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The Euler class for flows of isometries

1. FLOWS OF ISOMETRIES

1.1 Let (M,F) be a compact (n+1)-manifold provided with an orientable Riemannian foliation F of dimension one. The leaves of F are the orbits of a vector field without singularities. We say that F if a *flow of isometries* if there exists a Riemannian metric g on M and a vector field Z tangent to F which generates a group of isometries $(\psi_t)_{t\in\mathbb{R}}$. We always assume that Z is a unit vector field (we only have to replace g by $(g(Z,Z))^{-1}g$).

1.2 Recall that a differential form $\omega \in \Omega^*(M)$ is *base-like for* F if

$$i_Z\omega = 0 \text{ and } i_Z d\omega = 0.$$

The cohomology of the complex of base-like forms is the *base-like cohomology of* (M,F) denoted by $H^*(M/F)$; it depends only on (M,F) and not on the choice of (g,Z).

1.3 For a flow F of isometries the *characteristic 1-form* χ of F with respect to (g,Z) satisfies the equations

$$\chi(Z) = 1 \text{ and } i_Z d\chi = 0;$$

in particular, the form $d\chi$ is base-like for F. By *transverse volume form of* F (with respect to g) we mean the unique form $v \in \Omega^n(M/F)$ such that $v \wedge \chi$ is the volume form of (M,g).

1.4 According to [7], an orientable Riemannian foliation (M,F) of dimension one is a flow of isometries if and only if one of the two following equivalent conditions holds:

(a) $H^n(M/F) \neq 0$;

(b) $0 \neq [v] \in H^n(M/F)$.

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2. INTEGRATION ALONG THE LEAVES OF A FLOW OF ISOMETRIES

2.1 Let (M,F,g,Z) be as above and let $(\psi_t)_{t\in\mathbb{R}}$ be the one-parameter group of isometries generated by Z. Since the group of isometries $\mathrm{Isom}(M,g)$ of a compact Riemannian manifold is a compact Lie group, the closure T of $(\psi_t)_{t\in\mathbb{R}}$ in $\mathrm{Isom}(M,g)$ is a compact commutative Lie subgroup; thus, a torus.

2.2 Let $I\Omega^*(M) \subset \Omega^*(M)$ be the subcomplex of forms invariant by the action of T. According to the definition of T, it appears that

$$I\Omega^*(M) = \{\omega \in \Omega^*(M)/L_Z\omega = 0\}.$$

The inclusion of $I\Omega^*(M)$ in $\Omega^*(M)$ is a homotopy equivalence (see [4]), thus $H^*(I\Omega(M)) \cong H^*(M)$.

For any form $\omega \in I\Omega^r(M)$ the form $i_Z\omega$ is invariant by T and base-like for F. This enables us to construct an *integration operator along the leaves of* F (see [5]):

$$\int_g = (-1)^{r-1}\, i_Z : I\Omega^r(M) \to \Omega^{r-1}(M/F),$$

which commutes with d and satisfies the two following properties:

(a) it is onto; indeed, for any $\alpha \in \Omega^{r-1}(M/F)$ we get

$$L_Z(\alpha \wedge X) = L_Z\alpha \wedge X + \alpha \wedge L_ZX = \alpha \wedge di_ZX = 0,$$

thus $\alpha \wedge X \in I\Omega^r(M)$ and

$$\int_g \alpha \wedge X = (-1)^{r-1}(i_Z\alpha \wedge X + (-1)^{r-1}\alpha \wedge i_ZX) = \alpha.$$

(b) $\mathrm{Ker} \int_g = \Omega^r(M/F)$; indeed, if $\int_g \alpha = 0$ for $\alpha \subset I\Omega^r(M)$ then $i_Zd\alpha = L_Z\alpha - di_Z\alpha = 0$, and $\alpha \in \Omega^{r-1}(M/F)$. The other inclusion is obvious.

2.3 To sum up, we have constructed the following short exact sequence:

$$0 \to \Omega^r(M/F) \xrightarrow{i} I\Omega^r(M) \xrightarrow{\int_g} \Omega^{r-1}(M/F) \to 0.$$

3. GYSIN SEQUENCE AND EULER CLASS

3.1 From the above short sequence we get the following cohomology sequence:

$$\dots \to H^r(M/F) \xrightarrow{i^*} H^r(M) \xrightarrow{\int_g^*} H^{r-1}(M/F) \xrightarrow{\delta_g} H^{r+1}(M/F) \quad \dots$$

which we will call the *Gysin sequence of* F.

As for any $\alpha \in \Omega^{r-1}(M/F)$, we have $\int_g \alpha \wedge \chi = \alpha$, the definition of the connecting homomorphism δ_g gives

$$\delta_g[\alpha] = [d(\alpha \wedge \chi)] = (-1)^{r-1}[\alpha \wedge d\chi] = (-1)^{r-1}[\alpha] * [d\chi].$$

Then, as for Seifert fibrations, we define the *Euler class of* F *with respect to* g by: $e_g(F) = [d\chi] \in H^2(M/F)$.

3.2 Up to a non-zero factor, this class does not depend on the metric g. Indeed, let (M,F,g_1,Z_1) and (M,F,g_2,F_2) be two flows of isometries with the same underlying Riemannian foliation. The two Gysin sequences give

$$\dots \to H^0(M/F) \xrightarrow{\delta_{g_j}} H^2(M/F) \xrightarrow{i^*} H^2(M) \to \dots \quad j = 1,2.$$

The space $H^0(M/F)$ is of dimension one, thus, by exactness, dim Ker $i^* \leq 1$ and

$$\mathrm{Im}\ \delta_{g_1} = \mathrm{Ker}\ i^* = \mathrm{Im}\ \delta_{g_2}.$$

Now $e_{g_1}(F)$ and $e_{g_2}(F)$ are simultaneously zero or there exists $\lambda \in \mathbb{R} - \{0\}$ such that $e_{g_1}(F) = \lambda\, e_{g_2}(F)$. In particular, the fact that the Euler class of F with respect to the metric g vanishes does not depend on the choice of the metric g.

In the particular case of Seifert fibrations our Euler class coincides with the usual one, up to a non-zero factor. This factor is exactly the length of the generic leaf of F. Then for a suitable metric we obtain the usual Euler class of F.

4. VANISHING OF THE EULER CLASS

4.1 Next we obtain a geometrical interpretation of the vanishing of the Euler class of F which generalizes that of [8]. Some of our results are also proved in [7] by means of invariant currents and foliated cycles.

Theorem Let M be a compact manifold with an orientable Riemannian foliation

F: Then the following statements are equivalent:

(i) (M,F) is a foliated bundle;

(ii) F is a flow of isometries and $e_g(F) = 0$ for any suitable metric g;

(iii) $H^n(M/F) \neq 0$ and $i^* : H^n(M/F) \to H^n(M)$ is injective.

Proof (a) Suppose that (M,F) is a foliated bundle and let $F \to M \xrightarrow{p} B$ be the fibration transverse to F. We have the following commutative diagram:

$$\begin{array}{ccccc} & & F\times\mathbb{R} & \xrightarrow{pr} & \mathbb{R} \\ & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle q} \\ F & \longrightarrow & M & \xrightarrow{p} & S^1 \end{array}$$

where q is the canonical covering of S^1 and π is the covering given by the suspension of the holonomy diffeomorphism h of F. Because F is Riemannian there exists a metric g_F on F invariant by h. The metric $\hat{g} = g_F + (dt)^2$ is equivariant with respect to π, thus induces a metric g on M which is a bundle-like metric for π and makes F orthogonal to π. Therefore F is a flow of isometries and the characteristic form χ is equal to $p^*(dt)$. Then $d\chi = 0$ and the Euler class of F with respect to g vanishes. This proves (ii).

Conversely, let us suppose that F is a flow of isometries with respect to a metric g. If $e_g(F) = 0$ there exists $\gamma \in \Omega^1(M/F)$ such that $d\chi = d\gamma$. The form $\omega = \chi - \gamma$ satisfies

$$\omega(Z) = 1 \text{ and } d\omega = d\chi - d\gamma = 0.$$

By Tischler's theorem (see [9]) the foliation defined by the closed form ω can be approximated by a fibration which will again be transverse to F.

(b) Now we show that (ii) is equivalent to (iii). We first note that a Riemannian flow F is a flow of isometries if and only if $H^n(M/F) \neq 0$ (see 1.4). Consider the sequence

$$\ldots \to H^{n-2}(M/F) \xrightarrow{\delta_g} H^n(M/F) \xrightarrow{i^*} H^n(M) \to \ldots$$

for any suitable metric g. If (ii) holds, $e_g(F)$ is zero and the connecting homomorphism is zero. Therefore i^* is injective and (iii) is fulfilled.

On the other hand, recall that for any Riemannian foliation there exists, for the complex of base-like forms of F, a Hodge theory similar to that of the De Rham complex of a compact manifold (see [3]). It includes a base-like Hodge operator $*$ which enjoys the usual properties. Thus if we fix a suitable metric g on M, $d\chi$ is base-like cohomologous to a base-like harmonic form $\alpha \in \Omega^2(M/F)$, that is, there exists $\gamma \in \Omega^1(M/F)$ such that $d\chi = \alpha + d\gamma$.

To prove that (iii) implies (ii), it is enough to show that $\alpha = 0$. So assume that (iii) holds and α is different from zero. The base-like harmonic form $*\alpha \in \Omega^{n-2}(M/F)$ dual to α satisfies $\alpha \wedge *\alpha = \lambda v$, $\lambda \in \Omega^0(M/F)$ positive,

$$*\alpha \wedge d\chi = *\alpha \wedge \alpha + *\alpha \wedge d\gamma = \lambda v + (-1)^{n-2} d(*\alpha \wedge \gamma).$$

Then $[*\alpha \wedge d\chi] = [\lambda v]$ is a non-zero class in $H^n(M/F)$ (see 1.4). But, by exactness of the Gysin sequence of F, we also get $\delta_g(H^{n-2}(M/F)) = 0$, which implies $[*\alpha \wedge d\chi] = (-1)^{n-2}\, \delta_g[*\alpha] = 0$. This contradiction ends the proof.

4.2 Remark The condition (iii) is equivalent to the fact that the class $[v]$ is different from zero in $H^n(M)$, where v is the transverse volume form of (M,F,g) for a suitable metric g on M.

4.3 Corollary Let M be a compact $(n+1)$-manifold provided with a flow F of isometries. If $H_1(M) = 0$ then the Euler class of F is non-zero.

4.4 Corollary Let M be as above. If the Euler class of F is zero then there exists a finite covering $\tilde{M}$ of M which is diffeomorphic to the product $F \times S^1$.

Proof It follows from Theorem 4.1 that if the Euler class is zero then there exists $F \to M \xrightarrow{p} S^1$, a transverse fibration to F, defined by suspension of a diffeomorphism h of F. We can assume that F is a Seifert fibration (see [2]). Then the holonomy of any leaf is finite, i.e., h is a periodic map at any point. Now it is not difficult to see that there exists $p \in \mathbb{N}$ such that h^p is the identity. Consider $\psi:\pi_1 M \to \mathbb{Z}[h] \subset \mathrm{Diff}(F)$ the holonomy homomorphism of F. Because h^p is the identity we have an induced homomorphism $\bar{\psi}:\pi_1 M \to \mathbb{Z}/p\mathbb{Z}$. The associated covering $\pi:\tilde{M} \to M$ is a foliated bundle whose holonomy is generated by $h^p = \mathrm{Id}_F$; thus $\tilde{M} = F \times S^1$.

4.5 Remarks (i) The integration operator constructed in Section 2 is a particular case of an integration operator defined for any taut foliation.

(ii) The example of Carriere [1] shows that a Riemannian flow may admit a transverse foliation G without being a flow os isometries. In this case G is not Riemannian.

(iii) There exist flows of isometries which admit non-Riemannian transverse foliations and which have non-trivial Euler class [10].

5. CONTACT FLOWS AND FLOWS OF ISOMETRIES

A flow F (i.e., an orientable foliation of dimension one) defined on a compact $(2k+1)$-manifold M is a *contact flow* if there exists a form $\omega \in \Omega^1(M)$ such that:

(a) ω is a *contact form*, i.e., $\omega \wedge (d\omega)^k$ is a volume form on M;

(b) the unique vector field defined by $\omega(Y) = 1$ and $i_Y d\omega = 0$ is tangent to F.

5.1 By means of the Euler class, we get a partial characterization of the flows of isometries which are contact flows (see [8] for the compact case and also [6]).

<u>Theorem</u> Let M be a compact $(2k+1)$-manifold with a Riemannian contact flow F. Then F is a flow of isometries and the Euler class of F is different from zero.

<u>Proof</u> Let g be a bundle-like metric on (M,F). We can write $g = g_T + g_N$ where g_T (resp. g_N) is the restriction of g to the tangent bundle (resp. the normal bundle) of F. We define a new bundle-like metric on (M,F) by $\hat{g} = \omega \otimes \omega + g_N$, where ω is the contact form given by (a). It is not difficult to see that $L_Y \hat{g} = 0$ and therefore F is a flow of isometries with respect to $\hat{g}$. Furthermore, $d\omega$ is a base-like form and $[d\omega]$ belongs to $\operatorname{Ker} i^*$. If the Euler class of F is zero, then $[d\omega] = 0$ in $H^2(M/F)$ (see Theorem 4.1.), and there exists $\gamma \in \Omega^1(M/F)$ such that $d\omega = d\gamma$. Because $d\omega \wedge \gamma \wedge (d\omega)^{k-1} \in \Omega^{2k+1}(M/F) = \{0\}$, we get

$$\omega \wedge d\gamma \wedge (d\omega)^{k-1} = d(\omega \wedge \gamma \wedge (d\omega)^{k-1})$$

and

$$\omega \wedge (d\omega)^k = d(\omega \wedge \gamma \wedge (d\omega)^{k-1}).$$

Then $\omega \wedge (d\omega)^k$ is not a volume form which is in contradiction with a).

5.2 We find in [6] a converse statement for the case of geodesible flows in three-dimensional manifolds (including flows of isometries). Now this gives a complete characterisation.

Theorem Let (M,F,g) be a flow of isometries on a compact Riemannian 3-manifold, then the following statements are equivalent:

(i) the Euler class of F is zero;

(ii) F is not a contact flow.

Corollary If M is a compact Riemannian manifold with $H_1(M) = 0$, then any Riemannian flow F on M is a flow of isometries and a contact flow.

6. FLOWS OF ISOMETRIES ON S^3

The Euler class enables us to classify partially the flows of isometries on a given compact manifold. For example, consider the family of all flows of isometries on S^3. The Seifert fibrations have been analyzed in [8], therefore in order to get a complete description of these flows, it remains only to study a one-parameter family $\{F_\alpha, \alpha \in [0,1]\}$ which can be described as follows. For $\alpha \in]0,1]$, F_α is the foliation defined in complex coordinates by the flow

$$\psi_t^\alpha : S^3 \to S^3 \quad \text{with} \quad \psi_t^\alpha(z_1,z_2) = (e^{i\alpha t} z_1, e^{it} z_2).$$

For any α, $(\psi_t^\alpha)_{t\in\mathbb{R}}$ is a group of isometries of S^3 with respect to the usual metric g. As we pointed out in Section 1, F_α is also a flow of isometries with respect to the metric $g_\alpha = (g(Z_\alpha,Z_\alpha))^{-1}g$, where Z_α is the vector field defined by $(\psi_t^\alpha)_{t\in\mathbb{R}}$.

If v_α is the transverse volume form of (F_α,g_α) the Euler class $e_{g_\alpha}(F)$ is determined by the number $r_\alpha \neq 0$ such that $e_{g_\alpha}(F) = r_\alpha[v_\alpha]$. We compute r_α by the formula

$$r_\alpha = \frac{1}{\mathrm{vol}(S^3,g_\alpha)} \int_{S^3} d\chi_\alpha \wedge \chi_\alpha,$$

which gives

$$r_\alpha = (1+\alpha).$$

It is clear that this number r_α classifies completely the elements of the family $\{F_\alpha, \alpha \in]0,1]\}$.

If $\alpha = p/q$ is rational, F_α is a Seifert fibration and our Euler class $e_{g_\alpha}(F)$ is related to the Euler class $\varepsilon_\alpha(F)$ defined by Nicolau-Reventós by the formula

$$q\varepsilon_\alpha(F) = e_{g_\alpha}(F).$$

Indeed, $\varepsilon_\alpha(F) = [d\xi_\alpha] \in H^2(S^3/F_\alpha)$ for any one-form ξ_α on M whose integral along the fibres of F_α is the constant function 1. On the other hand, the integral of the characteristic form χ_α of F_α is the length of a regular leaf of F_α, that is q. The result follows by taking $\xi_\alpha = \frac{1}{q}\chi_\alpha$.

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An interpretation of the Weil operator $\chi(y_1)$

INTRODUCTION

In this article, we give a cohomology theoretic meaning to the differential form h_I corresponding to y_I of the secondary characteristic class $y_I c_I$ (see, e.g., [5, p. 154] of foliation. Let F be a C^∞-foliation on a manifold M. Let ∇^B (∇^R) be a Bott (Riemannian) connection on the normal bundle $\nu(F)$ of F. Let d_F denote the exterior differential along leaves and $H^{r,s}_{FDR}(M)$ the foliation de Rham cohomology vector space (cf. [8], [10] and [11]).

(Theorem 3.3) For any ∇^B and ∇^R, the $(0,2j-1)$-component $(h_j)_{0,2j-1}$ of h_j is a d_F-cocycle, and $[(h_j)_{0,2j-1}] \in H^{0,2j-1}_{FDR}(M)$ does not depend on the choice of ∇^B and ∇^R. The Weil operator $\chi(y_j)$ of [4] is regarded as a multiplication by $[(h_j)_{0,2j-1}]$.

In this sense, the operator $\chi(y_I)$ is essentially an element of $H^{0,|I|}_{FDR}(M)$. In other words, the notion of the Weil operators is expanded to that of cohomology classes of $H^{0,*}_{FDR}(M)$.

For a local foliation chart $(U_\alpha, x^\alpha_i, u^\alpha_j)$, C^∞-functions on $U_\alpha \cap U_\beta \neq \phi$

$$C_{\alpha\beta} = \log |\det(\partial u^\beta_j/\partial u^\alpha_k)|$$

are constant along leaves of $F|_{U_\alpha \cap U_\beta}$ and satisfy the cocycle condition

$$C_{\beta\gamma} - C_{\alpha\gamma} + C_{\alpha\beta} = 0$$

on $U_\alpha \cap U_\beta \cap U_\gamma$. Let C^∞_F denote the sheaf of germs of C^∞-functions constant along leaves $F|_{U_\alpha \cap U_\beta}$. The Čech cohomology class $m(M,F) \in \check{H}^1(M;C^\infty_F)$ determined by $\{C_{\alpha\beta}\}$ is called the modular cohomology class of F (see, e.g., [12]) which is closely related to the modular function δ of transverse measure on holonomy groupoid of F by [2, p.41]. One can eastablish a de Rham type isomorphism $\phi: H^{0,s}_{FDR}(M) \cong \check{H}^s(M;C^\infty_F)$. Then we have:

(Theorme 5.4) Let (M,F) be a C^∞-foliation on a compact Hausdorff manifold. Then, for $[(h_1)_{0,1}]$ corresponding to $\chi(y_1)$, we have

$$\phi(2\pi[(h_1)_{0,1}]) = -m(M,F).$$

The above formula is regarded as a new interpretation of $\chi(y_1)$ and also similar meaning of $\chi(y_j)$, j (odd) ≥ 3 is expected.

In Section 1, we review the Weil operator introduced by Heitsch and Hurder. In Section 2, we explain the foliation de Rham cohomology $H_{FDR}^{r,s}(M)$ and in Section 3, we prove that the homomorphism induced by leaf preserving transverse map is invariant under the leaf preserving homotopy through these maps. In particular, we obtain the Poincare lemma for $H_{FDR}^{r,s}(M)$. Then we prove Theorem 3.3.

In Section 4, the notion of F-simple cover is introduced and then de Rham type isomorphism for Čech cohomology $\check{H}^*(M;C_F^\infty)$ is proved. In the last section, a natural isomorphism from $H_{FDR}^{0,s}(M)$ to the differentiable singular cohomology $H_{FD}^s(M;\mathbb{R})$ restricted to leaves is obtained. Finally Theorem 5.4 is proved.

All manifolds, maps and foliations are assumed to be class C^∞.

1. THE WEIL OPERATORS

Let (M,F) be a C^∞-foliation of codimension q on a paracompact Hausdorff C^∞-manifold. For each point $m \in M$, there is an open neighbourhood U of m and we have linearly independent 1-forms $\omega_1^U,\dots,\omega_q^U$ on U defining F. Let A(M) be the vector space of C^∞-forms on M that is the de Rham complex of M and let $\eta|_U$ be the restriction of $\eta \in A(M)$ to U. We set

$$A(M,F) = \{\eta \in A(M) \mid (\eta_U) \wedge \omega_i^U = 0,\ i = 1,\dots,q\}.$$

One can see easily that

$$A(U,F) = A(U) \wedge \omega_1^U \wedge \dots \wedge \omega_q^U.$$

Let T(M) be the tangent bundle of M. By the integrability condition for tangent sub-bundle $T(F) = F$ of T(M) corresponding to F, we have 1-forms $\{\omega_{ij}^U\}$ on U such that

$$d\omega_i^U = \sum_{j=1}^{q} \omega_{ij}^U \wedge \omega_j^U.$$

Since we have

$$
\begin{aligned}
0 &= d((\eta|_U) \wedge \omega_i^U) \\
&= d(\eta|_U) \wedge \omega_i^U + (\eta|_U) \wedge (\sum_{j=1}^{q} \omega_{ij}^U \wedge \omega_j^U) \\
&= d(\eta|_U) \wedge \omega_i^U,
\end{aligned}
$$

the exterior differentiation is closed in $A(M,F)$ and thus $A(M,F)$ is a differential subcomplex of $A(M)$. We denote the cohomology vector space of $A(M,F)$ by $H^*(M,F)$.

Let K be a local curvature matrix of a connection ∇ on C^∞-vector bundle V on M. For any Chern monomial $c_J = c_1^{j_1} \ldots c_q^{j_q}$ of degree k on the Lie algebra $gl(q;\mathbb{R})$ of $GL(q;\mathbb{R})$, we set

$$c_J(\nabla) = c_J(K) \in A^{2k}(M).$$

It is well known that, for any connection ∇, $dc_J(\nabla) = 0$, (see, e.g., [7, pp. 296-298]).

Let $\Gamma(V)$ be the set of C^∞-sections of a C^∞-vector bundle V. Let $\nu(F)$ be the normal bundle of F, that is, $\nu(F) = T(M)/F$ and $\nu(F)^*$ the dual bundle of $\nu(F)$. Let $\nabla^B(\nabla^R)$ be a Bott (Riemannian) connection on $\nu(F)$. Then we have

$$c_J(\nabla^B) \in \Gamma(\wedge^k(\nu(F)^*) \wedge A^k(M),$$

which is the essential part of Bott vanishing [1, pp. 34-35].

