Linear Quadratic Guassian control (LQG)

How to straightforwardly compute K and L ?

Outline

- Control gain: K
- Kalman gain: L

State-feedback
Linear Quadratic Regulator (LQR)

Input-output dynamics:

$$\dot{w} = Aw + Bu$$
$$y = Cw$$
$$w(0) = w_I$$

Find $u(t)$ to minimize the following cost functional:

$$\mathfrak{J}(u) = \frac{1}{2}\int_0^T (y^2 + l^2u^2)dt$$

$$\mathfrak{J}(u) = \mathfrak{J}(w,u) = \frac{1}{2}\int_0^T (w^*Qw + u^*Ru)dt$$

Target specified by: $Q = C^*C$
Control cost specified by: $R = l^2$ (l large implies small u)

State-feedback
Lagrangian framework

Governing equation:

$$\dot{w} = Aw + Bu$$
$$y = Cw$$
$$w(0) = w_I$$

State: $w(t)$
Control: $u(t)$
Cost functional to be minimized (scalar):

$$\Im(u) = \Im(w,u) = \frac{1}{2}\int_0^T (w^* Q w + u^* R u)dt$$

Constraints :

$$F(w,u) = \dot{w} - (Aw + Bu) = 0$$
$$G(w) = w(0) - w_I$$

Scalar-product to express gradient: $\delta\Im = < \frac{\mathrm{d}\Im}{du}, \delta u >$

$$\langle a,b\rangle = \int_0^T (a^* b)dt$$

Linear Quadratic Regulator Lagrangian framework

Lagrangian:

$$\mathcal{L}(w,u,\widetilde{w}) = \Im(w,u) - \langle \widetilde{w}, F(w,u) \rangle = \frac{1}{2}\int_0^T (w^*Qw + u^*Ru)dt - \int_0^T \widetilde{w}^*(\dot{w} - (Aw+Bu))dt$$

Variation with respect to the state:

$$\lim_{\varepsilon \to 0} \frac{\mathcal{L}(w+\varepsilon\delta w, u, \widetilde{w}) - \mathcal{L}(w,u,\widetilde{w})}{\varepsilon} =$$

$$\lim_{\varepsilon \to 0} \frac{\begin{array}{c}\int_0^T \frac{1}{2}\left((w+\varepsilon\delta w)^* Q (w+\varepsilon\delta w) + u^*Ru\right)dt - \int_0^T \widetilde{w}^*\left(\dot{w} + \varepsilon\dot{\delta w} - (Aw + \varepsilon A\delta w + Bu)\right)dt \\ - \int_0^T (w^*Qw + u^*Ru)dt + \int_0^T \widetilde{w}^*(\dot{w} - (Aw+Bu))dt\end{array}}{\varepsilon}$$

$$= \int_0^T \frac{1}{2}(w^*Q\delta w + \delta w^* Q w)dt - \int_0^T \widetilde{w}^*\left(\dot{\delta w} - A\delta w\right)dt$$

$$= \int_0^T w^*Q\delta w dt - \int_0^T \left(\widetilde{w}^*\dot{\delta w} - \widetilde{w}^* A\delta w\right)dt$$

Linear Quadratic Regulator
Lagrangian framework

$$= \int_0^T [Qw + A^*\widetilde{w}]^* \delta w dt - [\widetilde{w}^* \delta w]_0^T + \int_0^T \left(\dot{\widetilde{w}}^* \delta w\right) dt$$

$$= \int_0^T \left[Qw + A^*\widetilde{w} + \dot{\widetilde{w}}\right]^* \delta w dt - \left(\widetilde{w}(T)^* \delta w(T) - \widetilde{w}(0)^* \delta w(0)\right)$$

Kill boundary term:
a/ $\delta w(0)$=0 since $w(0) = w_I$ is kept fixed ($w(0) + \delta w(0) = w_I$)

b/ We choose $\widetilde{w}(T) = 0$ to kill the boundary term at time T (since there is no condition
for $w(T)$, $\delta w(T)$ is not necessarily zero)

The adjoint is:

$$\dot{\widetilde{w}}(t) = -A^*\widetilde{w}(t) - Qw(t)$$
$$\widetilde{w}(T) = 0$$

