-function: $\tilde{P}(s)$.

Nominal performance (performance based on $P(s)$) is expected when $\tilde{P}(s) = P(s)$.
When $\tilde{P}(s) \neq P(s)$, the actual closed-loop transfer-functions read:

$$\tilde{T}_{uv}^{cl}(s) = \frac{\tilde{P}(s)}{1 - \tilde{P}(s)K(s)}, \qquad \tilde{T}_{ug}^{cl}(s) = \frac{1}{1 - \tilde{P}(s)K(s)}$$

$$\tilde{T}_{uv}^{cl}(s) = \frac{1}{1 - \tilde{P}(s)K(s)}, \qquad \tilde{T}_{ug}^{cl}(s) = \frac{K(s)}{1 - \tilde{P}(s)K(s)}$$

Three things may happen:
1/ The actual closed-loop system is stable and exhibits the nominal performances (expected situation)
2/ The actual closed-loop system is stable but displays weak performance (bad situation)
3/ The actual closed-loop system is unstable: there exists one zero of $1 - \tilde{P}(s)K(s)$ which displays a positive real part (catastrophic situation)

Robust controllers

A compensator $K(s)$ displays good **<u>stability robustness</u>** properties if it stabilizes the closed-loop system for systems $\tilde{P}(s)$ departing significantly from $P(s)$.

A compensator $K(s)$ displays good **<u>performance robustness</u>** properties if it stabilizes the closed-loop system and exhibits nominal performance for systems $\tilde{P}(s)$ departing significantly from $P(s)$.

Stability robustness analysis

The nominal closed-loop system is stable: the solutions of $1 - P(s)K(s) = 0$ all exihibt negative real parts.

We test the stability of the closed-loop system for two families of perturbed transfer functions:

$$\tilde{P}_g(s) = gP(s)$$
$$\tilde{P}_\phi(s) = e^{i\phi}P(s)$$

where g and ϕ are real numbers.

Physical interpretation:
- $\tilde{P}_g(s)$ represents an error in the estimate of the growth rate of the instabilities between u and y.
- $\tilde{P}_\phi(s)$ represents an error in the estimate of the group velocity of the instabilities between u and y. Note that a time delay is something more complex than just a constant phase-shift!

Stability robustness analysis

For $g = 1$ and $\phi = 0$, we have: $\tilde{P}_g = \tilde{P}_\phi(s) = P(s)$ and the closed-loop system is stable.

We now look for critical parameters g and ϕ which achieve marginal stability, i.e.: there exists $s = i\omega$ such that $1 - \tilde{P}_g(s)K(s) = 0$ or $1 - \tilde{P}_\phi(s)K(s) = 0$.
The system is therefore at the threshold of instability.

Definitions:
1/ the gain margin g^+ is defined as the smallest gain $\mathrm{g} > 1$, which achieves marginal stability.
2/ the downside gain margin g^- is the smallest gain $0 < \mathrm{g} < 1,$ which achieves marginal stability.
3/ ϕ^+ is the smallest positive phase shift, which achieves marginal stability.

Note: If $1 - e^{i\phi^+}P(i\omega)K(i\omega) = 0$, then $1 - e^{-i\phi^+}P(-i\omega)K(-i\omega) = 0$ since $P(s)$ and $K(s)$ are polynomials of s with real constants (the matrices (A, B, C) which define $P(s)$ and $K(s)$ are real). Hence, $\phi^- = -\phi^+$.

Stability robustness analysis

A compensator $K(s)$ displays good stability robustness if
1/ g^+ is large, say $a^+ \geq 2$
2/ g^- is small, say $a^- \leq 0.5$
3/ ϕ^+ is large, say $\phi^+ \geq 30^\circ$