Let $\pi : M \times \mathbb{R} \to M$ be the first factor projection and

$$\nabla^{BR} = (1 - t)\nabla^B + t\nabla^R \qquad t \in \mathbb{R}$$

which is a connection on the vector bundle $\pi^*\nu(F) \cong \nu(F) \times \mathbb{R}$. Define $h_j \in A^{2j-1}(M)$ by

$$
\begin{aligned}
h_j &= \int_0^1 i(\partial/\partial t) c_j(\nabla^{BR}) dt \\
&= \pi_*(c_j(\nabla^{BR})|_{M \times I}),
\end{aligned}
$$

where $i(\partial/\partial t)$ is the substitution operator of $\partial/\partial t$ and π_* is the integration over the fibre for $\pi|_{M \times I}$. A standard computation shows that $dh_j = c_j(\nabla^B) - c_j(\nabla^R)$. For j odd, we have $c_j(\nabla^R) = 0$ and hence $dh_j = c_j(\nabla^B)$.

For each k, let $[\wedge(y_1, y_3, \ldots, y_r)]_k$ ($r = 2[(q+1)/2]-1$) denote the homogeneous part of degree k of $\wedge(y_1, y_3, \ldots, y_r) \cong H^*(gl(q;\mathbb{R}), 0_q)$ (see, e.g., [5, p. 140]).

We define a homomorphism

$$\chi : [\wedge(y_1, y_3, \ldots, y_r)]_k \to \mathrm{Hom}(H^*(M;F),\ H_{DR}^{*+k}(M))$$

by the formula

$$\chi(y)[\eta] = [h \wedge \eta]$$

$$y = y_{j_1} \wedge \ldots \wedge y_{j_s},\ h = h_{j_1} \wedge \ldots \wedge h_{j_s}$$

$$\sum_{\ell=1}^{s} (2j_\ell - 1) = k$$

for $\eta \in A^*(M,F)$ with $d\eta = 0$.

The right side is well defined because we have

$$dh_j = c_j(\nabla^B) \in \Gamma(\nu(F)^*) \wedge A(M),$$

therefore

$$\begin{aligned} d(h \wedge \eta) &= dh \wedge \eta + (-1)^k h \wedge d\eta \\ &= 0, \end{aligned}$$

and for $\eta = d\lambda$ with $\lambda \wedge \omega_j^U = 0$, $j = 1, \ldots, q$,

$$\begin{aligned} d(h \wedge \lambda) &= dh \wedge \lambda + (-1)^k h \wedge d\lambda \\ &= (-1)^k h \wedge d\lambda. \end{aligned}$$

The latter formula means that $[h \wedge \eta]$ does not depend on the representative η of $[\eta]$. By making use of affine combinations of different Bott connections and Riemannian connections, one can also show that $[h \wedge \eta]$ does not depend on the choices of ∇^B and ∇^R. $\chi(y)$ is called the Weil operator associated to y (cf. [4]).

2. THE FOLIATION DR COHOMOLOGY

Let (M,F) be a codimension q C^∞-foliation on a paracompact Hausdorff manifold. By taking a Riemannian metric on M, one can split the tangent bundle $T(M)$ into the Whitney sum

$$T(M) = F \oplus V,$$

where V is the orthogonal complement of F and we have the splitting of the dual tangent bundle,

$$T^*(M) = V^* \oplus F^*.$$

V is clearly isomorphic to the normal bundle $\nu(F)$.

Let $(x,u): U \to \mathbb{R}^p \times \mathbb{R}^q$ be a local foliation chart of F. One can choose C^∞-1-forms on U, $\theta_1,\ldots,\theta_p \in \Gamma(F^*)|_U$ so that

$$\{\theta_1,\ldots,\theta_p,\ du_1,\ldots,du_q\}$$

is a basis of $T^*_m(M)$ for each $m \in U$. And one can also choose C^∞-vector fields $v_1,\ldots,v_q \in \Gamma(V)|_U$ so that

$$\{\partial/\partial x_1,\ldots,\partial/\partial x_p,\ v_1,\ldots,v_q\}$$

is the dual basis of $T_m(M)$ for $\{\theta_1,\ldots,\theta_p,\ du_1,\ldots,du_q\}$. Then we obtain

$$\theta_j = dx_j + \sum_{\alpha=1}^{q} a_{j\alpha} du_\alpha \qquad 1 \le j \le p,$$

$$v_\alpha = \partial/\partial u_\alpha + \sum_{j=1}^{p} b_{\alpha j}\partial/\partial x_j \quad 1 \le \alpha \le q,$$

where $a_{j\alpha}$, $b_{\alpha j}$ are C^∞-functions satisfying $a_{j\alpha} + b_{\alpha j} = 0$.

We set

$$A^{r,s}(M) = \wedge^r(\Gamma(V^*)) \wedge \wedge^s(\Gamma(F^*))$$

and we have

$$\begin{aligned} A(M) &= \sum_{k=0}^{n} \wedge^k(\Gamma(T^*(M)) \\ &= \sum_{k=0}^{n} \wedge^k(\Gamma(V^*) \oplus \Gamma(F^*)) \\ &= \sum_{k=0}^{n} \sum_{r+s=k} \wedge^r(\Gamma(V^*)) \wedge \wedge^s(\Gamma(F^*)) \\ &= \sum_{0\le r+s\le n} A^{r,s}(M). \end{aligned}$$

We denote $A^{r,s}(M)$ simply by $A^{r,s}$. An element of $A^{r,s}$ is the sum of differential forms of pure (r, s)-type,

$$\omega = f du_{j_1} \wedge \dots \wedge du_{j_r} \wedge \theta_{k_1} \wedge \dots \wedge \theta_{k_s}$$

and the exterior derivative $d\omega$ splits uniquely into the sum

$$d\omega = \omega_{r+2,\, s-1} + \omega_{r+1,\, s} + \omega_{r,\, s+1},$$

$$\omega_{r+2,\, s-1} \in A^{r+2,\, s-1}, \omega_{r+1,s} \in A^{r+1,\, s}, \omega_{r,\, s+1} \in A^{r,\, s+1}.$$

This splitting defines operators

$$d_1 : A^{r,s} \to A^{r+2,\, s-1}$$

$$d_2 : A^{r,s} \to A^{r+1,\, s},$$

$$d_F : A^{r,s} \to A^{r,\, s+1}.$$

From the relation $(d_1 + d_2 + d_F)^2 = d^2 = 0$, it follows that $d_1^2 = 0$, $d_F^2 = 0$ and others. For fixed $r (q \geq r \geq 0)$, one obtains a cochain complex,

$$0 \to A^{r,0} \xrightarrow{d_F} A^{r,1} \xrightarrow{d_F} \dots \xrightarrow{d_F} A^{r,p} \to 0.$$

We set $Z^{r,s} = \mathrm{Ker}(d_F: A^{r,s} \to A^{r,\, s+1})$, $B^{r,s} = \mathrm{Im}(d_F: A^{r,\, s-1} \to A^{r,s})$, then $B^{r,s} \subset Z^{r,s} \subset A^{r,s}$. We define the *foliation* DR (r, s)-*cohomology vector space* of (M,F) by

$$H^{r,s}_{FDR}(M) = Z^{r,s}/B^{r,s}.$$

For one leaf foliation $(q = 0)$ on M, we obtain clearly the ordinary s-dimensional de Rham cohomology of M,

$$H^{0,s}_{FDR}(M) = H^s_{DR}(M).$$

We define a homomorphism

$$\bar{\chi}: H^{0,s}_{FDR}(M) \to \mathrm{Hom}(H^*(M,F), H^{*+s}_{DR}(M))$$

by the formula $\bar{\chi}([z])[\eta] = [z \wedge \eta]$ where $z \in Z^{r,s}$ and $\eta \in A(M,F)$ with $d\eta = 0$. The right side is well defined as follows: we have

$$d(z \wedge \eta) = (d_1 + d_2 + d_F)z \wedge \eta + (-1)^{r+s} z \wedge d\eta$$

$$= (d_1 + d_2)z \wedge \eta$$

$$= 0.$$

If $\eta = d\lambda$, $\lambda \wedge du_j = 0$, $j = 1,\ldots,q$ then one obtains

$$d(z \wedge \lambda) = dz \wedge \lambda + (-1)^{r+s} z \wedge d\lambda$$

$$= (-1)^{r+s} z \wedge \eta$$

and if $z = d_F a$, then one gets

$$d(a \wedge \eta) = (d_1 + d_2 + d_F)a \wedge \eta + (-1)^{r+s-1} a \wedge d\eta$$

$$= z \wedge \eta \ .$$

Hence $[z \wedge \eta]$ does not depend on the choice of the representative η of $[\eta]$ and the representative z of $[z]$.

3. THE FOLIATION DR COHOMOLOGY CLASS OF WEIL OPERATOR

Let (M,F) and (M',F') be codimension q foliations, and let $f:M \to M'$ be a C^∞-map transverse to F' so that $F = f^*F'$. For any point $m \in M$, we set $m' = f(m)$. Let (x',u') be a local foliation chart around $m' \in M'$. One can choose a local foliation chart (x,u) around m such that $du_j = f^*du'_j$, $j = 1,\ldots,q$. We have

$$f^*(A^{r,s}(M')) \subset A^{r,s}(M).$$

Since $f^*d = df^*$, by comparing components of pure type $(r, s + 1)$, it follows that $f^*d_F = d_F f^*$.

Let f_0, f_1: $M \to M'$ be C^∞-maps transverse to F' so that $f_0^*F' = f_1^*F' = F$. If there is a C^∞-map $H:M \times R \to M'$ transverse to F' such that

$$f_i(m) = H(m,i) \qquad i = 0,1,$$

$$H^*F' = \pi^*F,$$

where $\pi:M \times R \to M$ is the first factor projection, then f_0, f_1 are called C^∞*-homotopic by leaf preserving map* and denoted by $f_0 \ {}_{F'}\!\simeq_{,F} f_1$. H is called a *leaf preserving* C^∞-*homotopy*.

<u>Lemma 3.1</u> If f_0, f_1: $M \to M'$ are C^∞-homotopic by leaf preserving map, then f_0^*, f_1^* are cochain homotopic.

<u>Proof</u> Let (x,u): $U \to \mathbb{R}^p \times \mathbb{R}^q$ be a local foliation chart. Local charts in $M \times \mathbb{R}$ of the type $((x,t),u)$: $U \times \mathbb{R} \to (\mathbb{R}^p \times \mathbb{R}) \times \mathbb{R}^q$ define a codimension q foliation $\bar{F}$ in $M \times \mathbb{R}$. It is clear that $\pi^*F = \bar{F}$. One can take a basis of tangent vector fields in the local foliation chart of $\bar{F}$,

$$\partial/\partial x_1, \ldots, \partial/\partial x_p, \quad \partial/\partial t, \; v_1, \ldots, v_q$$

and its dual basis of 1-forms

$$\theta_1, \ldots, \theta_p, \; dt, \; du_1, \ldots, du_q.$$

Let i_0, i_1: $M \to M \times \mathbb{R}$ be maps defined by

$$i_j(m) = (m,j) \quad j = 0,1.$$

Since i_j is transverse to $\bar{F}$, $i_j^*\bar{F}$ is defined and equal to F.
Any $\psi \in A^{r,s}(M \times \mathbb{R})$ is written uniquely as

$$\psi = \rho + \sigma \wedge dt,$$

where $\rho \in A^{r,s}(M \times \mathbb{R})$, $\sigma \in A^{r,s-1}(M \times \mathbb{R})$ do not contain dt. Define a homomorphism $\Psi: A^{r,s}(M \times \mathbb{R}) \to A^{r,s-1}(M)$ by

$$\Psi\rho = 0, \; \Psi(\sigma \wedge dt) = (-1)^{r+s-1} \int_0^1 \sigma \wedge dt.$$

Then we have

$$d_F\Psi\psi = d_F((-1)^{r+s-1} \int_0^1 \sigma \wedge dt),$$

$$\begin{aligned} \Psi d_{\bar{F}}\psi &= \Psi(d_F\rho + (-1)^{r+s}(\partial\rho/\partial t) \wedge dt + d_F\sigma \wedge dt) \\ &= [\rho]_0^1 + (-1)^{r+s} \int_0^1 d_F\sigma \wedge dt, \end{aligned}$$

and therefore

$$d_F\Psi\psi + \Psi d_{\bar{F}}\psi = [\rho]_0^1 = i_1^*\psi - i_0^*\psi.$$

Since $H: M \times I \to M'$ defines $f_0 \; {}_{F}\tilde{\,}_{,F} \; f_1$, we have $Hi_j = f_j$, $j = 0,1$ and $H^*F' = \tilde{F}$. Then we obtain a homomorphism $\Delta = \Psi H^*: A^{r,s}(M) \to A^{r,s-1}(M')$ satisfying for $\omega \in A^{r,s}(M')$

$$
\begin{aligned}
f_1^*\omega - f_0^*\omega &= i_1^*H^*\omega - i_0^*H^*\omega \\
&= d_F\Psi H^*\omega + \Psi H^* d_F\omega \\
&= d_F\Delta\omega + \Delta d_F\omega.
\end{aligned}
$$

As a corollary of Lemma 3.1, we get the following Poincare lemma which is considered as a detailed version of [10, Theorem 3.1]. A codimension q foliation (M,F) is called *F-contractible* if there exists a q-dimensional submanifold N of M transverse to F and a map $f:M \to N \subset M$ transverse to F such that $f \; {}_{F}\tilde{\,}_{,F} \; id_M$. The leaf preserving homotopy of this is called *F-contraction* to f.

Corollary 3.2 Suppose that (M,F) is *F-contractible*. If $\omega \in A^{r,s}(M)$ $(s \geq 1)$ and $d_F\omega = 0$, then there exists $\eta \in A^{r,s-1}(M)$ such that $\omega = d_F\eta$.

Proof By Lemma 3.1, it follows that

$$f^*\omega - \omega = d_F\Delta\omega.$$

But f is factored by $\hat{f}:M \to N$ and the inclusion map $i: N \to M$. Since $i^*F = F_0$ is the point foliation and $\omega \in A^{r,s}(M)$ $s \geq 1$, we have $i^*\omega = 0$ and hence $f^*\omega = \hat{f}^*i^*\omega = 0$. Therefore one obtains $\omega = d_F\eta$, $\eta = -\Delta\omega$.

In Section 2, we have constructed an operator

$$\bar{\chi}: H^{0,s}_{FDR}(M) \to \mathrm{Hom}(H^*(M, F), H^{*+s}_{FDR}(M)).$$

The Weil operator $\chi(y)$ and the homomorphism $\bar{\chi}$ are related by the following theorem.

Theorem 3.3 For any ∇^B and ∇^R on $\nu(F)$, the $(0, 2j-1)$-component $(h_j)_{0,\,2j-1}$ of h_j, j odd > 0 is a d_F-cocycle and the cohomology class $[(h_j)_{0,\,2j-1}] \in H^{0,\,2j-1}_{FDR}(M)$ does not depend on the choices of ∇^B and ∇^R. Clearly we have

$$\chi(y_j)[\eta] = \bar{\chi}([(h_j)_{0,\ 2j-1}])[\eta]$$

for each $[\eta] \in H^*(M,F)$.

Proof In Section 1, we have shown that

$$dh_j = c_j(\nabla^B) \in \Sigma A^{r,\ 2j-r} \quad r \geq j.$$

By the definition of d_F, it follows that

$$d_F(h_j)_{0,\ 2j-1} = (dh_j)_{0,\ 2j},$$

and hence $d_F(h_j)_{0,\ 2j-1} = 0$.

Let $h_j^{(k)}$ denote h_j for the Bott connection $\overset{(k)}{\nabla}{}^B$ $k = 0,1$ on $\nu(F)$ and $\bar{h}_j$ denote h_j for the Bott connection $(1 - t)\overset{(0)}{\nabla}{}^B + t\overset{(1)}{\nabla}{}^B$ on $\nu(\bar{F}) = \nu(F) \times \mathbb{R}$. Let $i_k : M \to M \times \mathbb{R}$ be maps defined by

$$i_k(m) = (m,k) \quad k = 0,1.$$

Then, by the proof of Lemma 3.1, we have

$$(h_j^{(1)})_{0,\ 2j-1} - (h_j^{(0)})_{0,\ 2j-1} = (i_1^* - i_0^*)(\bar{h}_j)_{0,\ 2j-1}$$

$$= (d_F\Psi + \Psi d_{\bar{F}})(\bar{h}_j)_{0,\ 2j-1}.$$

Since $d_{\bar{F}}(\bar{h}_j)_{0,\ 2j-1} = 0$, it follows that

$$(h_j^{(1)})_{0,\ 2j-1} - (h_j^{(0)})_{0,\ 2j-1} = d_F\Psi(\bar{h}_j)_{0,2j-1},$$

and hence $[(h_j)_{0,2j-1}] \in H_{FDR}^{0,\ 2j-1}(M)$ does not depend on the choice of ∇^B. By a similar method, $[(h_j)_{0,\ 2j-1}]$ also does not depend on the choice of Riemannian connection ∇^R on $\nu(F)$.

The last statement of the theorem is obvious.

4. FOLIATION DE RHAM ISOMORPHISM

Let (M,F) be a codimension q C^∞-foliation and $\mathcal{U} = \{U_\alpha\}$ a cover of M by open sets. If an intersection of finite open sets of $\mathcal{U}$ is F-contractible, we call $\mathcal{U}$ an *F-simple cover*.

<u>Lemma 4.1</u> Let (M,F) be a foliation on a paracompact Hausdorff manifold. Every open cover U of M admits a refinement $U' = \{U_i'\}$ which is F-simple.

<u>Proof</u> The tangent bundle $T(M)$ splits into the Whitney sum $T(M) = F \oplus V$, $F = T(F)$, $V = \nu(F)$. We take connection ∇^F (resp. ∇^V) on the vector bundles F (resp. V) and we define a connection ∇ on $T(M)$ by $\nabla = \nabla^F \oplus \nabla^V$. We call a curve $\gamma(t)$ in M ∇-geodesic if it satisfies $\nabla_{d\gamma/dt}(d\gamma/dt) = 0$. ∇^F-geodesic on a leaf is necessarily ∇-geodesic on M and hence ∇-geodesic tangent to a leaf is contained in the leaf.

One can assume that every U_α is a neighbourhood of local foliation chart $\phi_\alpha : U_\alpha \subset \mathbb{R}^p \times \mathbb{R}^q$ and that, for each $m \in M$, $\phi_\alpha(m) = (0,0)$ with some α. We take a small q-disk $D^q \subset \{0\} \times \mathbb{R}^q$ contained in U_α, and then take a sufficiently small normal open p-disk bundle E on D^q consisting of vectors tangent to leaves such that the image $\mathrm{Exp}(E)$ of E by the exponential map is contained in U_α.

Let $U' = \{U_j\}$ be an open cover by $\mathrm{Exp}(E)$ of M and $Q = U_{j_1}' \cap \ldots \cap U_{j_k}' \neq \phi$. By the property of ∇-geodesic stated in the above, a connected component of the intersection of a leaf and Q is ∇-geodesically convex (cf. [3, p. 34]). One can assume that $U_{j_1}' = \mathrm{Exp}(E) \subset U_\alpha$.

Let $\pi : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ be the natural projection. Obviously, $\pi(Q) = B$ is an open set of D^q. Since each fibre of $\pi : Q \to B$ is contractible, one can construct a cross-section $s : B \to Q$ of π and by C^∞-approximation argument, one can assume that s is a C^∞-map.

$N = s\pi(Q) \subset Q$ is a q-dimensional submanifold transverse to F. The contraction of each fibre of π to the point of N along ∇-geodesic with respect to its parameter gives a C^∞ F-contraction to $s\pi : Q \to N$.

Since $\{U_{j_1}', \ldots, U_{j_k}'\}$ is an arbitrary finite set of U' with non-empty intersection and $U_{j_1}' \cap \ldots \cap U_{j_k}'$ is F-contractible, $U' = \{U_j'\}$ is F-simple, and U' is obviously a refinement of U from its construction.

Let C_F^∞ denote the sheaf of germs of real valued C^∞-function on M, constant along leaves of F, and let $\check{H}^s(M; C_F^\infty)$ denote the s-dimensional Čech cohomology vector space of M with coefficient C_F^∞. We have the following de Rham type isomorphism which is a special case of [10, Theorem 3.2] and is proved here briefly by arguments of [1].

<u>Theorem 4.2</u> There is an isomorphism $\Phi : H_{FDR}^{0,s}(M) \underset{\to}{\cong} \check{H}^s(M; C_F^\infty)$.

Proof Let $A^{0,s}$ be the sheaf of germs of differential (0, s)-form. For open cover $\mathcal{U}$ of M, we have a k-dimensional Čech cochain vector space $\check{C}^k(\mathcal{U}; A^{0,s})$ of $\mathcal{U}$ with coefficient $A^{0,s}$, and we set

$$K^{k,s}(\mathcal{U}) = \check{C}^k(\mathcal{U}; A^{0,s}),$$

$$K^r(\mathcal{U}) = \sum_{k+s=r} K^{k,s}(\mathcal{U}).$$

Let $\check{\delta}: K^{k,s}(\mathcal{U}) \to K^{k+1,s}(\mathcal{U})$ denote the coboundary operator of Čech cochain.

On the other hand, $d_F: A^{0,s} \to A^{0,s+1}$ defines another operator $d_F: K^{k,s}(\mathcal{U}) \to K^{k,s+1}(\mathcal{U})$ such that $d_F^2 = 0$ and $d_F\check{\delta} = \check{\delta}d_F$. We set

$$D' = \check{\delta}$$

$$D'' = (-1)^k d_F \text{ on } K^{k,s}(\mathcal{U})$$

$$\bar{D} = D' + D''.$$

One can easily see that $\bar{D}: K^r(\mathcal{U}) \to K^{r+1}(\mathcal{U})$ is a coboundary operator, that is, $\bar{D}^2 = 0$.

Cochain maps

$$\alpha: A^{0,*}(M) = \sum_{s=0}^{p} A^{0,s} \to \sum_{s=0}^{p} K^{0,s}(\mathcal{U}) \subset \sum_r K^r(\mathcal{U}) = K(\mathcal{U}).$$

$$\beta: \check{C}(\mathcal{U}; C_F^\infty) = \sum_k \check{C}^k(\mathcal{U}; C_F^\infty) \to \sum_k K^{k,0}(\mathcal{U}) \to K(\mathcal{U})$$

are defined by the natural inclusion maps. By making use of Lemma 4.1 and by the parallel argument of [1, pp. 16-21], we obtain isomorphisms

$$\alpha^*: H_{FDR}^{0,*}(M) \underset{\to}{\cong} H^*(K(\mathcal{U}), \bar{D}),$$

$$\beta^*: H^*(C(\mathcal{U}; \check{C}_F^\infty)) \underset{\to}{\cong} H^*(K(\mathcal{U}), \bar{D}),$$

and then by taking limit of $H^*(C(\mathcal{U}; C_F^\infty))$ for $\mathcal{U}$, $(\beta^*)^{-1}\alpha^*$ defines the isomorphism

$$\phi: H_{FDR}^{0,*}(M) \underset{\to}{\cong} \check{H}^*(M; C_F^\infty).$$

5. MODULAR COHOMOLOGY CLASS AND $X(y_1)$

Let (M,F) be a codimension q foliation, D a positive C^∞-density along leaves of F and μ a positive C^∞-density on M. For a local foliation chart $(U_\alpha, x_i^\alpha, u_j^\alpha)$, $1 \le i \le p$, $1 \le j \le q$, we set

$$\mu_\alpha = \mu(\partial/\partial x_1,\ldots,\partial/\partial x_p, \partial/\partial u_1,\ldots,\partial/\partial u_q),$$

$$D_\alpha = D(\partial/\partial x_1,\ldots,\partial/\partial x_p).$$

Since we have

$$\mu_\alpha/D_\alpha = |\det(\partial u_j^\beta/\partial u_k^\alpha)|\mu_\beta/D_\beta$$

and $\partial u^\beta/\partial u^\alpha$ is constant along leaves, it follows that

$$d_F(\log(\mu_\alpha/D_\alpha)) = d_F(\log(\mu_\beta/D_\beta))$$

on $U_\alpha \cap U_\beta$. Therefore $\{d_F(\log(\mu_\alpha/D_\alpha))\}$ defines a global 1-form on M which is obviously d_F-closed. Therefore, we obtain $[d_F(\log(\mu_\alpha/D_\alpha)] \in H_{FDR}^{0,1}(M)$.