Linear Quadratic Regulator Lagrangian framework

Lagrangian:

$$\mathcal{L}(w,u,\widetilde{w}) = \frac{1}{2}\int_0^T (w^* Q w + u^* R u)dt - \int_0^T \widetilde{w}^*(\dot{w} - (Aw + Bu))dt$$

Variation with respect to the control:

$$\lim_{\varepsilon\to 0}\frac{\mathcal{L}(w,u+\varepsilon\delta u,\widetilde{w}) - \mathcal{L}(w,u,\widetilde{w})}{\varepsilon} =$$

$$\lim_{\varepsilon\to 0}\frac{\begin{array}{c}\int_0^T \frac{1}{2}(w^* Q w + (u+\varepsilon\delta u)^* R (u+\varepsilon\delta u))dt - \int_0^T \widetilde{w}^*\left(\dot{w} - \left(Aw + B(u+\varepsilon\delta u)\right)\right)dt \\ -\int_0^T \frac{1}{2}(w^* Q w + u^* R u)dt + \int_0^T \widetilde{w}^*(\dot{w} - (Aw + Bu))dt\end{array}}{\varepsilon} =$$

$$= \int_0^T \frac{1}{2}(u^* R\delta u + \delta u^* R u)dt + \int_0^T \widetilde{w}^* B\delta u dt = \int_0^T (Ru + B^*\widetilde{w})^*\delta u dt$$

$$\Rightarrow \frac{d\mathfrak{I}}{du} = \frac{\partial\mathcal{L}}{\partial u} = Ru + B^*\widetilde{w}$$

Linear Quadratic Regulator Lagrangian framework

<u>Conlusion:</u>
If the system is governed by

$$\dot{w}(t) = Aw(t) + Bu(t)$$

then the gradient of the cost functional can be obtained as:

$$\frac{d\Im}{du} = Ru + B^*\widetilde{w}$$

where:

$$\dot{\widetilde{w}}(t) = -A^*\widetilde{w}(t) - Qw(t)$$
$$\widetilde{w}(T) = 0$$

State-feedback
Optimal control

The optimal control law is obtained when:

$$\frac{d\mathfrak{I}}{du} = 0$$

This yields the following system for the determination of the optimal $u(t)$:

$$\dot{w}(t) = Aw(t) + Bu(t)$$
$$\dot{\widetilde{w}}(t) = -A^{*}\widetilde{w}(t) - Qw(t)$$
$$\widetilde{w}(T) = 0$$
$$Ru(t) + B^{*}\widetilde{w}(t) = 0$$

Eliminating $u(t)$, this can be rewritten as:

$$\dot{w}(t) = Aw(t) - BR^{-1}B^{*}\widetilde{w}(t)$$
$$\dot{\widetilde{w}}(t) = -A^{*}\widetilde{w}(t) - Qw(t)$$
$$\widetilde{w}(T) = 0$$

State-feedback

Let us try:

$$\widetilde{w}(t) = P(t)w(t)$$

Then:

$$\dot{\widetilde{w}}(t) = \dot{P}(t)w(t) + P(t)\dot{w}(t)$$
$$\Rightarrow -A^*\widetilde{w}(t) - Qw(t) = \dot{P}(t)w(t) + P(t)\big(Aw(t) - BR^{-1}B^*\widetilde{w}(t)\big)$$
$$\Rightarrow -A^*P(t)w(t) - Qw(t) = \dot{P}(t)w(t) + P(t)\big(Aw(t) - BR^{-1}B^*P(t)w(t)\big)$$

Should be valid for all $w(t)$:

$$\dot{P}(\mathrm{t}) = -A^*P(t) - P(t)A + P(t)BR^{-1}B^*P(t) - Q$$

The condition $\widetilde{w}(T) = 0$, yields the following final condition for P:

$$P(T) = 0$$

Hence:

$$u = -R^{-1}B^*\widetilde{w}(t) = \underbrace{-R^{-1}B^*P(t)}_{K(t)}w(t)$$

State-feedback

<u>Conclusion:</u>
The optimal control law is obtained through:

$$u = Kw(t), \qquad K = -R^{-1}B^*P(t)$$

where $P(t)$ is a solution of the following Riccati equation:

$$\dot{P}(t) = -A^*P(t) - P(t)A + P(t)BR^{-1}B^*P(t) - Q$$
$$P(T) = 0$$

<u>Steady-state controller:</u>
The steady-state controller obtained by setting $\dot{P}(t) = 0$ is optimal for $T \to \infty$. In this case, P verifies the algebraic Riccati equation (care in Octave):