On the other hand, we set

$$C_{\alpha\beta} = \log|\det(\partial u_j^\beta/\partial u_k^\alpha)|$$

on $U_\alpha \cap U_\beta$. Since $C_{\beta\gamma} - C_{\alpha\gamma} + C_{\alpha\beta} = 0$ on $U_\alpha \cap U_\beta \cap U_\gamma$, the Čech cochain $\{C_{\alpha\beta}\} \in \check{C}^1(\mathcal{U}; C_F^\infty)$ is a cocycle and, by taking limit for $\mathcal{U}$, its cohomology class defines an element $[\{C_{\alpha\beta}\}] \in \check{H}^1(M; C_F^\infty)$, which is called the *modular cohomology class* of F and is denoted by $m(M, F)$ (cf. [9, p. 9]).

Lemma 5.1 Let $\Phi: H_{FDR}^{0,1}(M) \overset{\cong}{\to} \check{H}^1(M; C_F^\infty)$ be the isomorphism of Theorem 4.2. Then we have

$$\Phi[\{d_F(\log(\mu_\alpha/D_\alpha))\}] = m(M, F).$$

Proof $\{\log(\mu_\alpha/D_\alpha)\}$ defines an element of $K^0(\mathcal{U})$ and we have

$$\begin{aligned}&\{d_F(\log(\mu_\alpha/D_\alpha))\} - \delta(\{\log(\mu_\alpha/D_\alpha)\})\\ &= \{d_F(\log(\mu_\alpha/D_\alpha))\} - \check{d}(\{\log(\mu_\alpha/D_\alpha)\}) - d_F(\{\log(\mu_\alpha/D_\alpha)\})\\ &= -\check{d}(\{\log(\mu_\alpha/D_\alpha)\}).\end{aligned}$$

By the definition of Čech coboundary operator, it follows that

$$(\check{\delta}(\{\log(\mu_\alpha/D_\alpha)\}))_{\alpha\beta} = \log(\mu_\beta/D_\beta) - \log(\mu_\alpha/D_\alpha)$$

$$= \log(\mu_\beta/D_\beta) - \log(\mu_\beta/D_\beta) - \log|\det(\partial u_j^\beta/\partial u_k^\alpha)|$$

$$= -C_{\alpha\beta}$$

and hence

$$\Phi[\{d_F\log(\mu_\alpha/D_\alpha)\}] = [\{C_{\alpha\beta}\}].$$

Let σ_s^F be a C^∞-singular s-simplex such that the image of σ_s^F is contained in a leaf of F. Let C_s^F denote the vector space over $\mathbb{R}$ with the basis $\{\sigma_s^F\}$. Then we have obviously $\partial C_s^F \subset C_{s-1}^F$ for the boundary operator ∂, and obtain a chain complex

$$(C_s^F, \partial): \ldots \xrightarrow{\partial} C_s^F \xrightarrow{\partial} C_{s-1}^F \xrightarrow{\partial} \ldots \xrightarrow{\partial} C_0^F \to 0,$$

$$C_0^F = \sum_{m \in M} \mathbb{R}_m, \qquad (\mathbb{R}_m \cong \mathbb{R}).$$

One can show a Stokes type formula for (C_s^F, ∂) and $(A^{0,s}, d_F)$.

<u>Lemma 5.2</u> Suppose that $\sigma^F \in C_{s+1}^F$ and $\omega \in A^{0,s}$, then one obtains

$$\int_{\partial\sigma^F} \omega = \int_{\sigma^F} d_F\omega.$$

<u>Proof</u> From the usual Stokes formula, it follows that

$$\int_{\partial\sigma^F} \omega = \int_{\sigma^F} d\omega$$

$$= \int_{\sigma^F} (d_1\omega + d_2\omega + d_F\omega).$$

But we have $d_1\omega \in A^{2,s-1}$, $d_2\omega \in A^{1,s}$ and hence

$$d_1\omega(X_1,\ldots,X_{s+1}) = d_2\omega(X_1,\ldots,X_{s+1})$$
$$= 0$$

for $X_j \in \Gamma(F)$ $j = 1,\ldots,s+1$. Therefore we get

$$\int_{\sigma^F} d_1\omega = \int_{\sigma^F} d_2\omega = 0$$

and the conclusion is shown.

Let $D^q \subset \mathbb{R}^q$ be an open ε-ball around the origin for a sufficiently small number $\varepsilon > 0$, Δ^s the standard s-simplex and $\hat{\sigma}_s^F : D^q \times \Delta^s \to M$ any differentiable map such that $\hat{\sigma}_s^F(x) = \hat{\sigma}_s^F | \{x\} \times \Delta^s \in C_s^F, x \in D^q$. A cochain $\xi \in C_s^F$ is called *differentiable*, if $\xi(\hat{\sigma}_s^F(x))$ is differentiable with respect to x. These cochains make a cochain complex (C_{FD}^s, δ) and its cohomology $H_{FD}^s(M; \mathbb{R})$ satisfies the Mayer-Vietoris sequence property for finite open covers of M. One can define a homomorphism $\lambda : (A^{0,s}, d_F) \to (C_{FD}^s, \delta)$ by

$$\lambda(\omega)(\sigma_s^F) = \int_{\sigma_s^F} \omega .$$

Lemma 5.2 shows that

$$\delta\lambda(\omega)(\sigma_{s+1}^F) = \lambda(\omega)(\partial\sigma_{s+1}^F)$$

$$= \lambda(d_F\omega)(\sigma_{s+1}^F),$$

that is, λ is a cochain map.

We have a natural isomorphism from $H_{FDR}^{0,s}(M)$ to $H_{FD}^s(M; \mathbb{R})$ as follows.

<u>Theorem 5.3</u> If F is a foliation on a compact Hausdorff manifold M, then λ induces an isomorphism

$$\lambda^* : H_{FDR}^{0,s}(M) \underset{\to}{\cong} H_{FD}^s(M; \mathbb{R}).$$

<u>Proof</u> Since the manifold M is compact, by Lemma 4.1, one can find a finite F-simple cover $\mathcal{U}$ of M by open sets. In exactly the same way as for the differentiable singular cochain complex, for F-contractible set E we have $H_{FD}^s(E;\mathbb{R}) = H_{FDR}^{0,s}(E) = 0$ for $s > 0$ and the natural isomorphism $H_{FDR}^{0,0}(E) \underset{\to}{\cong} H_{FD}^0(E;\mathbb{R})$.

By making use of Mayer-Vietoris exact sequences of $H_{FDR}^{0,*}$ and H_{FD}^* and by analogous arguments in the proof [6, Appendix Theorem 3.1] of the isomorphism $H_{DR}^*(M) \underset{\to}{\cong} H_D^*(M, \mathbb{R})$, one can see that the natural cochain map induces the isomorphism $H_{FDR}^{0,s}(M) \underset{\to}{\cong} H_{FD}^s(M; \mathbb{R})$.

<u>Theorem 5.4</u> Let (M, F) be a foliation on a compact Hausdorff manifold. Then we have

$$\chi(y_1) = \tilde{\chi}([(h_1)_{0,\ 1}]),$$

$$\phi(2\pi[(h_1)_{0,\ 1}]) = -m(M,F).$$

Proof The first equation is obvious by Theorem 3.3. Let $c:[0,1] \to M$ be a closed piecewise C^∞-curve on a leaf of F. By [9, Lemma 2.2 and Section 3], one obtains

$$\lambda(2\pi(h_1)_{0,\ 1})(c) = 2\pi h_1 (c)$$

$$= - \lambda\{d_F(\log(\mu_\alpha/D_\alpha))\}(c).$$

This means by Theorem 5.3, that

$$[2\pi(h_1)_{0,\ 1}] = -[\{d_F(\log(\mu_\alpha/D_\alpha))\}] \in H^{0,\ 1}_{FDR}(M).$$

Lemma 5.1 shows the conclusion.

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Lagrangian foliations and characteristic classes

This communication is a preliminary exposition concerning higher order generalizations of the Maslov class within the framework of the theory of secondary characteristic classes. A full version and complete proofs are expected to appear elsewhere.

The Maslov class appeared as an obstruction to the transversality of a Lagrangian submanifold to a fixed Lagrangian foliation in $\mathbb{R}^{2n}$ [3], and in [6] it has been remarked that it is the first of a certain series of secondary characteristic classes. Here, we consider all these classes (using the Chern-Simons-Bott approach) in the most general situation, and we discuss them as transversality obstructions. Then, we compute the classes considered for a Lagrangian submanifold of a Kähler manifold endowed with a parallel Lagrangian foliation, and we show that they are represented by means of various traces of the second fundamental form of the Lagrangian submanifold. This generalizes a result of J.M. Morvan [8].

1. REMARKS ON LAGRANGIAN FOLIATIONS

Though this is not our main object, we start with a few remarks about Lagrangian foliations.

A pair (V^{2n},Ω) where V is a 2n-dimensional differentiable manifold (we work in the C^{∞}-category), and Ω is a nondegenerate 2-form is an *almost symplectic manifold*, and if $d\Omega = 0$ it is a *symplectic manifold*. A submanifold M of V is *Lagrangian* if $\dim M = n$, and if Ω induces on M the zero form. A (distribution) foliation L_0 of V is *Lagrangian* if it consits of Lagrangian (planes) leaves, and we shall say that the pair (V,L_0) is an *(almost) Lagrangian manifold*. The typical example of a Lagrangian manifold is given by any cotangent bundle with the foliation defined by its fibres.

It is a basic fact that all the Lagrangian manifolds are locally equivalent [10], and this follows from

Theorem 1.1 (S. Lie). Let (V^{2n},Ω,L_0) be a Lagrangian manifold. Then, every point $x \in V$ has a neighbourhood U_x endowed with local coordinates

(x^α, y^α) $(\alpha = 1,\ldots,n)$ such that L_0 is given by x^α = const., and $\Omega = \Sigma_{\alpha=1}^n dx^\alpha \wedge dy^\alpha$.

The local coordinates of this theorem yield an atlas with transition functions of the local form

$$\tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta), \; \tilde{y}^\alpha = \sum_\gamma \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha}(x^\beta)y^\gamma + \phi^\alpha(x^\beta). \tag{1.1}$$

Hence, if $SpL(n,\mathbb{R})$ is the group of the symplectic 2n-matrices of the form

$$\begin{pmatrix} A & 0 \\ \underbrace{B}_{n} & \underbrace{C}_{n} \end{pmatrix} \begin{matrix} \}n \\ \}n \end{matrix} \quad ({}^tAC = Id., \; {}^tAB = {}^tBA), \tag{1.2}$$

we have

<u>Proposition 1.2</u> An almost Lagrangian manifold is a manifold endowed with a $SpL(n,\mathbb{R})$-structure, and the manifold is Lagrangian iff the $SpL(n,\mathbb{R})$-structure is integrable.

This remark allows for the utilization of the theory of G-structures in the study of Lagrangian manifolds.

On the other hand, the global equivalence of Lagrangian manifolds is a difficult open problem, and we should like to indicate a method of obtaining global invariants.

In view of (1.1), it makes sense to define, on the Lagrangian manifold (V,L_0), the *basic sheaf* S of germs of the functions $f = \Sigma_\alpha a_\alpha(x^\beta)y^\alpha + b(x^\beta)$, and it is clear that the cohomology spaces $H^i(V,S)$ will be global Lagrangian invariants.

Hopefully, these invariants could be computed as follows. Let Φ be the sheaf of germs of the functions $V \to \mathbb{R}$ that are constant on the leaves of L_0, and let ψ be the sheaf of germs of the projectable cross-sections of the transversal bundle of L_0. Then, there is an inclusion $i:\Phi \to S$, and an epimorphism $\sigma:S \to \psi$ given by

$$\sigma(\sum_\alpha a_\alpha(x^\beta)y^\alpha + b(x^\beta)) = \sum_\alpha a_\alpha(x^\beta)(\mathrm{sg}\, x^\alpha), \tag{1.3}$$

where sg h denotes the "Ω-gradient" of the function h, and it is easy to prove

<u>Proposition 1.3</u> The sequence

$$0 \to \Phi \xrightarrow{i} S \xrightarrow{\sigma} \psi \to 0 \tag{1.4}$$

is an exact sequence of sheaves.

Since general foliation theory yields computation methods for $H^*(V,\Phi)$ and $H^*(V,\psi)$ [9], the exact sequence (1.4) might provide the computation of $H^*(V,S)$.

2. SECONDARY CHARACTERISTIC CLASSES

Now, before defining Maslov classes, we need an adequate sketch of the Chern-Simons-Bott theory of secondary characteristic classes [2], [1].

Let G be a Lie group, let g be its Lie algebra, and let $I(G) = \bigoplus_{k\geq 0} I^k(G)$ be its Weil algebra of the multilinear, symmetric, ad$g$-invariant functions (or polynomials) $g \to \mathbb{R}$ [5]. Furthermore, let $\pi: P \to M$ be a G-principal bundle, and $\theta, \theta', \ldots$ be connection forms on P with the curvature forms $\Theta, \Theta', \ldots$. In the sequel, we shall sometimes identify the projectable forms on P with the corresponding forms on M.

Following [1], one takes a connection $\tilde{\theta} = \Sigma_0^r t_h\theta_h$, where $(t_h) \in \Delta^r$ = the standard r-symplex, with the curvature $\tilde{\Theta}$, on $P \times \Delta^r \to M \times \Delta^r$, and one defines

$$\Delta_{\theta_0 \ldots \theta_r} : I^k(G) \to \Lambda^{2k-r}(M)$$

(Λ denotes the exterior forms functor) by

$$\Delta_{\theta_0 \ldots \theta_r} f = \int_{\Delta^r} f(\tilde{\Theta}^{(k)}), \ (f \in I^k(G)). \tag{2.1}$$

This yields

$$d\Delta_{\theta_0 \ldots \theta_r} f = \sum_{h=0}^{r} (-1)^h \Delta_{\theta_0 \ldots \theta_{h-1}\theta_{h+1} \ldots \theta_r} f. \tag{2.2}$$

Then Δ_{θ_0} is the *Chern-Weil homomorphism*, and the forms in im Δ_{θ_0} represent the *principal characteristic classes* of P. The latter do not depend on the choice of the connection since (2.2) yields

$$d\Delta_{\theta_0\theta_1} f = \Delta_{\theta_1} f - \Delta_{\theta_0} f. \tag{2.3}$$

For further necessities let us also note the formula

$$\Delta_{\theta_0\theta_1} f = k \int_0^1 f(\theta_1-\theta_0, \tilde{\Theta}_t^{(k-1)})dt, \tag{2.4}$$

where $\tilde{\theta} = (1-t)\theta_0 + t\theta_1$, and $\tilde{\Theta}_t = \tilde{\Theta}(\text{mod. } dt)$.

On the other hand, there are the *transgression forms* on P [2]

$$T_\theta f = \int_{\Delta^1} f(\bar{\Theta}^{(k)}) = k \int_0^1 f(\Theta,\bar{\Theta}_t^{(k-1)})dt \tag{2.5}$$

where $\bar{\Theta} = d(t\theta) + [t\theta,t\theta]$ (the bracket is in g), and $\bar{\Theta}_t = \bar{\Theta}$ (mod. dt). These forms satisfy the basic relation

$$d(T_\theta f) = \pi^*\Delta_\theta f. \tag{2.6}$$

Accordingly, we have the following definitions. If $f \in \ker \Delta_\theta$, $T_\theta f$ is closed, and $[T_\theta f] \in H^{2k-1}(P,\mathbb{R})$ are the *Chern-Simons classes* of (P,θ). If $f \in \ker \Delta_{\theta_0} \cap \ker \Delta_{\theta_1}$, $\Delta_{\theta_0\theta_1} f$ is closed, and $[\Delta_{\theta_0\theta_1} f] \in H^{2k-1}(M,\mathbb{R})$ are the *secondary characteristic classes* of (P,θ_0,θ_1).

Furthermore, let be $I = \{s/0 \leq s \leq 1\}$, let ϕ be a connection on $P \times I \to M \times I$, and $\theta_s = \phi/_{P \times \{s\}}$. We shall say that ϕ is a *deformation* of θ_0 and a *link* of θ_0,θ_1. Analytically, one has $\phi = \theta_s + \alpha ds$ for some function $\alpha: P \times I \to g$, and its curvature is

$$\Phi = \Theta_s + \left(d\alpha + [\theta_s,\alpha] - \frac{\partial\theta_s}{\partial s}\right) \wedge ds, \tag{2.7}$$

where $\Theta_s = d\theta_s + [\theta_s,\theta_s]$, whence for $f \subset I^k(G)$

$$f(\Phi^{(k)}) = f(\Theta_s^{(k)}) + kf\left(d\alpha + [\theta_s,\alpha] - \frac{\partial\theta_s}{\partial s}, \Theta_s^{(k-1)}\right) \wedge ds. \tag{2.8}$$

Now, if we denote $i_s: P = P \times \{s\} \subset P \times I$, it is well known that one has for forms of $P \times I$

$$i_1^* - i_0^* = hd + dh, \tag{2.9}$$

where h is "fibre integration" on $P \times I$, and applied to $T_\phi f$, this gives, in view of (2.8)

$$T_{\theta_1} f - T_{\theta_0} f = k \int_0^1 f\left(\frac{\partial\theta_s}{\partial s} - d\alpha - [\theta_s,\alpha], \Theta_s^{(k-1)}\right)ds + \text{exact form}. \tag{2.10}$$

Particularly, by taking $\phi = \theta_0 + s(\theta_1 - \theta_0)$ we obtain

Theorem 2.1 The following relation always holds between Chern-Simons and secondary characteristic classes:

$$\pi^*[\Delta_{\theta_0\theta_1} f] = [T_{\theta_1} f] - [T_{\theta_0} f]. \tag{2.11}$$

Similarly, if (θ_0,θ_1), (θ_0',θ_1') are pairs of connections and $\phi_\lambda = \theta_s^\lambda + \alpha_\lambda ds$ are links of θ_λ, θ_λ' $(\lambda = 0,1)$, then (2.8) and (2.9) (on M) yield

$$\begin{aligned}\Delta_{\theta_0'\theta_1'} f - \Delta_{\theta_0\theta_1} f = k \int_0^1 f\left(\frac{\partial\theta_s^1}{\partial s} - d\alpha_1 - [\theta_s^1,\alpha_1],\ \Theta_s^{1(k-1)}\right) ds\ - \\ - k \int_0^1 f\left(\frac{\partial\theta_s^0}{\partial s} - d\alpha_0 - [\theta_s^0,\alpha_s],\ \Theta_s^{0(k-1)}\right) ds + \text{exact form.}\end{aligned} \tag{2.12}$$

Now, following D. Lehmann [7], we shall say that the connections θ_0,θ_1 of P are *f-homotopic* if there is a link ϕ of θ_0,θ_1 such that $f \in \ker \Delta_\phi$. If this happens, the integrals in (2.10) and (2.12) vanish, and we get

Theorem 2.2 If (θ_0,θ_0') and (θ_1,θ_1') are f-homotopic connections, respectively, then $[T_{\theta_0} f] = [T_{\theta_0'} f]$, and $[\Delta_{\theta_0\theta_1} f] = [\Delta_{\theta_0'\theta_1'} f]$.

This theorem clarifies the dependency of Chern-Simons and secondary characteristic classes on the choice of the connections.

Remark Formulas (2.10) and (2.12) yield easily the following generalizations of the Chern-Simons and Heitsch derivation formulas [2], [4]

$$\frac{\partial(T_{\theta_s} f)}{\partial s} = kf\left(\frac{\partial\theta_s}{\partial s} - d\alpha_s - [\theta_s,\alpha_s],\ \Theta_s^{(k-1)}\right) + \text{exact form,} \tag{2.13}$$

$$\begin{aligned}\frac{\partial(\Delta_{\theta_s^0\theta_s^1} f)}{\partial s} = k\Bigg\{ f\left(\frac{\partial\theta_s^1}{\partial s} - d\alpha_s^1 - [\theta_s^1,\alpha_s^1],\ \Theta_s^{1(k-1)}\right) - \\ - f\left(\frac{\partial\theta_s^0}{\partial s} - d\alpha_s^0 - [\theta_s^0,\alpha_s^0],\ \Theta_s^{0(k-1)}\right)\Bigg\} + \text{exact form,}\end{aligned} \tag{2.14}$$

where $\alpha_s = \alpha/_s = \text{const}\cdot$ (The original formulas were for $\alpha = 0$.)

3. GENERALIZED MASLOV CLASSES

Let $\pi: E \to M$ ($\dim M = m$, rank $E = 2n$) be a *symplectic vector bundle* with its structure defined by a nondegenerate cross section Ω of $\Lambda^2 E^*$. Then, a fibre basis $(e_1,\ldots,e_{2n})$ is *symplectic* if it assigns to Ω the canonical expression, and these bases yield the $Sp(n)$-principal bundle $\pi: S(E) \to M$ ($Sp(n)$ denotes the symplectic group).

It is classical that E admits $U(n)$-reductions ($U(n)$ is the unitary group), of the structure group defined by fibre complex structure operators J, and any two such structures are homotopically related by a family J_s. We shall choose one such reduction, and denote by $\pi: \mathcal{U}_J(E) \to M$ the $U(n)$-principal sub-bundle of $S(E)$ given by the unitary bases $(e_1,\ldots,e_n, Je_1,\ldots,Je_n)$ or, in the equivalent complex form, by the bases

$$\tilde{e}_i = (e_i - \sqrt{-1}\, Je_i)/\sqrt{2} \qquad (i = 1,\ldots,n). \tag{3.1}$$

The characteristic classes which we have in mind are then related to the *Chern polynomials* $c_k \in I(U(n))$ defined for $A \in u(n)$ = the unitary Lie algebra by [5]

$$c_k(A) = \left(\frac{-1}{2\pi\sqrt{-1}}\right)^k \operatorname{tr} \Lambda^k A. \tag{3.2}$$

First of all, using a connection θ on $\mathcal{U}_J(E)$ we obtain the *Chern classes* $[\Delta_\theta c_k] \in H^{2k}(M,\mathbb{R})$, and a simple homotopy argument shows that they depend only on the symplectic structure of E (i.e., they do not depend on the choice of J).

Furthermore, assume that we also have a *Lagrangian sub-bundle* L_0 of E. Then we can further reduce the structure group of E to the orthogonal group $O(n)$, and get the $O(n)$-principal sub-bundle $\pi: \mathcal{U}_J(E,L_0) \to M$ of $\mathcal{U}_J(E)$, defined by the unitary frames (3.1) such that $e_i \in L_0$ ($i = 1,\ldots,n$). Then, it is classical that $c_{2h-1} \in \ker \Delta_{\theta_0}$ for every $O(n)$-connection θ_0, and, therefore, we obtain Chern-Simons classes

$$\mu_h(E,L_0) = [T_{\theta_0} c_{2h-1}] \in H^{4h-3}(\mathcal{U}_J(E), \mathbb{R}). \tag{3.3}$$

The classes $\mu_h(E,L_0)$ will be called the *bundle Maslov classes* of (E,L_0). Since any two $O(n)$-connections are c_{2h-1}-homotopic [7], it follows from Theorem 2.2 that these classes do not depend on the choice of θ_0.

If θ_0 is represented by the local equations (*that use the Einstein summation convention*)

$$D\tilde{e}_i = \omega_i^j \tilde{e}_j \quad (\omega_i^j + \omega_j^i = 0; i,j = 1,\ldots,n) \tag{3.4}$$

with respect to bases (3.1) in $U_J(E,L_0)$, then the corresponding global connection form on $U_J(E)$ is defined by

$$\theta_i^j = \eta_h^j \, d\xi_i^h + \eta_h^j \xi_i^{\ell} \omega_\ell^h , \tag{3.5}$$

where $(\xi_i^j) \in U(n)$, and (η_i^j) is the inverse matrix of (ξ_i^j). Now, it follows from (2.5) and (3.2) that $\mu_h(E,L_0)$ are represented by

$$T_\theta c_{2h-1} = \frac{(-1)^{h+1}\sqrt{-1}}{(2h-2)!\,(2\pi)^{2h-1}} \int_0^1 \Big[\delta^{j_1 \ldots j_{2h-1}}_{i_1 \ldots i_{2h-1}} \, \theta^{i_1}_{j_1} \wedge (\bar{\theta}_t)^{i_2}_{j_2} \wedge \ldots \tag{3.6}$$

$$\ldots \wedge (\bar{\theta}_t)^{i_{2h-1}}_{j_{2h-1}}\Big]\, dt,$$

where $\bar{\theta}_i^j$ are computed as shown for (2.5), and using (3.5).

Particularly, we get

$$T_\theta c_1 = \frac{\sqrt{-1}}{4\pi}\, d \ln \det{}^2(\xi_i^j), \tag{3.7}$$

and it follows easily that $\mu_1(E,L)$ is the lift to $U_J(E)$ of $-(1/2)m(L)$, where $m(L)$ is the usual Maslov class on the bundle $L(E)$ of the Lagrangian subspaces of the fibres of E [3].

Now, let L_1 be one more Lagrangian sub-bundle of E, and let θ_1 be an $O(n)$-connection on $U_J(E)$ defined by the new reduction of the structure group to $O(n)$ given by L_1. Then, we clearly get secondary characteristic classes

$$\mu_h(E,L_0,L_1) = [\Delta_{\theta_0\theta_1} c_{2h-1}] \in H^{4h-3}(M,\mathbb{R}), \tag{3.8}$$

and these will be called the (*generalised*) *Maslov classes* of L_1 with respect to L_0. Using again the c_{2h-1}-homotopy argument, it follows that $\mu_h(E,L_0,L_1)$ do not depend on the choice of the $O(n)$-connections θ_0, θ_1 and, also, the homotopy of any two adapted complex structures J allows us to prove that $\mu_h(E,L_0,L_1)$ do not depend on the choice of J.

In order to compute the Maslov classes, we represent again θ_0 by (3.4), and we represent θ_1 by similar equations $D\tilde{e}'_i = \omega'^j_i \tilde{e}'_j$, where primes denote that we have the similar quantities associated to L_1 instead of L_0. Then, let us take some fixed unitary bases (ε_i) in $U_J(E)$. We shall have transition relations of the form

$$\varepsilon_i = \gamma^j_i \tilde{e}_j , \ \varepsilon_i = \gamma'^j_i \tilde{e}'_j, \tag{3.9}$$

and new connection forms

$$\begin{aligned} (\theta_0) \ \pi^j_i &= \beta^j_h d\gamma^h_i + \beta^j_h \gamma^\ell_i \omega^h_\ell , \\ (\theta_1) \ \pi'^j_i &= \beta'^j_h d\gamma'^h_i + \beta'^j_h \gamma'^\ell_i \omega'^h_\ell , \end{aligned} \tag{3.10}$$

where the matrices β, β' are inverse to γ, γ', respectively. From (3.10), we can further compute the curvature $\tilde{\Theta}_t$ needed in (2.4), and, accordingly, write down

$$\begin{aligned} \Delta_{\theta_0\theta_1} c_{2h-1} = \frac{(-1)^{h+1}}{(2h-2)!} \frac{\sqrt{-1}}{(2\pi)^{2h-1}} \int_0^1 \Big[& \delta^{j_1 \ldots j_{2h-1}}_{i_1 \ldots i_{2h-1}} (\pi'^{i_1}_{j_1} - \pi^{i_1}_{j_1}) \wedge \\ & \wedge (\tilde{\Theta}_t)^{i_2}_{j_2} \wedge \ldots \wedge (\tilde{\Theta}_t)^{i_{2h-1}}_{j_{2h-1}} \Big] dt, \end{aligned} \tag{3.11}$$

thereby obtaining representatives of the Maslov classes [3], [6].

Particularly, by taking $\varepsilon_i = \tilde{e}_i$, and in view of (3.7), we get $\mu_1(E,L_0,L_1) = (1/2)m(L_0,L_1)$ where $m(L_0,L_1)$ is the usual Maslov class of L_1 with respect to L_0.

Now, we can obtain some basic properties of the Maslov classes defined above.

<u>Theorem 3.1</u> If the Lagrangian sub-bundles L_0, L_1 are everywhere transversal then all $\mu_h(E,L_0,L_1) = 0$.

Indeed, in this case we may choose J such that $L_1 = JL_0$, and we may choose bases such that $e'_i = Je_i$ $(i = 1,\ldots,n)$, and $\varepsilon_i = \tilde{e}_i$ (see formulas (3.1) and (3.9) for notation). The forms of (3.10) will then be related by $\omega'^j_i = \omega^j_i$, $\pi^j_i = \omega^j_i$, $\pi'^j_i = \omega^j_i$. Consequently, the first factor in (3.11) vanishes, and we get the conclusion.

<u>Remark</u> Theorem 3.1 shows that $\mu_h(E,L_0,L_1)$ are obstructions to the transversality of L_0, L_1, but it is clear that the conclusion also holds if we assume

only that L_0, L_1 can be deformed via Lagrangian sub-bundles to transversal bundles L_0', L_1'.

Theorem 3.2 For Maslov classes, the following relations always hold

(a) $\pi^*\mu_h(E,L_0,L_1) = \mu_h(E,L_1) - \mu_h(E,L_0)$;

(b) $\mu_h(E,L_0,L_1) = -\mu_h(E,L_1,L_0)$;

(c) $\mu_h(E,L_0,L_1) + \mu_h(E,L_1,L_2) + \mu_h(E,L_2,L_0) = 0$.

Indeed, (a) follows from (2.11); (b) follows from either (2.1) or (2.4), and (c) follows from (2.2) or, more precisely, from

$$d(\Delta_{\theta_0\theta_1\theta_2}\, c_{2h-1}) = \Delta_{\theta_1\theta_2}\, c_{2h-1} - \Delta_{\theta_0\theta_2}\, c_{2h-1} + \Delta_{\theta_0\theta_1}\, c_{2h-1}.$$

Remark Property (c) above shows that $\mu_h(E,L_0,L_1) = 0$ if L_0, L_1 admit a (global) common transversal Lagrangian sub-bundle L_2. Hence, these classes are obstructions to the existence of the latter.

4. MASLOV CLASSES AND THE SECOND FUNDAMENTAL FORM

As seen in the introduction, an important transversality problem is that of the transversality between a Lagrangian submanifold M^n of a symplectic manifold V^{2n}, and a Lagrangian foliation L_0 of the latter. In this case, $TV/_M$ is a symplectic vector bundle $E \to M$, $L_0 = L_0/_M$, $L = TM$ are Lagrangian sub-bundles of E, and we are interested in the transversality of these two sub-bundles. From Section 3, we know that the *Maslov classes* $\mu_h(M,L_0) \overset{\text{def.}}{=} \mu_h(E,L_0,L)$ provide obstructions to the transversality of M and L_0.

Generally, the computation of these classes is difficult, but we can compute them in a particular case where the results are both nice and important since it includes $V = \mathbb{R}^{2n}$. Namely, we shall assume that V admits a compatible Kähler structure (J,g) such that L_0 is parallel with respect to the metric g. One can prove that g is then, necessarily, a flat Kähler metric. Clearly, $\mathbb{R}^{2n} = \mathbb{C}^n$ with the "horizontal" n-dimensional distribution L_0 is of this type, and also, if N is any locally flat Riemannian manifold, the cotangent bundle $V = T^*N$ has a natural flat Kähler structure (J,g) such that the fibres of T^*N are g-parallel.

Now, let M be a Lagrangian submanifold of the manifold (V,L_0) considered above, and let us take the bases e'_i $(i = 1,\dots,n)$ needed for (3.9), (3.10), etc. to be orthonormal tangent bases of M; then take ε_i of (3.9) to be

$$\varepsilon_i = (e'_i - \sqrt{-1}\ \ Je'_i)/\sqrt{2} \tag{4.1}$$

Since L_0 is parallel, the Levi-Civita connection ∇ of g induces a connection in L_0, which extends to a connection θ_0 in $U_J(E)$ usable in the computation of the Maslov classes, and which must have some local equations

$$D\varepsilon_i = \pi_i^j \varepsilon_j \tag{4.2}$$

(hence, in this case, we do not need the bases $\tilde{e}_j$ of (3.9) for the computation).

On the other hand, it is also clear that we may take the connection θ_1 needed in the computation of the Maslov classes to be defined by the connection induced by ∇ on M. The latter is determined by the Gauss equations of M, which can be written as

$$\nabla e'_i = \pi'^j_i\ e'_j + \beta_i^j(Je'_j), \tag{4.3}$$

since (Je'_i) is a normal basis of M. In (4.3), π' is the matrix of the induced connection, and β is a matrix of 1-forms which defines the *second fundamental form* of M. Accordingly, θ_1 has the local equations

$$D\varepsilon_i = \pi'^j_i\ \varepsilon_j. \tag{4.4}$$

Now, we obtain from (4.1), (4.2) and (4.3) that

$$\pi'^j_i - \pi_i^j = -\sqrt{-1}\ \beta_i^j\ , \tag{4.5}$$

and furthermore, the curvature needed in (3.11) can be computed from the Gauss-Codazzi integrability conditions of (4.3) together with the fact that ∇ has zero curvature.

After this computation, we shall get from (3.11) that the representative forms of the Maslov classes of M and L_0 are given by

$$\Delta_{\theta_0\theta_1} c_1 \doteq \frac{1}{2\pi} \beta_i^i\ , \tag{4.6}$$

which can be seen to be equivalent to the interpretation of J.M. Morvan [8], and

$$\Delta_{\theta_0\theta_1} c_{2h-1} = \frac{1}{(2\pi)^{2h-1}} \nu_h \, \delta^{j_1 \dots j_{2h-1}}_{i_1 \dots i_{2h-1}} \beta^{i_1}_{j_1} \wedge \beta^{k_2}_{j_2} \wedge \beta^{i_2}_{k_2} \wedge \dots \wedge \beta^{k_{2h-1}}_{j_{2h-1}} \wedge \beta^{i_{2h-1}}_{k_{2h-1}} \tag{4.7}$$

where ν_h are constants given by

$$\nu_h = \sum_{i=0}^{2h-2} \frac{(-1)^{h+i+1}}{(2h-2)!} \frac{2^i}{4h-i-3} \binom{2h-2}{i} \tag{4.8}$$

In other words, the Maslov classes $\mu_h(M,L_0)$ are given by various traces of the second fundamental form, and we have

Theorem 4.1 Let V be a Kähler manifold endowed with a parallel Lagrangian foliation L_0, and let M be a Lagrangian submanifold of V. Then the Maslov classes $\mu_h(M,L_0)$ depend only on the second fundamental form of M in V, and they vanish if M is a totally geodesic submanifold of V (and, moreover, $\mu_1 = 0$ if M is a minimal submanifold).

We may expect to be able to use a similar method of computation for any cotangent bundle $V = T^*N$ of a Riemannian manifold N, by replacing ∇ with an adequate metric almost complex connection $\bar{\nabla}$, and by replacing β with a $\bar{\nabla}$-second fundamental form. The results (except for μ_1) will be more complicated since they will involve the (non-vanishing) curvature of $\bar{\nabla}$.

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Examples of circle foliations on open 3-manifolds

INTRODUCTION

In [3] D.B.A. Epstein showed that every C^r-foliation F ($1 \leq r \leq \infty$; for $r = 0$ see [7]) of a compact 3-manifold M by circles is C^r-diffeomorphic to a Seifert fibration on M, i.e. to a foliation which near each leaf is given by the orbits of a locally free circle action. For non-compact M the situation is more complicated. In [9, pp. 113-115], G. Reeb produces a C^∞-foliation F of codimension 2 on an open subset M of $S^{n-2} \times S^1 \times S^1$ with all leaves compact such that $B_1(F) = \{x \in M: \lambda \text{ is not locally bounded in } x\}$ is not empty. Here $\lambda: M \to (0,\infty)$ is the function assigning to $x \in M$ the volume of the leaf through x with respect to some Riemannian metric. Reeb assumes $n \geq 4$, but his formulae also work for $n = 3$. A slight variant of Reeb's example is the real analytic example of D.B.A. Epstein in [3].

$B_1(F)$ is the "obstruction" for F being a Seifert fibration, i.e. a foliation F by circles is a Seifert fibration iff $B_1(F) = \emptyset$. (A completely analogous result is true for higher dimensional foliations. See [4].)

$B_1(F)$ is the first set in the (coarse) Epstein hierarchy of bad sets of a foliation with all leaves compact. One defines by transfinite induction for every ordinal $\alpha > 1$

$$B_\alpha = B_\alpha(F) = \begin{cases} \bigcap\limits_{\beta<\alpha} B_\beta\,, & \text{if } \alpha \text{ is a limit ordinal.} \\ \{x \in B_{\alpha-1} : \lambda|B_{\alpha-1} \text{ is not locally bounded in } x\}, & \\ & \text{if } \alpha-1 \text{ exists.} \end{cases}$$

$|EF| = \sup\{\alpha : B_\alpha \neq \emptyset\}$ is a countable ordinal called the length of the Epstein hierarchy $EF = \{B_1(F) \supset B_2(F) \supset \ldots\}$. $|EF|$ measures how complicated F is: B_α is the obstruction to $F|B_{\alpha-1}$ being a Seifert fibration.

This paper is one in a series of three. Of the other two, one is mainly expository and is concerned with the structure of the bad sets B_α. In the last paper we show that $\tilde{H}_*(M) \neq 0$ if M supports a circle foliation with Epstein hierarchy of finite length.

Our interest in the study of circle foliations on open 3-manifolds stems from the following question raised by D.B.A. Epstein in [5]: Can $\mathbb{R}^3$ be foliated by circles? The result above shows that such a foliation must have infinite Epstein hierarchy.

The purpose of our examples is to show the following:

(1) Many interesting spaces which do not admit Seifert fibrations admit circle fibrations, e.g. complements of some wild knots in S^3.

(2) $B_1(F)$ may have rather unpleasant topological properties even if $B_2(F) = \emptyset$.

(3) There are circle foliations with Epstein hierarchy of infinite length (this contradicts the remark on p. 28 of [2] claiming that $B_2(F) = \emptyset$ for any foliation with all leaves compact of codimension 2).

(4) Describe a possible approach to put a circle foliation on $\mathbb{R}^3$. This approach necessitates a countable number of extensions of certain circle foliations from a solid torus V to a larger solid torus in which V is trivially embedded. We show that the first two extensions can be made. The result is a (very complicated) circle foliation F on an open solid torus with $|EF| = 2$.

2. VARIANTS OF REEB'S EXAMPLE

Let E' be $S^1 \times S^1 \times [-1,1]$ with coordinates (x_1,x_2,ϕ,t), $(x_1,x_2) \in \mathbb{R}^2$, $x_1^2 + x_2^2 = 1$, $\phi \in \mathbb{R}$ mod 1, and $t \in [-1,1]$. Let $f:[-1,1] \to \mathbb{R}$ be a C^r-map, $1 \le r \le \omega$, such that $f^{-1}(0) = \{0\}$. We consider on E' the Pfaffian system

$$\Omega(f) = \{\omega_1,\omega_2\} = \{dt,\ dx_1 + f(t)d\phi\}.$$

$\Omega(f)$ is non-singular on

$$E = E' \setminus \{(x_1,x_2,\phi,t) \in E' : |x_1| = 1,\ t = 0\}$$

and completely integrable. Therefore $\Omega(f)$ defines a C^r-foliation $F(f)$ on E. (For $f(t) = t$, $t \in [0,1]$, we obtain Epstein's example in [3].) Here is a more geometric description of $F(f)$. Consider the level $E_s = E \cap (S^1 \times S^1 \times \{s\})$. Because of $dt = 0$ on $F(f)$ each E_s is saturated, and for $s \ne 0$ E_s is a 2-torus foliated by the graphs of the maps $\phi_c(x_1,x_2) = -\frac{1}{f(x)} x_1 + c$ from $S^1 \subset \mathbb{R}^2$ to $\mathbb{R}$ mod 1. For s (and therefore f(s)) close to 0, a leaf on E_s (for $f(s) > 0$) is shown in Figure 1. On E_0 the leaves are of the form

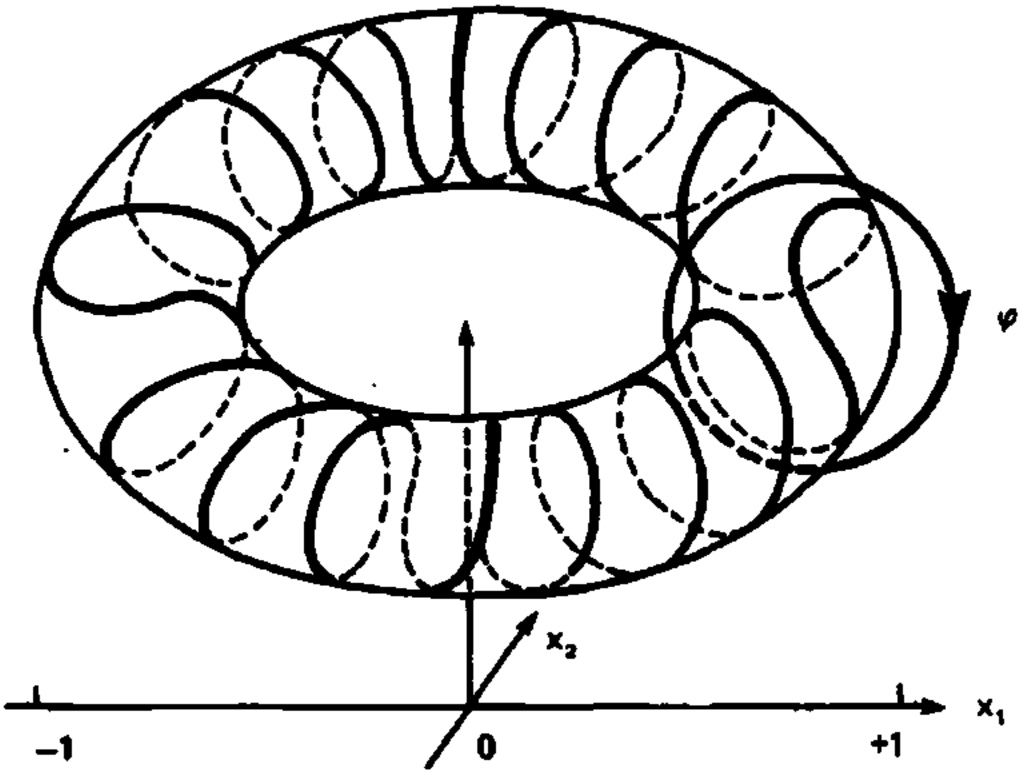

Figure 1

$(x_1,x_2) \times S^1 \times \{0\}$ for $-1 < x_1 < 1$ (Figure 2). The union of these leaves is $B_1(F(f))$, and $B_2(F(f)) = \emptyset$.

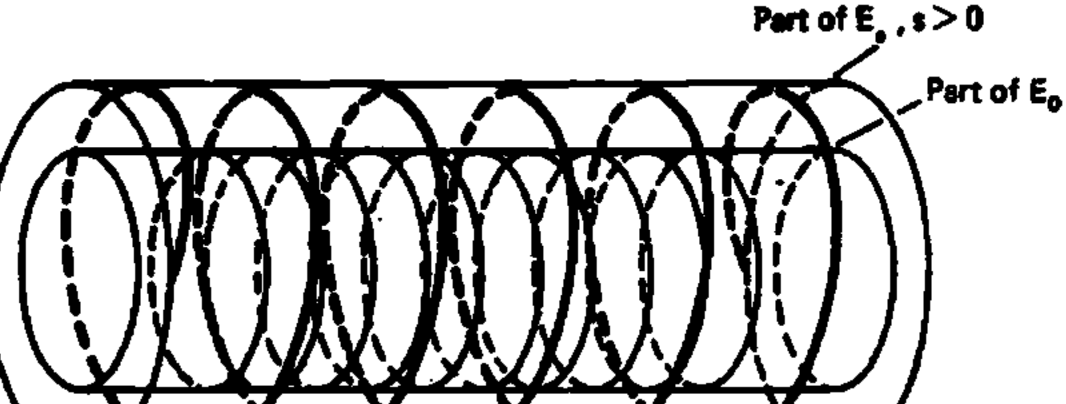

Figure 2

By piecing together various parts of some $F(f)$ with Seifert fibre spaces one obtains already some interesting circle foliations. We will describe foliations on the complement of (possibly infinite) products of torus knots. It is not hard to see that no complement of a product of two non-trivial knots admits a Seifert fibration and thus the total spaces of our examples will not carry any Seifert fibre space structure.