$$A^*P + PA - PBR^{-1}B^*P + Q = 0$$

In the case of a **discrete in time system**,

$$K = -(R + B^*PB)^{-1}B^*PA$$

where P is the algebraic Riccati equation (dare in Octave):

$$P = A^*PA - A^*PB(R + B^*PB)^{-1}B^*PA + Q$$

State-feedback
Interpretation as 2-norm

Theorem: The LQR controller K minimizes the 2-norm of the following input-output system:

$$\dot{w} = (A + BK)w + Iv, \qquad y = \begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix} w$$

where the input v is a vector of dimension n. The 2-norm of this system is:

$$\|g\|_2 = \sqrt{tr\left\{\int_0^\infty g(t)^* g(t) dt\right\}} = \sqrt{\sum_{i,j} \int_0^\infty \left|g_{ij}\right|^2 dt}$$

where $g(t)$ is the impulse response :

$$g(t) = \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}}K \end{bmatrix} e^{(A+BK)t}$$

The 2-norm corresponds to the square-root of the sum of the costs linked to the n initial conditions e_i. It may also be rewritten as:

$$\|g\|_2 = \sqrt{\mathrm{tr}\left\{\int_0^\infty e^{(A^*+K^*B^*)t}[Q + K^*RK]e^{(A+BK)t} dt\right\}}$$

State-feedback
Interpretation as 2-norm

Proof: Let $G = \int_0^{\infty} g(t)^* g(t) dt$. We have:

$$e_i^* G e_i = \int_0^{\infty} e_i^* g(t)^* g(t) e_i dt = \int_0^{\infty} \sum_{k,j} \delta_{ki} \left(\sum_l g_{lk}^* g_{lj} \right) \delta_{ji}\, dt = \sum_l \int_0^{\infty} |g_{li}|^2\, dt$$

$$= \int_0^{\infty} (w_i^* Q w_i + u_i^* R u_i) dt$$

where $w_i(t) = e^{(A+BK)t} e_i$ and $u_i\ (t) = K w_i(t)$. Hence the i^{th} diagonal element of G represents the cost linked to the initial condition e_i. The trace of G corresponds to the sum of the costs linked to all initial conditions e_i.

The 2-norm of the input-output system may be written as:

$$\|g\|_2 = \sqrt{\mathrm{tr}\left\{ \int_0^{\infty} e^{(A^*+K^*B^*)t} \begin{bmatrix} Q^{1/2} & K^* R^{1/2} \end{bmatrix} \begin{bmatrix} Q^{1/2} \\ R^{1/2} K \end{bmatrix} e^{(A+BK)t} dt \right\}}$$

$$= \sqrt{\mathrm{tr}\left\{ \int_0^{\infty} e^{(A^*+K^*B^*)t} [Q + K^* R K] e^{(A+BK)t} dt \right\}}$$

Outline

- Control gain: K

- Kalman gain: L

Dynamic observer
Kalman filter

The governing equations of the physical system with a control law based on an estimated state ($u = Kw_e + v$) are:

$$\begin{cases} \dot{w} = Aw + BKw_e + Bv \\ y = Cw + g \end{cases}$$

where v and g are white noises of PSD S_v and S_g that excite the system.

Introducing the unit PSD noises $v = \sqrt{S_v}v'$ and $g = \sqrt{S_g}g'$, the system may be rewritten as:

$$\begin{cases} \dot{w} = Aw + BKw_e + \sqrt{S_v}Bv' \\ y = Cw + \sqrt{S_g}g' \end{cases}$$

The estimator is looked for under the form:

$$\begin{cases} \dot{w}_e = Aw_e + BKw_e - L(y - y_e) \\ y_e = Cw_e \end{cases}$$