Let p,q be relatively prime integers. $F(p,q)$ denotes the foliation of $S^3 = \{(z_1,z_2) \in \mathbb{C}^2 : z_1\bar{z}_1 + z_2\bar{z}_2 = 1\}$ given by the orbits of the restriction to S^3 of the circle action $t \to ((z_1,z_2) \to (e^{2\pi itp} z_1, e^{2\pi itq} z_2))$, $t \in \mathbb{R}$ mod 1, on $\mathbb{C}^2$. The standard torus knot $k_{p,q}$ is the orbit through $(1/\sqrt{2}, 1/\sqrt{2})$ of this action. $F_{p,q}$ induces a Seifert fibration on the complement $K_{p,q}$ of any invariant tubular neighbourhood of $k_{p,q}$ in S^3. By taking $(1/\sqrt{2}, 1/\sqrt{2})$ to be ∞ in S^3 we may obtain an embedding of $S^3 \setminus k_{p,q} \subset \mathbb{R}^3$, with the knot being the x_1-axis in $\mathbb{R}^3 \setminus \{|x_1| \geq 1\}$ and $K_{p,q}$ contained in $\{|x_1| < 1\}$.

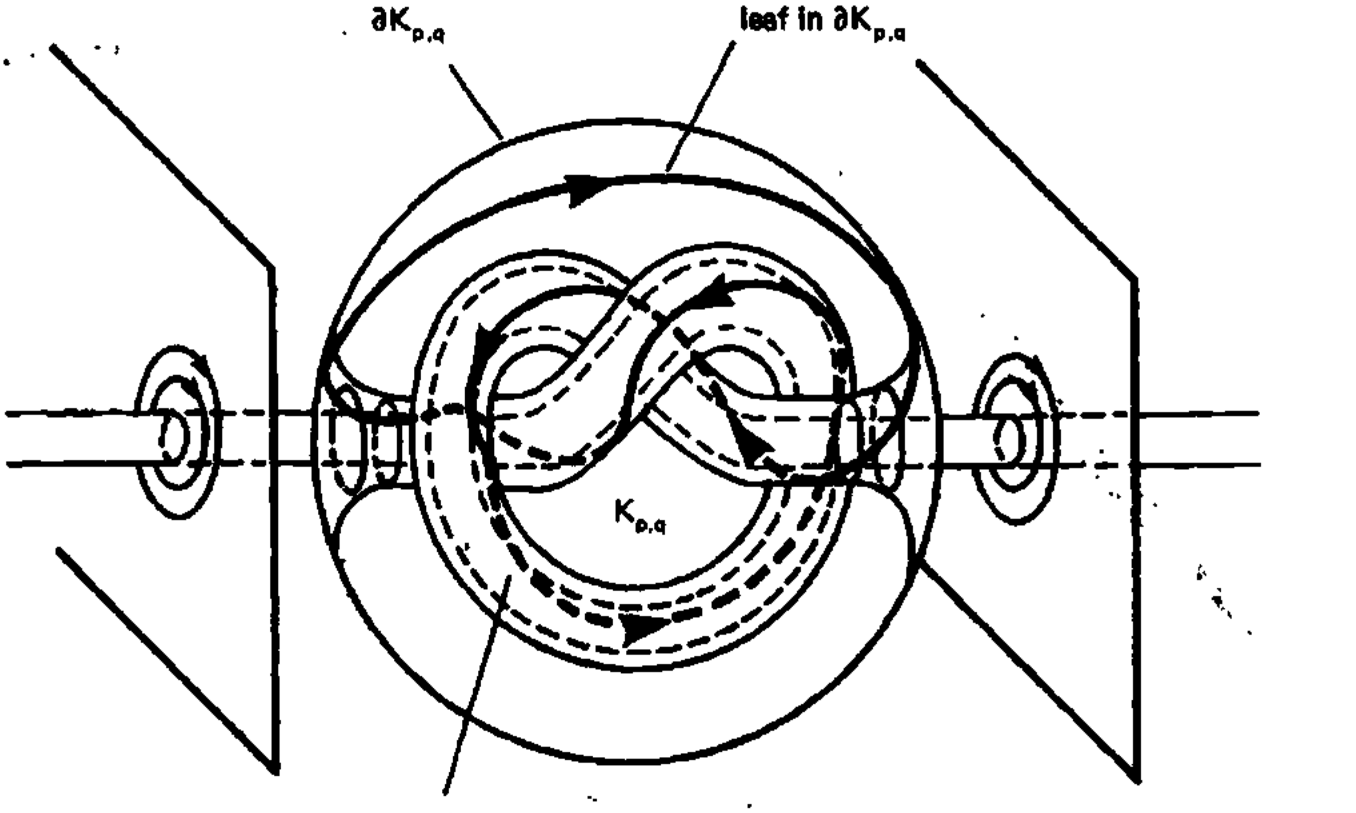

Figure 3

Let Z' be a small tubular cylinder around $k_{p,q} \cap \{|x_1| \leq 1\}$ whose top and bottom are two circular disks of radius r in the planes $|x_1| = 1$ around the intersections of these planes with the x-axis (see Figure 3). Let Z be the union of $\mathring{Z}'$ with the interior of these disks and consider the 3-manifold with boundary

$$D = \{|x_1| \le 1\} \setminus (\mathring{K}_{p,q} \cup Z).$$

∂D consists of two components: $\partial K_{p,q}$ which is a solid torus with a foliation induced from $F_{p,q}$, and an annulus. Let $E^+ \subset E$ be the union of levels E_s, $0 \le s \le 1$, and remove the cylinder $E_0 \cap \{x_2 < 0\}$ from E^+. Denote the resulting space by E^*. There exists a homeomorphism $h: E^* \to D$ mapping E_1 leaf preserving onto $\partial K_{p,q}$ and having the following property: remember that each circle $(x_1,x_2) \times S^1 \times \{0\}$, $|x_1| < 1$, $x_2 > 0$ is a leaf of $F(f)$ in E^*. Then h maps for $-1 < x_1 \le -\frac{1}{2}$ each one of these leaves to a concentric circle around $(-1,0,0)$ in the plane $x_1 = -1$ in $\mathbb{R}^3$, and it maps for $\frac{1}{2} \le x_1 < 1$ these leaves to concentric circles around $(1,0,0)$ in the plane $x_1 = +1$. The remaining leaves $(x_1,x_2) \times S^1 \times \{0\}$, $-\frac{1}{2} < x_1 < \frac{1}{2}$, of E_0 are mapped onto the annulus $\partial Z'$. (To see that such a homeomorphism exists, note that D is just a thickened 2-torus with an annulus in one of its boundary tori removed and that any leaf of $F_{p,q}$ in $\partial K_{p,q}$ together with a concentric circle around $(1,0,0)$ in $\{x_1 = 1\}$ of radius greater than r represents free abelian generators for $H_1(D)$.) It is clear that one can choose f and h in such a way that the foliations $F_{p,q}$ and $h(F(f))$ fit together across $\partial K_{p,q}$ to form a C^∞ circle foliation $F(p,q,f)$ on $M_{p,q} = \{|x_1| \le 1\} \setminus Z$, and that $F(p,q,f)$ has a smooth extension to

$$M_{p,q} \cup \{(x_1,x_2,x_3): |x_1| \ge 1,\ x_2^2 + x_3^2 \ge r^2\}$$

where the new leaves are simply concentric circles of radius $\zeta \ge r$ around the x_1-axis in the planes $\{x_1 = c\}$ with $|c| \ge 1$.

Now let (p_1,q_1), (p_2,q_2),... be any finite or countably infinite sequence of pairs of coprime integers. We can put a C^∞-circle foliation on the complement K of the product of the torus knots k_{p_1,q_1}, k_{p_2,q_2},... in the following way: K is diffeomorphic to

$$M_- \cup M_{p_1,q_1} \cup g\, M_{p_2,q_2} \cup g^2\, M_{p_3,q_3} \cup \ldots$$

where $g(x_1,x_2,x_3) = (x_1 + 2,x_2,x_3)$ and $M_- = \{(x_1,x_2,x_3): x_1 \le -1,\ x_2^2 + x_3^2 \ge r\}$. Foliate $g^i\, M_{p_i,q_i}$ by $g^i(F(p_i,q_i,f))$ and foliate M_- by concentric circles around the x-axis. If the sequence (p_1,q_1),... ends with (p_n,q_n), then also foliate $\{x_1 \ge 2n-1,\ x_2^2 + x_3^2 \ge r\}$ by concentric circles around the x-axis.

The second example in this section is intended to give some indication of the possible unpleasant topological properties of the set $B_1(F)$.

Let K be a continuum in the closed unit disk D^2 of $\mathbb{R}^2$. K is supposed to have the following properties:

(1) $\partial D^2 \setminus K$ is the disjoint union of $n > 1$ open arcs $A_1, \ldots, A_n$;

(2) $\mathring{D}^2 \setminus K$ is the union of $n+1$ disjoint simply connected domains $E_0, E_1, \ldots, E_n$ such that $\mathrm{Fr}\, E_0 = K$ and $\mathrm{Fr}\, E_i = K \cup A_i$, $i = 1, \ldots, n$. (Fr denotes the set theoretic boundary with respect to the topology of $\mathbb{R}^2$.)

Such a continuum can easily be constructed and it has rather peculiar topological properties [8], §62, VI, Theorem 11, and §48]. We want to describe an example of a circle foliation F_K on $\mathring{D}^2 \times S^1$, such that $B_1(F_K) = (K \cap \mathring{D}^2) \times S^1$ and F_K restricted to B_1 is the product foliation.

Let $\hat{E}_i$ be Caratheodory's prime end compactification of E_i ([1], [6]). By [6], Theorem 6.6 , there exists a homeomorphism $\phi_i : \hat{E}_i \to D^2$ such that $\phi_i | E_i : E_i \to \mathring{D}^2$ is a diffeomorphism and $\phi_i | A_i$ embeds A_i onto the open southern hemisphere D^1_- of the unit circle in $\mathbb{R}^2$ (note that A_i is canonically a subspace of $\hat{E}_i$), $i = 1,2,\ldots,n$. On $(E_0 \cup K) \times S^1$ we let F_K be the product foliation. $F_K | E_i \times S^1$ will be the pullback under $\phi_i \times \mathrm{id}$ of a foliation F_i on $\mathring{D}_2 \times S^1$ which we will now describe.

Let $E^* \subset E^+$ be the submanifold of the total space E of the Reeb example $F(f)$ described above and let $\frac{1}{2} D^2 \subset D^2$ be the circle of radius $\frac{1}{2}$ around the origin. Then $[(\mathring{D}^2 \setminus \frac{1}{2} \mathring{D}^2) \cup D^1_+] \times S^1$ is diffeomorphic to E^* under the map $F: (r, \Psi, \phi) \to (\cos 2\pi\Psi, \sin 2\pi\Psi, \phi - \Psi, 2 - 2r)$ where (r, Ψ), $r \in [0,1]$, $\Psi \in \mathbb{R} \bmod 1$, are polar coordinates on D^2, $\phi \in \mathbb{R} \bmod 1$. The coordinates on E^* are the ones from above. If $g_i(s)$ is the distance of the circle $\phi_i^{-1}(\{r = s\}) \subset E_i$ from K with respect to the standard metric of $\mathbb{R}^2$, let $f_i : [0,1] \to [0,\infty)$ be a C^∞-map with the following properties:

(3) $f_i^{-1}(0) = 0$;

(4) $f_i(2-s) \le \exp(-g_i^{-2}(s))$ for $\frac{1}{2} \le s < 1$;

(5) f_i is equal to some constant c_i ($\le \exp(-g_i^{-2}(\frac{1}{2}))$ in a neighbourhood of 1.

We take $F_i | [(\mathring{D}^2 \setminus \frac{1}{2}\mathring{D}^2) \cup D^1_+] \times S^1$ to be $F^*(F(f_i))$. $F_i | \partial(\frac{1}{2} D^2) \times S^1$ is the union of circles

$$C_s = \{(\tfrac{1}{2}, \Psi, \Psi - \tfrac{1}{c_i} \cos 2\pi\Psi + s) \in \partial(\tfrac{1}{2} D^2) \times S^1 : 0 \le \Psi \le 1\}.$$

They are the fibres of a trivial S^1-bundle over $\partial(\frac{1}{2} D^2)$ which can be smoothly extended to an S^1-bundle over $\frac{1}{2} D^2$ to define F_i on all of $\mathring{D}_2 \times S^1$. The choice of f_i will guarantee that the $(\phi_i \times \mathrm{id})^*F_i$ define circle foliations on $E_i \times S^1$ which fit smoothly together with the product foliation on $(K \cup E_0) \times S^1$ to define a circle foliation F_K. Obviously $F_K|(\mathring{D}^2 \setminus K) \times S^1$ is a (trivial) S^1-bundle. Therefore $B_1(F_K) \subset K \times S^1$. For the converse inclusion note that a leaf on $\phi_i^{-1}(\{r = s\}) \times S^1$ has at least length $\exp(g_i^{-2}(s))$ if we give $D^2 \times S^1 = D^2 \times \mathbb{R} \bmod 1$ the obvious product metric.

3. ITERATIONS OF REEB'S EXAMPLE (an example with infinite hierarchy)

In this section we will prove:

Proposition For any $0 \leq \alpha \leq \omega$ there exists a C^∞ circle foliation F_α on a connected open 3-manifold such that $|EF_\alpha| = \alpha$.

Proof We already know that the statement is true for $\alpha = 0,1$. To construct a foliation of length 2, we first modify our standard example a little. The manifolds E, E^+, E^*, E_s, $-1 \leq s \leq 1$, are defined as in the previous section. We also use the same coordinates (x_1,x_2,ϕ,t) for E. This time we want $f\colon [-1,1] \to [0,1]$ to be a C^∞-map with $f^{-1}(0) = [-1,0]$. The formulae of the previous section define again a foliation $F(f)$ on E such that each of the sets E_s, E^+, E^* is saturated. Notice that every leaf of $F(f)$ in $E_{(0,1]} = \bigcup_{t\in(0,1]} E_t$ intersects the annulus $A = \{(x_1,x_2)\} \times S^1 \times (0,1]$ for any $(x_1,x_2) \in S^1$ at exactly one point. To fix notation we take $(x_1,x_2) = (0,1)$ to define A. Let $B \subset A$ be the open disk of points $(0,1,\phi,t) \in E$ with $0 < \phi < 1/2$ and $0 < t < 1$. The closure $\bar{B}$ of B in E is a smooth disk with 4 corners (see Figure 4). Let $h\colon\bar{B} \to D^2$ be a homeomorphism which is a diffeomorphism in the complement of the corners and which maps the upper half circle C^+ of the outside boundary of A, i.e. the set $(0,1,\phi,1) \in E$ with $0 \leq \phi \leq 1/2$, to the lower half circle in the boundary of D^2. Let B' be the inverse image of $\frac{1}{4} \mathring{D}^2$ under h.

For a subset C of E let $F(C)$ be the union of leaves of $F(f)$ through points of C. Then we define a diffeomorphism H from $N = (\mathring{D}^2 \setminus \frac{1}{4} \mathring{D}^2) \times S^1$ onto $F(B \setminus B')$ as follows. For $(y,\xi) \in N$, $H(y,\xi)$ is the intersection of the leaf $F(\{h^{-1}(y)\})$ with the annulus

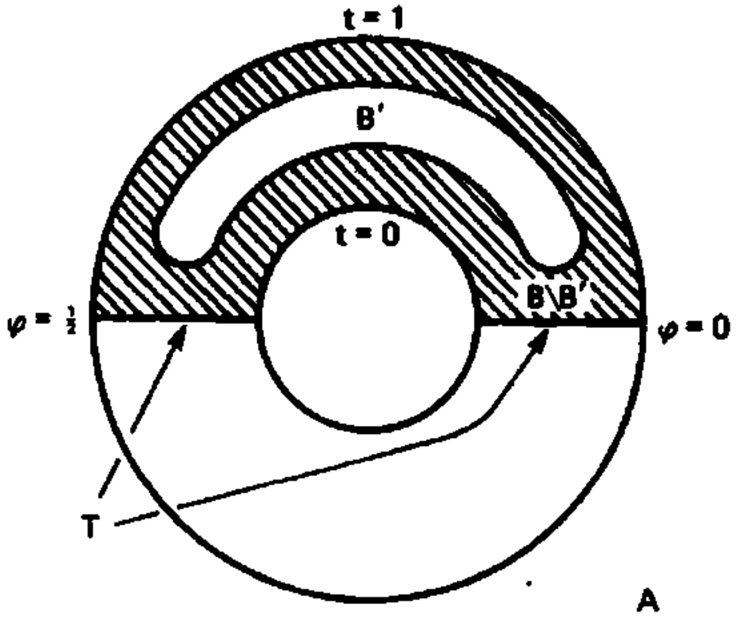

Figure 4

$$(\cos 2\pi\xi,\ \sin 2\pi\xi) \times S^1 \times [0,\ 1)$$

in E. Furthermore, let $K\colon N \to E_{(0,1]}$ be the diffeomorphism given by

$$K((r,\Psi),\xi) = (\cos 2\pi\Psi,\ \sin 2\pi\Psi,\ \ \xi\smallsetminus\Psi,\ 4/3 - 4r/3)$$

where now $(r,\Psi) \in \mathring{D}^2 \smallsetminus \frac{1}{4}\,\mathring{D}^2$ are polar coordinates, $\Psi \in \mathbb{R} \bmod 1$.

We now define a foliation $F(f(2))$ on $M_2 = E^+ \smallsetminus F(B' \cup C^+)$. On $E^+ \smallsetminus F(B \cup C^+)$ we take the restriction of $F(f)$, and on $F(B \smallsetminus B')$ we take $(K \circ H^{-1})^*(F(f)|E_{(0,1]})$. The two foliations fit together. To see this, note that $H^*(F(f))$ is the product foliation on $N = (\mathring{D}^2 \smallsetminus \frac{1}{4}\,\mathring{D}^2) \times S^1$, and $K^*(F(f)|E_{(0,1]})$ is a foliation on N which, by the choice of f, can be extended to a foliation on $(2D^2 \smallsetminus \frac{1}{4}\mathring{D}^2) \times S^1$ by taking the product foliation on $(2D^2 \smallsetminus \mathring{D}^2) \times S^1$.

Since $F(f)$ is an S^1-bundle in a neighbourhood of $F(B' \cup C^+) = R_1$, M_2 is a 3-manifold (with boundary). The Epstein hierarchy of $F(f(2))$ is given by $B_1(F(f(2))) = \Gamma_1(F(f)) \cup F(T)$, $B_2(F(f(2))) = B_1(F(f))$, where $T = \{(0,1,\phi,t)\colon \phi = 0 \text{ or } 1/2,\ 0 < t < 1\}$ is a union of two arcs (see Figure 4).

This concludes the construction of a circle foliation with Epstein hierarchy of length 2. We note that we can plug the hole $F(B')$ in M_2 by glueing in a trivially fibred solid torus. But $H_2(M_2 \cup F(B')) \cong \mathbb{Z}$ will remain nontrivial.

To obtain a foliation of length 3 we repeat the above process. We obtained $F(f(2))$ by first removing $F(B)$ from our standard example $F(f)$ on E^+, and then adding $F(B\setminus B')$ with a new foliation which, up to the diffeomorphism $K \circ H^{-1}$, is just another copy of $E_{(0,1]} \subset E^+$ with its foliation $F(f)$. This foliation fits into the foliation on $E^+\setminus F(B)$ (with the length of the leaves growing to infinity when approaching $F(\partial B)$) except for the points on the two leaves of $F(f)|(E^+\setminus F(B))$ through the points $h^{-1}((\pm 1,0)) = \{(0,1,\phi,1) \in E^+ : \phi = 0 \text{ or } 1/2\}$. This is the reason why we also remove $F(C^+)$ to obtain M_2.

So to continue, we will simply replace $F(B\setminus B')$ by $H \circ K^{-1}(M_2 \cap E_{(0,1]})$ with its foliation $(H \circ K^{-1})_* F(f(2))$. We obtain a manifold (with boundary) $M_3 \subset M_2$ with a circle foliation $F(f(3))$. The Epstein hierarchy of $F(f(3))$ is given as

$$B_1 = B_1(F(f)) \cup H \circ K^{-1}(B_1(F(f(2)))),$$

$$B_2 = B_1(F(f)) \cup H \circ K^{-1}(B_1(F(f))), \quad B_3 = B_1(F(f)).$$

Continuing in this way, we obtain a sequence $E = M_1 \supset M_2 \supset M_3 \supset \ldots$ of 3-manifolds and a sequence of circle foliations $F(f), F(f(2)), F(f(3)),\ldots$, with $F(f(i))$ living on M_i and $|EF(f(i))| = i$.

It remains to construct a circle foliation with infinite hierarchy. This is done by simply piecing all the M_i, $F(f(i))$ together. Notice that the annulus $Z = \{(x_1,x_2,\phi,0) \in E : x_2 > 0\}$ is contained in all M_i, and $F(f(i))|Z = F(f)|Z$ is the trivial S^1-bundle with fibres $L(x_1,x_2) = \{(x_1,x_2,\phi,0) : 0 \le \phi \le 1\}$, where $x_1^2 + x_2^2 = 1$, $x_2 > 0$. Let Y be the closed upper half plane on $\mathbb{R}^2$ with the points $(2i-1,0)$, $i = 1,2,\ldots$ removed. Then consider the manifold

$$M_\omega = Y \times S^1 \cup \coprod_{i=1}^{\infty} M_i/\sim .$$

Here $\coprod$ denotes "disjoint union" and each M_i is attached to $Y \times S^1$ by identifying $Z \subset M_i$ with $(2i-1, 2i+1) \times S^1$ in $Y \times S^1$ via the diffeomorphism $(x_1,x_2,\phi,0) \to (2i + x_1, 0, \phi)$. Our choice of f allows us to extend the foliations $F(f(i))$ to $Y \times S^1$ by simply putting on $Y \times S^1$ the product foliation. Denote the resulting foliation by $F(f(\omega))$. Obviously $\{(y_1,0) \in Y : y_1 > 2i-1\} \times S^1 \subset B_i(F(f(\omega)))$. Therefore $|EF(f(\omega))| = \omega$.

<u>Remark 1</u> In our example $B_\omega = \emptyset$. I do not know whether one can construct a

circle foliation F on a 3-manifold with $B_\omega(F) \neq \emptyset$.

<u>Remark 2</u> Note that all manifolds M_i, $1 \leq i \leq \omega$, can be embedded in $\mathbb{R}^3$.

<u>Remark 3</u> Rank $H_2(M_i) = i$. We can close off in each M_i one boundary component by adding a trivially foliated solid torus to the single compact boundary component of $M_{i-1} \setminus M_i$. This shows that we may construct circle foliations F_i on 3-manifolds W_i such that $|EF_i| = i$ and rank $H_2(W_i) = i-1$. But with the above methods the rank of $H_2(W_i)$ cannot be lowered any further. This is because at each stage we have to remove the annulus corresponding to $F(T)$. In the next section we suggest a program to foliate $\mathbb{R}^3$ by circles. This program necessitates the construction of circle foliations F_i on open solid tori with $|EF_i| = i$, $i = 1,2,\ldots$. Up to now I can only complete this program up to $i = 2$.

4. ONE-PARAMETER FAMILIES OF DIFFEOMORPHISMS OF THE OPEN DISK

In this section we construct a rather complicated circle foliation F with $|EF| = 2$ on an open solid torus. A motivation for this example is the fact that it is the second storey in the construction of a building with infinitely many storeys which would, if completed, result in a circle foliation of $\mathbb{R}^3$.

$\mathbb{R}^3$ can be written as the union of an ascending sequence $V_0 \subset V_1 \subset V_2 \subset \ldots$ of open solid tori such that V_i is unknotted and contractible in V_{i+1} and such that $V_i \setminus V_{i-1}$ is a closed solid torus minus a closed annulus in its boundary. To be more explicit, let V_0 and W_1 be as in Figure 5.

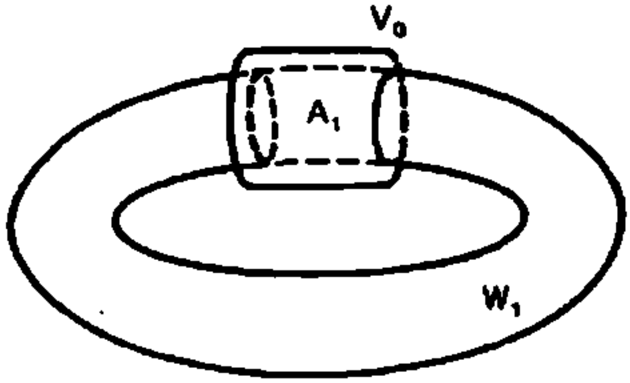

Figure 5

W_1 is a 3-manifold with boundary A_1, where A_1 is a meridianal annulus of the closed solid torus $\bar{W}_1$, the closure being taken in $\mathbb{R}^3$. V_0 is an open solid torus such that $\bar{V}_0 \cap \bar{W}_1 = \bar{A}_1$. V_0 is contractible and unknotted in the open solid torus $V_\phi = V_0 \cup W_1$. In particular, V_0 and V_1 are unknotted in $\mathbb{R}^3$. Therefore there exists a homeomorphism $h:\mathbb{R}^3 \to \mathbb{R}^3$ which is a diffeomorphism in $\mathbb{R}^3 \setminus$(2 simple closed curves on ∂V_0) mapping V_0 onto V_1. Since $V_0 \subset V_1$, we obtain an ascending sequence $V_0 \subset V_1 = h(V_0) \subset V_2 = h^2(V_0) \subset \ldots$ and it is not hard to prove that $V_\omega = \bigcup_{i=0}^{\infty} h^i(V_0)$ is homeomorphic to $\mathbb{R}^3$.

The results of the third paper in this series show the following: if there is a circle foliation F on $\mathbb{R}^3$ then $|EF| \geq \omega$, and if $B_\omega(F) = \emptyset$ there exists an ascending sequence $U_0 \subset U_1 \subset \ldots$ of open saturated sets, each U_i a component of some $\mathbb{R}^3 \setminus B_{\alpha_i}(F)$ such that $H_*(U_i) \cong H_*$(solid torus), and $H_1(U_i) \to H_1(U_{i+1})$ is the 0-map. So the simplest possible circle foliation on $\mathbb{R}^3$ might be obtained in the following way: start with a circle foliation on V_0, extend this to a circle foliation on V_1, extend this to a circle foliation on V_2 (after possibly some deformation of the foliation on V_1). Continue this process ad infinitum. Whether this works I do not know. Below we shall show that there exists a circle foliation with the required properties on V_2. (The main difficulty comes from the fact that we are not allowed, as in the examples in the preceding sections, to drill holes to remove the points where the tangent spaces to the foliation do not converge; also note that a circle foliation on V_i with the required property - i.e., such that each V_j for $0 \leq j < i$ is saturated - will have Epstein hierarchy of length at least i.)

To begin with the construction of the circle foliation on V_2 we observe that we already have an example with the required properties on V_1. For this we take the foliation $F(f)$, where $f:[-1,1] \to [0,1]$ is a C^∞-map with $f^{-1}(0) = [-1,0]$, and restrict it to the invariant set

$$E \setminus \{(x_1,x_2,\phi,t) \in E: -1 < t \leq 0,\ x_2 \leq \frac{1}{\sqrt{2}}\} \setminus E_{-1}.$$

We attach to this space, along its boundary E_1, a solid torus V with the product foliation in such a way that the leaves of $F(f)$ in E_0 (they are the sets $\{(x_1,x_2)\} \times S^1 \times \{0\}$) are homologous in E to meridians of V. The resulting space is an open solid torus which we may identify with V_1, where V_0 corresponds to $\{(x_1,x_2,\phi,t) \in E: -1 < t < 0,\ x_2 > \frac{1}{\sqrt{2}}\}$ and W corresponds to $E^* \bigcup_{E_1=\partial V} V$, where E^* denotes now the union of $E_{(0,1)}$ with

$\{(x_1,x_2,\phi,0) \in E: x_2 > \frac{1}{\sqrt{2}}\}$. This example also suggests a general procedure for passing from a foliation on V_{i-1} to one on V_i: deform the given foliation on V_{i-1} in such a way that it extends to a foliation of a longitudinal annulus A_i in ∂V_{i-1}; then attach $E^*{}_{E_1}\underline{\cup}_{\partial V} V$ to $V_{i-1} \cup A_i$ by identifying A_i with the annulus $\{(x_1,x_2,\phi,0) \in E: x_2 > \frac{1}{\sqrt{2}}\}$. The result will be an open solid torus V_i with a foliation meeting our requirements. The hard part is to find the deformation of the foliation on V_{i-1}. We do this now for $i = 2$. Since it suffices to deform the foliation only near the boundary of V_1, it suffices to consider the thickened-up 2-torus

$$V_1 \setminus \mathring{V} = E^* \cup \{(x_1,x_2,\phi,t) \in E: -1 < t < 0,\ x_2 > \frac{1}{\sqrt{2}}\}$$

which is diffeomorphic to $[\frac{1}{2},1) \times S^1 \times S^1$.

It will be convenient to use angular coordinates $\theta \in \mathbb{R} \bmod 1$ for the first two coordinates (x_1,x_2) of E (i.e., $x_1 = \cos 2\pi\theta$, $x_2 = \sin 2\pi\theta$). For $[\frac{1}{2},1) \times S^1 \times S^1$ we take coordinates (r,Ψ,s), $r \in [\frac{1}{2},1)$, $\Psi,s \in \mathbb{R} \bmod 1$. (r,Ψ) are polar coordinates for the annulus $C = \{(y_1,y_2): \frac{1}{4} \leq y_1^2 + y_2^2 < 1\}$ of $\mathbb{R}^2$.

Our plan is the following. Transport $F(f)$ from $V_1 \setminus \mathring{V}$ to $[\frac{1}{2},1) \times S^1 \times S^1$ via a diffeomorphism b to have better coordinates for $V_1 \setminus \mathring{V}$. Then use a diffeomorphism d of $[\frac{1}{2},1) \times S^1 \times S^1$ to deform the foliation $b(F(f))$ into a circle foliation which has an extension to a longitudinal annulus in $\{1\} \times S^1 \times S^1$. Choose d to be the identity near $\{\frac{1}{2}\} \times S^1 \times S^1$ so that the construction can be extended over V.

We first describe $b: V_1 \setminus \mathring{V} \to [\frac{1}{2},1) \times S^1 \times S^1$. It will map each circle $\{\theta\} \times S^1 \times \{t\}$ "identically" onto the circle $\{r(\theta,t),\ \Psi(\theta,t)\} \times S^1$. Thus it suffices to describe the maps $r(\theta,t)$ and $\Psi(\theta,t)$. We will do this first for $-1 < t \leq 0$. Then $\frac{1}{8} < \theta < \frac{3}{8}$, and we map the corresponding half-open disk to the diffeomorphic set $R = \{(r,\Psi): 1 > r \geq \frac{1}{\sqrt{2}\sin 2\pi\Psi},\ \frac{1}{8} < \Psi < \frac{3}{8}\}$ in $[\frac{1}{2},1) \times S^1$ such that $r(\theta,0) = \frac{1}{\sqrt{2}\sin 2\pi\theta}$ and $\Psi(\theta,0) = \theta$. The map on the remaining set of points $(\theta,t) \in S^1 \times (0,1)$ is more easily described with the help of Figure 6.

We fill up $[\frac{1}{2},1) \times S^1 \setminus R$ by a family K_t, $1 \geq t > 0$, of disjoint simple closed curves such that the following holds:

(1) For each $\theta \in \mathbb{R} \bmod 1$, the radius $\{(r,\Psi): \Psi = \theta\}$ intersects each K_t in

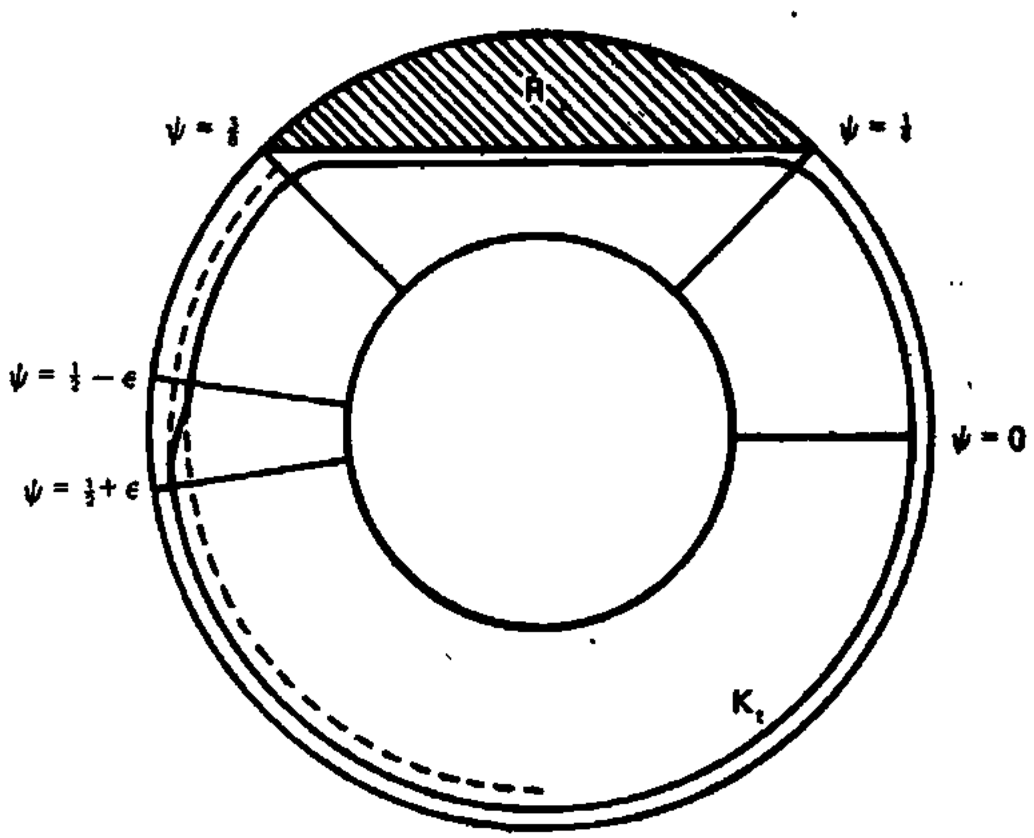

Figure 6

exactly one point $(r(\theta,t),\theta)$ such that $S^1 \times (0,1] \to [\frac{1}{2},1) \times S^1 \setminus R$, $(\theta,t) \to (r(\theta,t),\theta)$ is a diffeomorphism.

(2) For each fixed t, $r(\theta,t)$ is constant on the intervals $\frac{3}{8} + \varepsilon \le \theta \le \frac{1}{2} - \varepsilon$ and $\frac{1}{2} + \varepsilon \le \theta \le \frac{9}{8} - \varepsilon$; for $\frac{1}{4} \le \theta \le \frac{9}{8} - \varepsilon$ it is (non-strictly) increasing.

(3) For each $t \le \frac{1}{2}$, the set $\{(r(\theta,t),\theta): \frac{1}{8} + \varepsilon \le \theta \le \frac{3}{8} - \varepsilon\}$ is a straight horizontal line in $\mathbb{R}$.

(4) For each $t \le \frac{1}{2}$ and $\frac{1}{2} - \frac{\varepsilon}{2} \le \theta \le \frac{1}{2} + \frac{\varepsilon}{2}$, $\frac{1}{\sqrt{(1-r)^2 + 1}}\left((1-r)\frac{\partial}{\partial r} + \frac{\partial}{\partial \Psi}\right)$ is the positively oriented unit tangent vector to K_t in $(r(t,\theta),\theta)$.

Such a family of circles exists, and it is reasonably unique once $r(0,t)$ is fixed. We define $r(\theta,t)$ by (1) and $\Psi(\theta,t) = \theta$ for the points $(\theta,t) \in S^1 \times (0,\frac{1}{2}]$. Notice that $r(\theta,1) = \frac{1}{2}$ and $r(\theta,t) \to 1$ as $t \to 0$ for $\frac{3}{8} \le \theta \le \frac{9}{8}$. This finishes the description of b.

The diffeomorphism d of $[\frac{1}{2},1) \times S^1 \times S^1$ will be defined by a smooth 1-parameter family of diffeomorphisms, $d_s:[\frac{1}{2},1) \times S^1 \to [\frac{1}{2},1) \times S^1$, $s \in [0,1]$, such that $d_s = d_0$ and $d_{1-s} = d_s$ for $0 \leq s < \delta$ and such that d_s is the identity near $\{\frac{1}{2}\} \times S^1$. The map d corresponding to the 1-parameter family d_s is then defined by $d(r,\Psi,s) = (d_s(r,\Psi),s)$.

To motivate the somewhat complicated formulae let us consider another way to describe Reeb examples. Let $\alpha:[0,1] \to [0,1]$ be the restriction of a $\mathbb{Z}$-periodic C^∞-function from $\mathbb{R}$ to $[0,1]$ of the form shown in Figure 7. Let

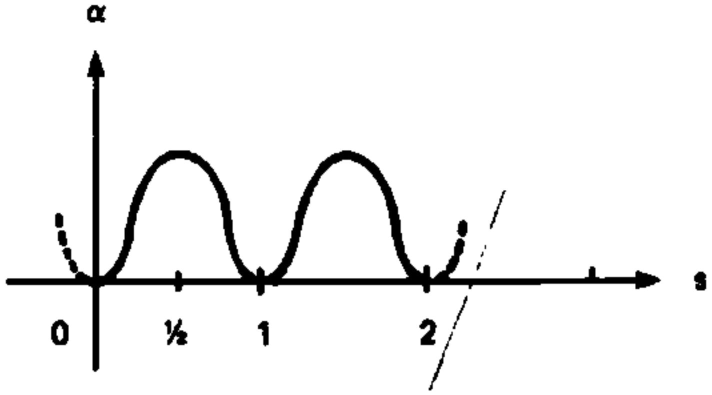

Figure 7

$\beta:[\frac{1}{2},1) \to [0,\infty)$ be a C^∞-map such that $\beta^{-1}(0) = [\frac{1}{2}, \frac{1}{2} + \delta]$ for some small δ and $\beta(r) \nearrow \infty$ as $r \nearrow 1$. Let F be the product foliation on $[\frac{1}{2},1) \times S^1 \times S^1$ with leaves $(r,\Psi) \times S^1$, $(r,\Psi) \in [\frac{1}{2},1) \times S^1$. Let $d(r,\Psi,s) = (r,\Psi+\alpha(s)\cdot \beta(r),s)$. Then the image $d(F)$ of F under d is a circle foliation on $[\frac{1}{2},1) \times S^1 \times S^1$. The tori $\{r\} \times S^1 \times S^1$ are saturated with respect to $d(F)$, and the leaves on $\{r\} \times S^1 \times S^1$ become longer and longer as r goes to 1. $d(F)$ can be extended to the union of the two annuli $\{1\} \times S^1 \times (S^1 \setminus \{0,\frac{1}{2}\})$, where the foliation on these annuli is the obvious product S^1-foliation. The resulting foliation will be smooth if β grows fast enough (any exponential growth suffices).

We would like to do the same, but with the foliation $b(F(f))$ instead of F. It is an instructive exercise to show that with the simple minded 1-parameter families of diffeomorphisms

$$d_s(r,\Psi) = (r, \Psi + \alpha(s) \cdot \beta(r))$$

of $[\frac{1}{2},1) \times S^1$ above, there will be no point in $\{1\} \times S^1 \times S^1$ such that, in a neighbourhood $U \subset \{1\} \times S^1 \times S^1$ of this point, there is a nowhere vanishing vector field towards which $(d\circ b)(F(f))$ converges. The reason for this lies in the fact that, for points $(r,\Psi,s) \in [\frac{1}{2},1) \times S^1 \times S^1$ with $0 < \Psi < \frac{1}{2}$, the positive unit tangent vector to $b(F)$ (after we have chosen an orientation for b(F)) will converge to $\frac{\partial}{\partial s}$ as r approaches 1, while for $\frac{1}{2} < \Psi < 1$ the limit will be $-\frac{\partial}{\partial s}$. $\frac{\partial}{\partial s}$ is just the positive unit tangent vector field to F in the example above and the positive unit vectors of d(F) in (r,Ψ,s) converge to $\frac{\partial}{\partial \Psi}$ if $r \to 1$ and if $0 < s < \frac{1}{2}$. Since in the same domain the positive unit vectors of d(-F) converge to $-\frac{\partial}{\partial \Psi}$, we find in the neighbourhood of any point in $\{1\}\times S^1\times(0,\frac{1}{2})$ positive unit tangent vectors V_1, V_2 of $d\circ b(F(f))$ with V_1 arbitrarily close to $\frac{\partial}{\partial \Psi}$ and V_2 arbitrarily close to $-\frac{\partial}{\partial \Psi}$. This indicates that we have to make a special effort to find a deformation which also forces the unit tangent vectors of b(F(f)) at points (r,Ψ,s) with $\frac{1}{2} < \Psi < 1$ and $0 < s < \frac{1}{2}$ into the positive Ψ direction as r approaches 1.

Let $g:[0,1] \to [-1,1]$ be the restriction of a $\mathbb{Z}$-periodic C^∞-function as indicated in Figure 8. Let $a(r)\frac{\partial}{\partial r}$ be a C^∞-vector field on $[\frac{1}{2},1]$ with

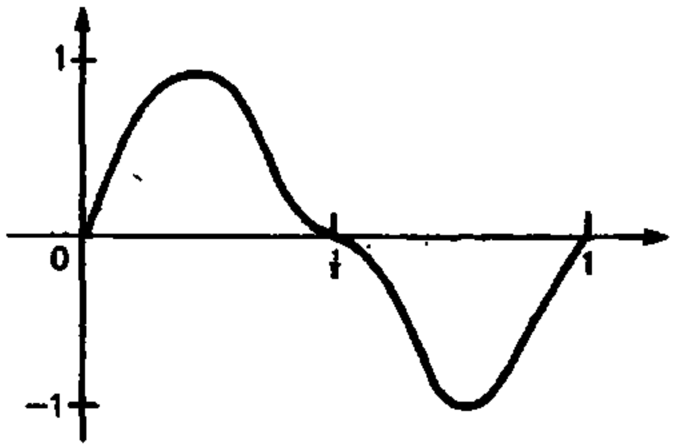

Figure 8

$a(r) = 0$ for r near $\frac{1}{2}$, and $a(r) = r(1-r)$ for $r \geq \frac{3}{4}$. On $[\frac{1}{2},1) \times S^1$ consider the vector field $g(\Psi)\cdot a(r)\cdot\frac{\partial}{\partial r}$, where (r,Ψ) are the coordinates on $[\frac{1}{2},1) \times S^1$, and let e_s be the corresponding 1-parameter family of diffeomorphisms. Let $\alpha:[0,1] \to [0,\infty)$ be the restriction of a $\mathbb{Z}$-periodic C^∞-map with $\alpha(0) = 0$ and $\alpha(s) = s$ for $\frac{1}{8} \leq s \leq \frac{3}{8}$. Then we let $d_s = h \circ e_{\alpha(s)}$ be our 1-parameter family of diffeomorphisms of $[\frac{1}{2},1) \times S^1$ where

$$h(r,\Psi) = \left(r, \Psi + e^{\frac{\beta(r)}{1-r}}\right).$$

Here $\beta(r)$ is equal to 0 for r near $\frac{1}{2}$ and equal to r for $r \geq \frac{3}{4}$. We claim that $(d\circ b)(F(f))$ has the required properties. More specifically we prove:

<u>Proposition</u> For proper choices of b, f and g and for an orientation of F(f), the positive unit tangent vectors of $(d\circ b)(F(f))$ in (r,Ψ,s) will converge to

$\frac{\partial}{\partial\Psi}$ if $(r,\Psi,s) \to (1,\Psi_0,s_0)$ with $\frac{1}{8} < s_0 < \frac{3}{8}$.