Dynamic observer
Minimization problem

<u>Theorem:</u> the input-output equation governing the estimation error:

$$\dot{e} = \dot{w} - \dot{w}_e = (A + LC)e + \begin{bmatrix} B\sqrt{S_v} & L\sqrt{S_g} \end{bmatrix} \begin{bmatrix} v' \\ g' \end{bmatrix}$$

$$y = Ie$$

is characterized by the impulse response:

$$g(t) = Ie^{(A+LC)t} \begin{bmatrix} B\sqrt{S_v} & L\sqrt{S_g} \end{bmatrix}$$

The 2-norm of this sytem:

$$\|g\|_2 = \sqrt{\mathrm{tr}\left\{\int_0^\infty g(t)^* g(t) dt\right\}}$$

corresponds to the standard deviation of the estimation error $\sqrt{E(e^*e)}$ in presence of unit PSD white noise in g' and v'. It may also be rewritten as:

$$\|g\|_2 = \sqrt{\mathrm{tr}\left\{\int_0^\infty \left(e^{(A+LC)t}\left[BS_vB^* + LS_gL^*\right]e^{(A^*+C^*L^*)t}\right)dt\right\}}$$

Dynamic observer
Minimization problem

<u>Standard deviation of an output signal.</u>

Let us consider a stable system:

$$\dot{w} = A'w + B'v$$
$$y = C'w$$

If v is white-noise characterized by a PSD (Power Spectral Density) S, then the standard deviation of the output y is equal to:

$$\sqrt{E(y^* y)} = \left|\left|Z'(t)\right|\right|_2 \sqrt{S}$$

where $Z'(t) = C' e^{A't} B'$ is the impulse response of the system.

Dynamic observer
Minimization problem

Proof:

$$\|g\|_2 = \sqrt{\mathrm{tr}\left\{\int_0^\infty g(t)^* g(t) dt\right\}} = \sqrt{\mathrm{tr}\left\{\int_0^\infty g(t) g(t)^* dt\right\}}$$

$$= \sqrt{\mathrm{tr}\left\{\int_0^\infty \left(e^{(A+LC)t} \begin{bmatrix} B\sqrt{S_v} & L\sqrt{S_g} \end{bmatrix} \begin{bmatrix} \sqrt{S_v} B^* \\ \sqrt{S_g} L^* \end{bmatrix} e^{(A^*+C^*L^*)t} \right) dt \right\}}$$

$$= \sqrt{\mathrm{tr}\left\{\int_0^\infty \left(e^{(A+LC)t} \left[B S_v B^* + L S_g L^* \right] e^{(A^*+C^*L^*)t} \right) dt \right\}}$$

Dynamic observer
Minimization problem

Estimation problem: $\|g\|_2 = \sqrt{\mathrm{tr}\left\{\int_0^\infty \left(e^{(A+LC)t}\left[BS_vB^* + LS_gL^*\right]e^{(A^*+C^*L^*)t}\right)dt\right\}}$

Control problem: $\|g\|_2 = \sqrt{\mathrm{tr}\left\{\int_0^\infty e^{(A^*+K^*B^*)t}[Q + K^*RK]e^{(A+BK)t}dt\right\}}$

Estimation problem is linked to a control problem:

$$\begin{cases} \text{estimation} \Leftrightarrow \text{control} \\ \mathrm{A}^* \Leftrightarrow \mathrm{A} \\ C^* \Leftrightarrow B \\ L^* \Leftrightarrow K \\ S_g \Leftrightarrow R \\ BS_vB^* \Leftrightarrow Q \end{cases}$$

Kalman gain:

$$\mathrm{K} = -R^{-1}B^*P \Longrightarrow L^* = -S_g^{-1}CP$$
$$\Longrightarrow L = -P^*C^*S_g^{-1} = -PC^*S_g^{-1}$$

Riccati equation:

$$A^*P + PA - PBR^{-1}B^*P + Q = 0$$
$$\Longrightarrow AP + PA^* - PC^*S_g^{-1}CP + BS_vB^* = 0$$

Dynamic observer
Kalman gain

The steady-state Kalman gain is:

$$L = -PC^*S_g^{-1}$$

where P verifies the algebraic Riccati equation (care in Octave):

$$AP + PA^* - PC^*S_g^{-1}CP + BS_vB^* = 0$$

In the case of a discrete in time system ($\Sigma_g = \mathrm{S_g}/\Delta t$ and $\Sigma_v = S_v/\Delta t$ are the noise variances, Δt the sampling time, S_g and S_v the PSDs):

$$L = -APC^*\left(\Sigma_g + CPC^*\right)^{-1}$$

where P verifies the discrete algebraic Riccati equation (dare in Octave):

$$P = APA^* - APC^*\left(\Sigma_g + CPC^*\right)^{-1}CPA^* + B\Sigma_vB^*$$