Proof We first investigate the effect of d_*, the differential of d, on the vector fields $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial\Psi}$, $\frac{\partial}{\partial s}$ in $[\frac{1}{2},1) \times S^1 \times S^1$ for the points (r,Ψ,s) with $\frac{1}{8} < s < \frac{3}{8}$ and $r \geq \frac{7}{8}$. Notice that in this domain $a(r) = r(1-r)$ and $\alpha(s) = s$, and $r\cdot e^{s\cdot g(\Psi)}/(1+r(e^{s\cdot g(\Psi)}-1)) \geq \frac{3}{4}$. Therefore

$$d(r,\Psi,s) = \left(\frac{r\cdot e^{s\cdot g(\Psi)}}{1+r(e^{s\cdot g(\Psi)}-1)}, \Psi + e^{\frac{r}{1-r} e^{s\cdot g(\Psi)}}, s\right)$$

Differentiating, we obtain

$$d_*\left(\frac{\partial}{\partial r}\Big|_{(r,\Psi,s)}\right) = \frac{e^{s\cdot g(\Psi)}}{(1+r(e^{s\cdot g(\Psi)}-1))^2}\frac{\partial}{\partial r} + \frac{e^{s\cdot g(\Psi)}}{(1-r)^2} e^{\frac{r}{1-r} e^{s\cdot g(\Psi)}} \frac{\partial}{\partial\Psi},$$

$$d_*\left(\frac{\partial}{\partial\Psi}\Big|_{(r,\Psi,s)}\right) = \frac{r(1-r)\cdot s\cdot g'(\Psi)\cdot e^{s\cdot g(\Psi)}}{(1+r(e^{s\cdot g(\Psi)}-1))2}\frac{\partial}{\partial r}$$

$$+ \left[1 + e^{\frac{r}{1-r} e^{s\cdot g(\Psi)}} \cdot \frac{r}{1-r} \cdot sg'(\Psi)\cdot e^{sg(\Psi)}\right]\frac{\partial}{\partial\Psi},$$

$$d_*\left(\frac{\partial}{\partial s}\Big|_{(r,\Psi,s)}\right) = \frac{r(1-r)\cdot g(\Psi)\cdot e^{s\cdot g(\Psi)}}{(1+r(e^{s\cdot g(\Psi)}-1))^2}\frac{\partial}{\partial r}$$

$$+ \frac{r}{1-r} g(\Psi)\cdot e^{s\cdot g(\Psi)}\cdot e^{\frac{r}{1-r} e^{s\cdot g(\Psi)}} \frac{\partial}{\partial\Psi} + \frac{\partial}{\partial s}.$$

We have a fairly explicit description of $b(F(f))$. On $R \times S^1$ it is the product flow, and we will choose the orientation of $F(f)$ so that the positive unit tangent vector of $b(F(f))$ in $R \times S^1$ is $\frac{\partial}{\partial s}$. Furthermore, the 2-tori $K_t \times S^1 \subset ([\frac{1}{2},1) \times S^1 \setminus R) \times S^1$ correspond to the saturated 2-tori E_t in $V_1 \setminus \mathring{V}$, $0 < t \leq 1$, and for any $\delta > 0$ the positive unit tangent vector of $b(F(f))$ on points (r,Ψ) on K_t with $\delta < \Psi < \frac{1}{2} - \delta$ will be close to $\frac{\partial}{\partial s}$ for small t, while on points (r,Ψ) with $\frac{1}{2} + \delta < \Psi < 1 - \delta$ it will be close to $-\frac{\partial}{\partial s}$, so it will have the form

$$X(r,\Psi,s) = A(r,\Psi,s)\frac{\partial}{\partial r} + B(r,\Psi,s)\frac{\partial}{\partial\Psi} + C(r,\Psi,s)$$

with $C(r,\Psi,s)$ close to $+1$ or -1 depending on whether $\delta < \Psi < \frac{1}{2} - \delta$ or $\frac{1}{2} + \delta < \Psi < 1 - \delta$. Since in this region also $\frac{s \cdot g'(\Psi)}{g(\Psi)}$ is bounded, the only problem for convergence of $d_*(X(r,\Psi,s))$ to $\frac{\partial}{\partial\Psi}$ as $r \to 1$ can occur if $A(r,\Psi,s)$ is negative and r is close to 1. This happens only for Ψ close to $\frac{1}{8}$, and here for the first time we have to specify the diffeomorphism $b: V_1 \setminus \vartheta \to [\frac{1}{2},1) \times S^1 \times S^1$. If $\frac{\partial}{\partial k}(r(\Psi,t),\Psi)$ is the positive unit tangent vector to K_t in $(r(\Psi,t),\Psi)$, then the $\frac{\partial}{\partial s}$-component of $X(r(\Psi,t),\Psi,s)$ is somewhat more than the $\frac{\sin 2\pi\Psi}{f(t)}$-fold of the $\frac{\partial}{\partial k}$ component. For $\frac{3}{8} \le \Psi \le \frac{9}{8}$, $r(\Psi,t)$ fluctuates between $r_0(t)$ and $r_1(t)$ with $r_0(t) \to 1$ as $t \to 0$. We choose b in such a way that $1-r_1(t) \ge e^{1/t} \cdot f(t)$ for small t. (This obviously puts a restraint on f. If we choose $f(t) = e^{-1/t^2}$ for small t, the above inequality can be satisfied.) Then the negative contribution to $\frac{\partial}{\partial\Psi}$ coming from $d_*(A(r,\Psi,s)\frac{\partial}{\partial r})$ will be negligible when compared with the contribution to $\frac{\partial}{\partial\Psi}$ coming from $d_*(C(r,\Psi,s)\frac{\partial}{\partial s})$ for points (r,Ψ,s) with Ψ close to $\frac{1}{8}$, r close to 1, and as always $\frac{1}{8} < s < \frac{3}{8}$.

So it remains to analyze $d_*(X(r,\Psi,s))$ for Ψ close to 0 and $\frac{1}{2}$. When Ψ is close to 0, $A(r,\Psi,s) = 0$. Since $B(r,\Psi,s) > 0$ everywhere and $g'(\Psi) > k$ for some $k > 0$, if Ψ is close to 0, $d_*(X(r,\Psi,s)) \to \frac{\partial}{\partial\Psi}$ as long as Ψ stays in some neighbourhood of 0, $s \in (\frac{1}{8}, \frac{3}{8})$, and $r \to 1$.

Finally, for $\Psi \in (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ the shapes of the curves K_t come into play. $X(r(\Psi,t),\Psi,s)$ will be of the form $N(r(\Psi,t),\Psi,s)(a(\Psi,t)\frac{\partial}{\partial k}(r(\Psi,t),\Psi) + \frac{\sin 2\pi\Psi}{f(t)} \cdot \frac{\partial}{\partial s})$, where N is chosen to make sure that X is a unit vector, and $a(\Psi,t)$ is a function close to 1. It takes care of the stretching of the curve $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon) \times \{\phi_0\} \times t$ under the map b. By property (3) of the curves K_t, the $\frac{\partial}{\partial r}$ component of $\partial k(r(\Psi,t),\Psi)$ will be the $(1-r(\Psi,t))$-fold of the $\frac{\partial}{\partial\Psi}$ component, as long as $\frac{1}{2} - \frac{\varepsilon}{2} \le \Psi \le \frac{1}{2} + \frac{\varepsilon}{2}$. Therefore the contribution from $d_*(\frac{\partial}{\partial k}(r(\Psi,t),\Psi))$ to $\frac{\partial}{\partial\Psi}$ will be

$$\left(\frac{1}{(1+(1-r)^2)}\right)^{\frac{1}{2}} \cdot \left(1 + (1-r \cdot s \cdot g'(\Psi))\,\frac{e^{sg(\Psi)}}{1-r} \cdot e^{\frac{r}{1-r}e^{sg(\Psi)}}\right).$$

If ε is small enough, $|r \cdot s \cdot g'(\Psi)| < 1$ and so this number rapidly goes to $+\infty$ as $r \to 1$. So a problem for convergence can develop only when $|\frac{\sin 2\pi\Psi}{f(t)}|$ is not small in comparison with 1 while $\frac{r}{1-r}|g(\Psi)| \cdot e^{r/(1-r)}$ (which is the coefficient of $\frac{\partial}{\partial\Psi}$ in $d_*(\frac{\partial}{\partial s})$ up to a bounded factor) is not large in comparison with 1 (which is the coefficient of $\frac{\partial}{\partial s}$ in $d_*(\frac{\partial}{\partial s})$). Here we introduce the second restraint on b: $1-r_1(t) \le e^{-1/t}$ for small t. With our choice of

$f(t) = e^{-1/t^2}$), this is possible. Then, for ε small enough, we have $1-r_0(t) \leq 2\, e^{-1/t}$. If we further demand that, near $\frac{1}{2}$, $g(\Psi) = (\Psi - \frac{1}{2})^3 \cdot \gamma(\Psi)$, where $\gamma(\frac{1}{2}) < 0$, then

$$|g(\Psi)| \cdot \frac{r(\Psi,t)}{1-r(\Psi,t)} \cdot e^{\frac{r(\Psi,t)}{1-r(\Psi,t)}}$$

$$\geq |g(\Psi)|\, \frac{1}{4}\, e^{\frac{1}{t}} \cdot e^{\frac{1}{4}e^{1/t}} \;\geq\; \text{const.}\ |(\Psi - \tfrac{1}{2})^3|\ e^{\frac{1}{4}\, e^{1/t}} .$$

Since, for Ψ near $\frac{1}{2}$, $|\sin 2\pi\Psi|$ is close to $2\pi|\Psi - \frac{1}{2}|$ and $\frac{1}{f(t)} = e^{1/t^2}$ for small t, we obtain

$$\left|\frac{\sin 2\pi\Psi}{f(t)}\right| \leq \left(\frac{r}{1-r}\, |g(\Psi)| \cdot e^{\frac{r}{1-r}}\right)^{\frac{1}{3}} \cdot \left(\text{const.}\ e^{1/t^2 - \frac{1}{12}e^{1/t}}\right)$$

Therefore, if the $\frac{\partial}{\partial s}$ component of $X(r,\Psi,s)$ cannot be neglected in comparison with the $\frac{\partial}{\partial k}$ component, i.e., if $\left|\frac{\sin 2\pi\Psi}{f(t)}\right|$ is not very small, then the $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial s}$ components of $d_*(\frac{\partial}{\partial s})$ are neglibly small in comparison with the $\frac{\partial}{\partial \Psi}$ component.

Up till now we have completely ignored the $\frac{\partial}{\partial r}$ -component of $d \circ b(F(f))$. But, for r close to 1, the $\frac{\partial}{\partial r}$ -component of $d_*(\frac{\partial}{\partial r})$, $d_*(\frac{\partial}{\partial \Psi})$, and $d_*(\frac{\partial}{\partial s})$ can be neglected in comparison with the other components. Therefore the above discussion shows that the positive unit tangent vector field of $d \circ b(F(f))$ at (r,Ψ,s), $\frac{1}{8} < s < \frac{3}{8}$ converges (rapidly) to $\frac{\partial}{\partial \Psi}$ as r goes to 1.

As we have indicated above, the proposition implies the main result of this section.

Corollary There exists a circle foliation F on an open solid torus V having the following properties. $B_2(F)$ is an annulus splitting V into two open solid tori V_1, U_1 invariant under F, where $V_1 \subset V$ is contractible and unknotted. $B_1(F|V_1)$ is again an annulus splitting V_1 into two invariant open solid tori V_0, U_0, and $V_0 \subset V_1$ is contractible and unknotted.

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Some remarks on ∇-G-foliations

In this short note we present two methods for dealing with some problems for transverse structures of G-foliations and in particular for ∇-G-foliations. As examples of applications of these methods we prove two theorems. We construct the graph of a ∇-G-foliation, and then study the transverse structure on the graph. Its properties allow us to prove the following theorem.

Theorem A Let $\mathcal{F}$ be a transversely complete ∇-G-foliation. Then the leaves of the foliation $\mathcal{F}$ have the same universal covering space.

In the second part we give definitions of transverse structures of higher order. They admit a foliation of the same dimension as the initial one, and its leaves are covering spaces of the leaves of the initial foliation. We show the following.

Theorem B A G-foliation $\mathcal{F}$ admits a transversely projectible G-connection if and only if the foliation $\mathcal{F}^r$ of the normal bundle of order r admits a transversely projectible G^r-connection.

1. PRELIMINARIES

Let M be a connected n-manifold, N a q-manifold ($n \geq q$) with a G-structure B(N,G). Let ∇ be a G-connection in the G-structure B(N,G). Let $(M,\mathcal{F})$ be a ∇-G-foliation modelled on B(N,G), and defined by a cocylce $\{U_i, f_i, g_{ij}\}$, i.e. $f_i: U_i \to N$ is a submersion, $f_j | U_i \cap U_j = g_{ji} \circ f_i$, and g_{ij} are local automorphisms of the G-structure B(N,G) and affine automorphisms of the connection ∇. Let H be a sub-bundle of the tangent bundle TM supplementary to the foliation $\mathcal{F}$. Let L(H) be the frame bundle of H and denote by $B(M,\mathcal{F},G)$ the G-reduction of L(H) defined by the G-foliation $\mathcal{F}$. A G-structure obtained in this way is called a transverse G-structure. We denote by $\tilde{f}_i$ the mapping of $B(M,\mathcal{F},G)|U_i$ into B(N,G) defined by f_i. Let ω be the connection form of the connection ∇. Then the forms $\tilde{f}_i^*\omega$ glue together to define a connection $\bar{\omega}$ on L(H), which is a connection in the transverse G-structure $B(M,\mathcal{F},G)$. By $\bar{\nabla}$ we denote the corresponding differential operator on H.

The fundamental form $\bar{\theta}$ of the transverse G-structure $B(M,\mathcal{F},G)$ is defined as follows:

Let p be a transverse frame of $B(M,\mathcal{F},G)$. Then

$$\bar{\theta}_p : T_p B(M,\mathcal{F},G) \xrightarrow{d\pi} T_{\pi p} M \xrightarrow{p_{\mathcal{F}}} H_{\pi p} \xrightarrow{p^{-1}} R^q,$$

where $\pi : B(M,\mathcal{F},G) \to M$ is the natural mapping, $p_{\mathcal{F}} : TM \to H$ is the projection of TM onto H along $\mathcal{F}$, and p^{-1} is the inverse of the linear isomorphism defined by the frame p.

For any vector $\xi \in R^q$ we can define a vector field $\bar{B}(\xi)$ on $B(M,\mathcal{F},G)$ called the fundamental horizontal vector field. We demand that:

(i) $\bar{\theta}\bar{B}(\xi) = \xi$;

(ii) $\bar{B}(\xi)$ is a horizontal vector field;

(iii) for any $p \in B(M,\mathcal{F},G)$, $\bar{B}(\xi)_p \in \bar{H}_p$, where $\bar{H}$ is the supplementary distribution in ker $\bar{\omega}$ to the lifted foliation $\bar{\mathcal{F}}$:

$$\bar{H}_p = (d_p\pi \mid \ker \bar{\omega})^{-1}(H_{\pi p}).$$

One can easily check that $\bar{\theta} \mid \bar{\pi}^{-1}(U_i) = \tilde{f}_i^*\theta$, where θ is the fundamental form of the G-structure $B(N,G)$. The mapping $d_p\tilde{f}_i|\bar{H}_p : \bar{H}_p \to \Gamma_{\tilde{f}_i(p)}$ is an isomorphism, where Γ is the horizontal space of the connection ω. Additionally, if $B(\xi)$ is the fundamental horizontal vector field on $B(N,G)$ defined by the vector $\xi \in R^q$, then

$$d_p\tilde{f}_i(\bar{B}(\xi)) = B(\xi)_{\tilde{f}_i(p)}.$$

<u>Proposition 1</u> Let S be a section of the bundle H on U_i, constant along the leaves of $\mathcal{F}$. Then

$$\bar{\nabla}_X S = (df_i|H)^{-1}\nabla_{df_iX}\tilde{S},$$

where $S = (df_i|H)^{-1}\tilde{S}\circ f_i$.

<u>Proof</u> This formula is obtained directly from the definition via the Christofel symbols.

Let $\alpha:[0,1] \to M$ be a curve in a leaf of the foliation $\mathcal{F}$, $\alpha(0) = x$, $\alpha(1) = y$. Then the curve α defines a mapping T_α of H_x into H_y. Let γ be a curve such that $\gamma(0) = x$, $d/dt\gamma(0) = v \in H_x$. Let α_t be the curve starting at $\gamma(t)$, the holonomy lift of α to $\gamma(t)$. Then $t \to \alpha_t(1)$ is a curve at y transverse to $\mathcal{F}$. The H-component of the vector $d/dt\alpha_t(1)(0)$ we assume as the value of T_α on the vector v. One can easily check that it does not depend on the choice of the curve γ. The mapping T_α is a linear isomorphism, and in its turn defines an isomorphism $\tilde{T}_\alpha : L(H)_x \to L(H)_y$.

Let (U_x,ϕ_x) be an adapted chart at the point x and let (U_y,ϕ_y) be an adapted chart at the point y such that each plaque is contractible. Let D_x be the transverse submanifold at x and let D_y be the transverse submanifold at y. For any points $u \in U_x$ and $v \in U_y$ there exist the unique points $u' \in D_x$ and $v' \in D_y$, respectively, such that the points u and u', v and v' belong to the same plaque of U_x or U_y, respectively.

Let V_x be a neighbourhood of x in D_x such that, for any point x' of V_x, there exists the holonomy lift of the curve α to x'.

Let us denote by V_y the image by the holonomy mapping of V_x in D_y. Let $\hat{U}_x$ and $\hat{U}_y$ be the saturations of V_x and V_y in U_x and U_y, respectively. By α_{uv} denote the curve $s_v^{-1} * \alpha_{u'} * s_u$, where $\alpha_{u'}$ is the holonomy lift of the curve α to u', s_u is a curve in the plaque linking u to u', s_v is a curve in the plaque linking v to v'. The holonomy mapping T along the curve α_{uv} does not depend on the choice of the curves s_u and s_v. We can choose curves s_u and s_v in such a way that the mapping

$$\tilde{\alpha} : \hat{U}_x \underline{\times} \hat{U}_y \times I \to M : \tilde{\alpha}(u,v,t) = \alpha_{uv}(t)$$

is smooth, where $\hat{U}_x \underline{\times} \hat{U}_y = \{(u,v) \in \hat{U}_x \times \hat{U}_y : T_\alpha(u') = v'\}$.

For any pair of points $(u,v) \in \hat{U}_x \underline{\times} \hat{U}_y$, the curve α_{uv} defines the mapping $\tilde{T}_{\alpha_{uv}} : L(H)_u \to L(H)_v$, and the mapping

$$\tilde{T}_{\tilde{\alpha}} : L(H)|\pi^{-1}(\hat{U}_x) \underline{\times} \pi^{-1}(\hat{U}_y) \to \pi^{-1}(\hat{U}_x) \underline{\times} L(H)|\pi^{-1}(\hat{U}_y)$$

is smooth.

<u>Lemma 1</u> $d_p\tilde{T}_\alpha(\bar{B}(\xi)_p) = \bar{B}(\xi)_{\tilde{T}_\alpha(p)}$.

<u>Proof</u> Let $x = \alpha(0)$ and $y = \alpha(1)$, and let V_x and V_y be two transverse manifolds at x and y, respectively. Additionally, we assume that the manifold V_x is contained in some U_i and V_y in some U_j. Then the mappings $f_i|V_x$ and $f_j|V_y$ are local diffeomorphisms. The mapping

$$f_j \circ T_\alpha \circ f_i^{-1}|_{V_x} : f_i(V_x) \to f_j(V_y)$$

is just the composition $g_{i_0 i_1} \circ \ldots \circ g_{i_{k-1} i_k}$ for some indices $i_0, \ldots, i_k$. Thus the mapping

$$\tilde{f}_j \circ \tilde{T}_\alpha \circ \tilde{f}_i^{-1}|_{V_x} : L(f_i(V_x)) \to L(f_j(V_y))$$

is equal to $\tilde{g}_{i_0 i_1} \circ \ldots \circ \tilde{g}_{i_{k-1} i_k}$, and therefore it is an affine mapping.

The vector fields $\bar{B}(\xi)$ on $L(H)$ or $B(M,\mathcal{F},G)$ are mapped by $\tilde{f}_i$ onto the vector fields $B(\xi)$. Then

$$\tilde{f}_j \circ \tilde{T}_\alpha(\bar{B}(\xi)) = \tilde{f}_j \circ \tilde{T}_\alpha \circ \tilde{f}_i^{-1} \circ \tilde{f}_i(\bar{B}(\xi)) = \tilde{g}_{i_0 i_1} \circ \ldots \circ \tilde{g}_{i_{k-1} i_k}(B(\xi))$$
$$= B(\xi).$$

Thus $\tilde{T}_\alpha^* \bar{B}(\xi) = \bar{B}(\xi)$.

2. GRAPH OF A ∇-G-FOLIATION

For the convenience of the reader we recall the construction of the graph of a foliation, due to Ch. Ehresmann and later developed by H.E. Winkelnkemper [4].

Let x and y be two points of the manifold M lying in the same leaf and let α be a piecewise smooth curve linking x to y and contained in the same leaf. We say that two such curves α and β are equivalent if the holonomy along the curve $\alpha\beta^{-1}$ is trivial. The space of all such triples $(x,y,[\alpha])$, where $[\alpha]$ denotes the equivalence class of α in the above equivalence relation, is called the graph of the foliation $\mathcal{F}$, and we denote it by $GR(\mathcal{F})$.

The topology is introduced in the following way. Let $z = (x_0, y_0, [\alpha])$ be any point of $GR(\mathcal{F})$. Let α be a representative of $[\alpha]$. Take an adapted chart (U_i, ϕ_i) at x_0 and an adapted chart (U_j, ϕ_j) at y_0 such that $\phi_i : D^k \times D^q \to M$, $\phi_i(0,0) = x_0$, and $\phi_j : D^k \times D^q \to M$, $\phi_j(0,0) = y_0$. By W_i we denote the transverse submanifold $\phi_i(0 \times D^q)$ and by W_j the submanifold $\phi_j(0 \times D^q)$. For any point $x \in U_i$, by x_i we denote the point of the plaque of x belonging to W_i, and for any point y of U_j by y_j the point of the plaque of y belonging to W_j. By V_i denote all the points of U_i whose plaques can be linked with a plaque of U_j by a chain of plaques following the curve α. By V_j denote the set of points of U_j which lie in the end plaques of the above chains. Let

$W_{z,a}$ be a subset of $GR(\mathcal{F})$ defined as follows:

$$W_{z,\alpha} = \{(x,y,[\beta]) \in GR(\mathcal{F});\ x \in V_i,\ y \in V_j,\ \beta = s_y^{-1} * \tilde{\alpha} * s_x\}$$

where s_x, s_y are curves linking the points x and y with x_i and y_j, respectively, in the corresponding plaques, and $\tilde{\alpha}$ is the holonomy lift of α to x_i. The set is well defined as the end point of $\tilde{\alpha}$ must be y_j, and the equivalence class of β does not depend on the choices of the curves s_x and s_y. The sets $W_{z,\alpha}$ we take as a sub-base of the topology of $GR(\mathcal{F})$.

In our case, by means of these sets we can introduce a differentiable structure (cf. [4]). In this differentiable structure they are adapted charts for a 2k-dimensional foliation $\tilde{\mathcal{F}}$. The foliation $\tilde{\mathcal{F}}$ can also be defined as the inverse image of the foliation $\mathcal{F}$ by the canonical projection $\bar{p}_1 : GR(\mathcal{F}) \to M$ or $p_2 : GR(\mathcal{F}) \to M$, where $\bar{p}_1(x,y,[\alpha]) = x$ and $\bar{p}_2(x,y,[\alpha]) = y$.

The tangent bundle of the manifold $GR(\mathcal{F})$ is isomorphic to the sum $\bar{p}_1^* \mathcal{F} \oplus \bar{p}_2^* \mathcal{F} \oplus \tilde{H}$, where the bundle $\tilde{H}$ is given by

$$\tilde{H} = \{v \in TGR(\mathcal{F});\ d\bar{p}_1(v) \in H \text{ and } d\bar{p}_2(v) \in H\};$$

or, in more detail, let $z = (x,y,[\alpha]) \in GR(\mathcal{F})$, then any tangent vector $Z \in T_z GR(\mathcal{F})$ is equal to $(X_x, Y_y, [\alpha])$, where $X_x \in T_x M$, $Y_y \in T_y M$. In particular, we can consider $\tilde{H}$ as

$$\{X \in TGR(\mathcal{F});\ X = (v, T_\alpha(v), [\alpha]),\ v \in H\}.$$

Let us consider the reduction L_0 of the frame bundle $L(GR(\mathcal{F}))$ defined by the decomposition $\bar{p}_1^* \mathcal{F} \oplus \bar{p}_2^* \mathcal{F} \oplus \tilde{H}$ of the tangent bundle $TGR(\mathcal{F})$. Let $z = (x,y,[\alpha])$. A frame $v = (v_1,\dots,v_k,w_1,\dots,w_q,\tilde{v}_1,\dots,\tilde{v}_k)$ at z is given by the following vectors:

$$v^1 = (v_1,\dots,v_k) \text{ a frame at } x \text{ of } \mathcal{F},$$

$$v^2 = (\tilde{v}_1,\dots,\tilde{v}_k) \text{ a frame at } y \text{ of } \mathcal{F},$$

$w_i \in \tilde{H}$, $i = 1,\dots,q$, $d\bar{p}_1(w_i) \in H$, $d\bar{p}_2(w_i) \in H$ and $T_\alpha\, d\bar{p}_1(w_i) = d\bar{p}_2(w_i)$. Thus $(v_1,\dots,v_k, d\bar{p}_1(w_1),\dots,d\bar{p}_1(w_q))$ is a frame at x, and $(\tilde{v}_1,\dots,\tilde{v}_k, d\bar{p}_2(w_1),\dots,d\bar{p}_2(w_q))$ is a frame at y. Therefore any vector tangent to L_0 is given by a curve γ

$$\gamma : t \to (v_1(t),\dots,v_k(t),w_1(t),\dots,w_q(t),\tilde{v}_1(t),\dots,\tilde{v}_k(t))$$

such that

$$d\bar{p}_2(w_i(t)) = T_\alpha d\bar{p}_1(w_i(t)) \text{ for any } i = 1,\dots,q.$$

Let $\bar{\omega}$ be the connection on H defined by the connection ∇. Directly from the definition of the foliation $\tilde{\mathcal{F}}$ is a ∇-G-foliation, and the bundle $\tilde{H}$ can be considered as the normal bundle of the foliation $\tilde{\mathcal{F}}$.

<u>Lemma 2</u> The mapping $\tilde{T}_\alpha$ is an affine transformation of the connection $\bar{\omega}$.

<u>Proof</u> Let $\alpha(0) = x$ and $\alpha(1) = y$, and let $x \in U_i$ for some i and $y \in U_j$ for some j. Then

$$\tilde{T}^*_\alpha\bar{\omega} = \tilde{T}^*_\alpha\tilde{f}^*_j\omega = (\tilde{f}_j\tilde{T}_\alpha)^*\omega = (\tilde{f}_j\tilde{T}_\alpha\tilde{f}_i^{-1}\tilde{f}_i)^*\omega = \tilde{f}^*_i(\tilde{f}_j\tilde{T}_\alpha\tilde{f}_i^{-1})^*\omega$$

$$= f^*(\tilde{g}_{i_0 i_1} \dots \tilde{g}_{i_{k-1} i_k})^*\omega = \tilde{f}^*_i\omega = \bar{\omega}.$$

Thus $\tilde{T}_\alpha$ is an affine transformation of $\bar{\omega}$.

<u>Definition</u> A ∇-G-foliation $\mathcal{F}$ is called *transversely complete* if fundamental horizontal vector fields $\bar{B}(\xi)$ are complete for any $\xi \in R^q$.

<u>Lemma 3</u> Let a ∇-G-foliation $\mathcal{F}$ be transversely complete. Then the foliation $\tilde{\mathcal{F}}$ on the graph manifold $GR(\mathcal{F})$ is transversely complete.

<u>Proof</u> Let $\bar{B}(\xi)$ be a fundamental horizontal vector field on $B(M,\mathcal{F},G)$, and ϕ_t its global flow. Let $w = (w_1,\dots,w_q)$ be a frame of $\tilde{H}$ at $z = (x,y,[\alpha])$. Then each w_i, $i = 1,\dots,q$ is equal to $(w_i^1, w_i^2, [\alpha])$, where $w_i^2 = dT_\alpha w_i^1$, and $w^1 = (w_1^1,\dots,w_q^1)$ is a frame at x of H, and $w^2 = (w_1^2,\dots,w_q^2)$ is a frame at y. We put

$$\tilde{\phi}_t(w) = (\phi_t(w^1), \phi_t(w^2), [\phi_t(\alpha)])$$

where the curve $\phi_t(\alpha)$ is obtained in the following way. Since α is a leaf curve, we can lift it to a leaf curve $\tilde{\alpha}$ at $w^1 \in B(M,\mathcal{F},G)$, then we take $\phi_t(\tilde{\alpha})$, which is a leaf curve as the vector fields $\bar{B}(\xi)$ are infinitesimal automorphisms of the foliation $\tilde{\mathcal{F}}$ of the fibre bundle. Next, we project this curve back to M and obtain a leaf curve, which we denote by $\phi_t(\alpha)$. We have to show that the homotopy class of the curve $\phi_t(\alpha)$ does not depend on the choice of the lift $\tilde{\alpha}$.

The segment $\phi[0,t]\tilde{\alpha}(0)$ of the flow is a transverse curve to the foliation $\bar{\mathcal{F}}$ of $B(M,\mathcal{F},G)$. It projects to a transverse curve τ to the foliation $\mathcal{F}$ on the manifold M. Since the flow preserves the foliation $\bar{\mathcal{F}}$, the mapping

$$\beta:[0,1]\times[0,t]\ni(s,v)\to\phi_v\tilde{\alpha}(s)$$

is the holonomy lift of the curve $\tilde{\alpha}$ along $\phi[0,t]\tilde{\alpha}(0)$, and the projection $\tilde{\beta}$ of β onto the manifold M is the holonomy lift of the curve α along the curve τ.

To complete the checking that $\tilde{\phi}_t$ is well defined, we recall that $d\tilde{T}_\alpha\bar{B}(\xi)=\bar{B}(\xi)$. Then the mapping $\tilde{T}_\alpha$ will commute with the flow of $\bar{B}(\xi)$, and indeed $\tilde{\phi}_t(w)$ will be an element of $B(M,\mathcal{F},G)$, i.e.

$$\tilde{T}_\alpha\phi_t(w^1)=\phi_t\tilde{T}_\alpha(w^1)=\phi_t(w^2).$$

Since the foliation $\tilde{\mathcal{F}}$ on the manifold $GR(\mathcal{F})$ is defined by a cocycle $\{\bar{p}_1^{-1}(U_i),f_i\bar{p}_1,g_{ij}\}$, and the tangent vector to $\tilde{\phi}_t$ is the vector $(\bar{B}(\xi),\bar{B}(\xi),[\alpha])$, the flow $\tilde{\phi}_t$ is the flow of a fundamental horizontal vector field, as, of course, this vector is horizontal, since, locally, the connection on $B(GR(\mathcal{F}),\tilde{\mathcal{F}};G)$ is given by $(\tilde{f}_i\circ\bar{\bar{p}}_1)^*\omega$.

3. PROOF OF THEOREM A

Any two points of the manifold M can be joined by a piecewise smooth curve whose segments are either leaf curves or projected segments of integral curves of fundamental horizontal vector fields. We would like to lift these curves to $GR(\mathcal{F})$. As the horizontal bundle we assume the bundle $\tilde{H}\oplus\bar{p}_1^*\mathcal{F}$, which is transverse to the fibre of the submersion $\bar{p}_1:GR(\mathcal{F})\to M$. The fibres are covering spaces of the leaves of the foliation $\mathcal{F}$ (cf. [4]).

First of all, we lift leaf curves. Let γ be a leaf curve. The lift of this curve has to be tangent to the bundle $\tilde{H}\oplus\bar{p}_1^*\mathcal{F}$. It cannot be tangent to $\tilde{H}$, thus it must be tangent to $\bar{p}_1^*\mathcal{F}$. Therefore the tangent vector has to be of the form $(X,0,[\alpha])$. Since $d\bar{p}_1(X,0,[\alpha])=X$, the lift of the curve γ to the point $z=(x,y;[\alpha])$, where, $x=\alpha(0)$ is the curve $\tilde{\gamma}:[0,1]\ni t\to(\gamma(t),y,[\alpha*\gamma^{-1}[0,t]])$.

To lift a transverse curve, we need a more subtle construction. Any such curve γ is a projection of a segment of an integral curve $\phi[0,1](p_0)$ of a fundamental horizontal vector field $\bar{B}(\xi)$ on the manifold $B(M,\mathcal{F},G)$. Take a corresponding segment $\tilde{\phi}_t[0,1](\tilde{p}_0)$ of an integral curve of the vector field $\bar{B}(\xi)$ on the manifold $B(GR(\mathcal{F}),\mathcal{F},G)$ starting from a point $\tilde{p}_0$ in the fibre $\bar{p}_1^{-1}(p_0)$. The projection of the curve onto the manifold $GR(\mathcal{F})$ is tangent to the bundle $\tilde{H}$, as $d/dt\tilde{\phi}_t = (\bar{B}(\xi),\bar{B}(\xi),[\tilde{\alpha}])$ and $d\pi(\bar{B}(\xi)) \in H$ directly from the definition of the vector field $\bar{B}(\xi)$. Additionally, as $d\bar{T}_\alpha\bar{B}(\xi) = \bar{B}(\xi)$ and $d\bar{T}_\alpha d\pi\bar{B}(\xi) = d\pi d\tilde{T}_\alpha\bar{B}(\xi) = d\pi\bar{B}(\xi)$, it follows that, indeed, the vector tangent to $\pi\tilde{\phi}_t[0,1](\tilde{p}_0)$ is tangent to $\tilde{H}$. The choice of $\tilde{p}_0$ in the fibre $\bar{p}_1^{-1}(p_0)$ corresponds to the choice of the point in the fibre $\bar{p}_1^{-1}(\gamma(0))$.

As we have shown, we can lift horizontally from M to $GR(\mathcal{F})$ any curve of the chosen type. We shall call such curves ∇-curves.

__Lemma 4__ Any ∇-curve γ in M defines a diffeomorphism of $\bar{p}_1^{-1}(\gamma(0))$ onto $\bar{p}_1^{-1}(\gamma(1))$.

__Proof__ The horizontal lifts of ∇-curves depend smoothly on the initial condition; thus, lifting the curve γ to the points of the fibre $\bar{p}_1^{-1}(\gamma(0))$ and taking the end points of the lifts, we define the mapping of $\bar{p}_1^{-1}(\gamma(0))$ into $\bar{p}_1^{-1}(\gamma(1))$ which is a diffeomorphism.

Using the same methods as in the proof of [1, Theorem 1, p. 239], we show the following theorem.

__Theorem 1__ Let G be a connected Lie group and let M be a connected manifold. Let $\mathcal{F}$ be a transversely complete ∇-G-foliation. Then $\bar{p}_1: GR(\mathcal{F}) \to M$ is a locally trivial fibre bundle, with the structure group $\mathrm{Diff})\bar{p}_1^{-1}(x_0))$, where x_0 is any point of M.

As a corollary of this theorem we get our main theorem as the fibres of $\bar{p}_1$ are covering spaces of the leaves of the foliation $\mathcal{F}$.

4. TRANSVERSE STRUCTURES OF FOLIATIONS

We present the definitions of some transverse structures and their basic properties. More details can be found in [6].

__Example 1__ Transverse (p,r)-velocities (p^r-jets).

Let m be a point of the manifold M. Let $f:(R^p,0) \to (M,m)$ be any local smooth mapping of R^p into M mapping 0 onto m. Let f,g be two such mappings and let (U,ϕ) be any adapted chart at m such that $\phi: U \to R^{n-q} \times R^q$, $\phi(x) = (\phi_1(x), \phi_2(x))$, thus ϕ_2 is constant along the leaves. We shall also use the notation

$\phi_1 = (y_1,\ldots,y_{n-q})$, $\phi_2 = (x_1,\ldots,x_q)$. We say that the mappings f and g are equivalent if $j_0^r\phi_2 f = j_0^r\phi_2 g$. This is equivalent to

$$\partial^{|v|}/\partial x^v(x_i\cdot f) = \partial^{|v|}/\partial x^v(x_i\cdot g)$$

for any multi-index $v \in N^p$, $|v| \le r$, $i = 1,\ldots,q$. We shall denote the number of such indices by $p(r)$. This equivalence relation does not depend on the choice of an adapted chart at the point m. The equivalence class of a mapping f we denote by $[f]_p^r$. The set of all equivalence classes at a point m we denote by $N_m^{p,r}(M,\mathcal{F})$ and the space $\bigcup_{m\in M} N_m^{p,r}(M,\mathcal{F})$ by $N^{p,r}(M,\mathcal{F})$. By π_p^r let us denote the natural projection of $N^{p,r}(M,\mathcal{F})$ into M, i.e. $\pi_p^r([f]_p^r) = f(0)$. One can easily check that for any adapted chart (U,ϕ) the set $\bigcup_{m\in U} N_m^{p,r}(M,\mathcal{F})$ is isomorphic to $U \times R^{q\cdot p(r)}$ and that the isomorphism is given by the mapping $[f]_p^r \to \partial^{|v|}/\partial x^v(x_i\cdot f)_{i=1,\ldots,q}$. Thus, if we denote the mapping defined above by ϕ_p^r, $\phi_p^r:(\pi_p^r)^{-1}(U) \to R^{n-q} \times R^q \times R^{q\cdot p(r)}$, the collection of all such ϕ_p^r defined by an adapted atlas on M defines an atlas on the space $N^{p,r}(M,\mathcal{F})$. To see this, one has only to notice that if ϕ_i,ϕ_j are two adapted charts for a foliated manifold $(M,\mathcal{F})$, the composition $\phi_i\phi_j^{-1}:R^{n-q} \times R^q \to R^{n-q} \times R^q$ is of the form $(f_1(y,x),f_2(x))$, where y denotes the first n-q coordinates, x the last q, $f_1:R^{n-q} \times R^q \to R^{n-q}$, and $f_2:R^{n-q} \times R^q \to R^q$; then $\phi_{ip}^r(\phi_{jp}^r)^{-1}$: $R^{n-q} \times R^q \times R^{q\cdot p(r)} \to R^{n-q} \times R^q \times R^{q\cdot p(r)}$ is equal to $(f_1,T_p^r(f_2))$, where $T_p^r(f_2)$ is the mapping of $T^r(R^q) = R^q \times R^{q\cdot p(r)}$ induced by f_2.

Summing up, we have proved that $N^{p,r}(M,\mathcal{F})$ is a locally trivial bundle, whose total space admits a codimension $q\cdot p(r) + q$ foliation $\mathcal{F}_p^r$ projecting by π_p^r onto the initial foliation.

If p = q and we take only transverse embeddings of R^q into M, the above construction gives a bundle called the transverse frame bundle of the foliated manifold $(M,\mathcal{F})$ and is denoted by $L^r(M,\mathcal{F})$. It is a principal fibre bundle with the fibre L_q^r.

<u>Example 2</u> The bundle of transverse A-points of $(M,\mathcal{F})$

Let A be an associative algebra over the field R with the unit 1. The algebra A is called local if it is commutative, of finite dimension over R, and if it admits the unique maximal ideal $\mathfrak{m}$ of codimension 1 such that $\mathfrak{m}^{h+1} = 0$ for some non-negative integer h. The smallest such h is called the height of A.

Let $R[p] = R[x_1,\ldots,x_p]$ be the algebra of all formal power series in $x_1,\ldots,x_p$, and let $\mathfrak{m}_p$ be the maximal ideal of $R[p]$ of all formal power series without constant terms. Let $\mathfrak{A}$ be a non-trivial ideal of $R[p]$ such that $R[p]/\mathfrak{A}$ is of finite dimension. Then $A = R[p]/\mathfrak{A}$ is a local algebra with the maximal ideal $\mathfrak{m} = \mathfrak{m}_p/\mathfrak{A}$. Any local algebra is isomorphic to such a local algebra (cf. [2]).

Let $C^\infty_m(M,\mathcal{F})$ be the algebra of germs of smooth functions constant on the leaves of the foliation $\mathcal{F}$ at the point m of the manifold M. An algebra homeomorphism $\mathcal{L}:C^\infty_m(M,\mathcal{F}) \to A$ will be called an A-point of $(M,\mathcal{F})$ near to m (or an infinitely near transverse point to m of kind A) if $\mathcal{L}(f) \equiv f(m) \bmod \mathfrak{m}$ for every $f \in C^\infty_m(M,\mathcal{F})$. We denote by $A_m(M,\mathcal{F})$ the set of all A-points of $(M,\mathcal{F})$ near to m, and by $A(M,\mathcal{F}) = \bigcup_{m\in M} A_m(M,\mathcal{F})$. The mapping $A_m(M,\mathcal{F}) \ni \alpha \to m \in M$ is denoted by π_A.

One can prove that the set $A(M,\mathcal{F})$ admits a differentiable structure such that $\pi_A:A(M,\mathcal{F}) \to M$ is a fibre bundle over M with fibre A, and that there exists a canonically defined foliation $\mathcal{F}_A$ of the same dimension as the foliation $\mathcal{F}$.

5. PROOF OF THEOREM B

To prove Theorem B we shall use Cl. Roger's definition of the universal Atiyah-Molino class, and therefore we consider foliations as Γ-structures. First consider a pair of groupoids Γ_1 and Γ_2. Let M_1 and M_2 be two smooth manifolds, Γ_1 and Γ_2 two groupoids on M_1 and M_2, respectively. Assume that there exist two homeomorphisms of the groupoids Γ_1 and Γ_2

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\bar f} & \Gamma_2 \\ {\scriptstyle \alpha_1,\beta_1}\downdownarrows & & \downdownarrows{\scriptstyle \alpha_2,\beta_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array} \qquad \begin{array}{ccc} \Gamma_2 & \xrightarrow{i} & \Gamma_1 \\ {\scriptstyle \alpha_2,\beta_2}\downdownarrows & & \downdownarrows{\scriptstyle \alpha_1,\beta_1} \\ M_2 & \longrightarrow & M_2 \end{array} \tag{1}$$

such that $\bar f\cdot i = \mathrm{id}_{\Gamma_2}$, $f\cdot i = \mathrm{id}_{M_2}$ and $\Gamma_1 \mathrm{im}\, i \subset \mathrm{im}\, i$, $\gamma i(\gamma_1) = i(\bar f\gamma)i(\gamma'_1)$.

The need to consider such a pair of groupoids is explained by the following. Let $M = M_2$ be a smooth manifold, M_1 be the r-tangent bundle of M, $M_1 = T^rM$. Let $P(M,G)$ be a given G-structure on M. Let Γ_2 be a groupoid of germs of automorphisms of the G-structure $P(M,G)$, let Γ_1 be the groupoid of germs of lifts of elements of Γ_2 to T^rM, and $P^r(M,G)$ be the r-prolongation of the G-structure $P(M,G)$. It is a G^r-structure on T^rM, where G^r is the r-prolongation of the group G. Let $i:M \to T^rM$ be the zero section. Then the mapping

$i:\Gamma_2 \to \Gamma_1$ is defined as follows:

$$\Gamma_2 \ni (f)_x \longrightarrow (T^r f)_{i(x)} \in \Gamma_1.$$

The mapping $\bar{f}$ we define as the natural projection. Of course, these mappings are homeomorphisms of groupoids.

If $\mathcal{F}$ is a G-foliation modelled on B(N,G), then the foliation $\mathcal{F}^r$ is a G^r-foliation modelled on $B^r(N,G)$. Thus a G-foliation is a $\Gamma_2 = \Gamma$-structure for a suitable choice of the groupoid Γ, and then the foliation $\mathcal{F}^r$ is a Γ^r-structure.

Let $(\underline{F}_1, g_1)$ be a Γ_1-sheaf over M_1, and $(\underline{F}_2, g_2)$ be a Γ_2-sheaf over M_2. A cohomeomorphism F of $\underline{F}_1$ into $\underline{F}_2$ over i is called a (Γ_1, Γ_2)-cohomeomorphism if, for any $x \in M_2$, $v \in \underline{F}_{1i(x)}$, $\gamma \in \Gamma_1$, $\alpha_1(\gamma) = i(x)$,

$$F(g_1(\gamma)(v)) = g_2(\bar{f}(\gamma))F(v).$$

After long computations one can show the following.

<u>Lemma 5</u> Any (Γ_1, Γ_2)-cohomeomorphism F of the sheaves $\underline{F}_1$ and $\underline{F}_2$ induces a mapping in cohomology:

$$\tilde{F}^*: H^*(B\Gamma_1, \underline{F}_1) \longrightarrow H^*(B\Gamma_2, \underline{F}_2). \tag{2}$$

As the next step of the proof let us consider the following situation.

Let $\pi: X \to Y$ be a continuous surjection, $\mathfrak{U} = \{U_i\}$ be a covering of Y, and $\mathfrak{V} = \{\pi^{-1}(U_i)\}$ be a covering of X. Let $\mathcal{F}_2$ be a continuous functor, $\mathcal{F}_2: Y_{\mathfrak{U}} \to \Gamma_2$, let $\mathcal{F}_1$ be another continuous functor, $\mathcal{F}_1: X_{\mathfrak{V}} \to \Gamma_1$, such that the following diagram is commutative

$$\begin{tikzcd} X \arrow[r, "\mathcal{F}_1"] \arrow[d, "\pi"'] & \Gamma_1 \arrow[d, "\frac{i}{f}"] \\ Y \arrow[r, "\mathcal{F}_2"] & \Gamma_2 \end{tikzcd} \tag{3}$$

Additionally, we assume that there exists a section of π, $i_Y: Y \to X$ such that $\pi i_Y = id_Y$. We require that the diagram

$$\begin{tikzcd} X \arrow[r, "\mathcal{F}_1"] & \Gamma_1 \\ Y \arrow[u, "i_Y"] \arrow[r, "\mathcal{F}_2"] \arrow[ur, phantom, "\bullet"] & \Gamma_2 \arrow[u, "i"'] \end{tikzcd} \tag{4}$$

is commutative.

Any G-foliation $\mathcal{F}$ on M modelled on B(M,G) defines such a pair of functors. We have to take $Y = M$, $\mathfrak{U}$ the open covering of the defining cocycle, $\Gamma_2 = \Gamma$, $X = N^r(M,\mathcal{F})$, $\Gamma_1 = \Gamma^r$, $\mathfrak{V} = \{(\pi^r)^{-1}(U_i) = V_i\}$.

A Γ_1-sheaf $(\underline{F}_1, g_1)$ defines via $\mathcal{F}_1$ a sheaf $\mathcal{F}_1^* \underline{F}_1$ on the space X, and a Γ_2-sheaf $(\underline{F}_2, g_2)$ defines via $\mathcal{F}_2$ sheaf $\mathcal{F}_2^* \underline{F}_2$ on Y. For the details see [3].

<u>Lemma 6</u> For any (Γ_1,Γ_2)-cohomomorphism F of the sheaves $\underline{F}_1$ and $\underline{F}_2$ and any two cofunctors $\mathcal{F}_1$ and $\mathcal{F}_2$ such that the diagrams (3) and (4) are commutative, the following diagram is commutative:

$$\begin{tikzcd} H^*(X,\mathcal{F}_1^*\underline{F}_1) \arrow[d, "\hat{F}^*"'] & H^*(B\Gamma_1,\underline{F}_1) \arrow[l, "\hat{\mathcal{F}}_1^*"'] \arrow[d, "F^*"] \\ H^*(Y,\mathcal{F}_2^*\underline{F}_2) & H^*(B\Gamma_2,\underline{F}_2) \arrow[l, "\hat{\mathcal{F}}_2^*"'] \end{tikzcd} \tag{5}$$

Having proved the properties contained in Lemmas 5 and 6, we can complete the proof of Theorem B. Let us remark first that as a model G-structure we can always take a trivial G-structure, but then the manifold N does not need to be connected.

Let N be a manifold and let P(N,G) be a trivial principal G-fibre bundle. Let Γ be a groupoid of germs of automorphisms of P(N,G). The tangent bundle TP admits a natural action of the group G; let Q = TP/G, let T be the tangent bundle to N, and let L be the associated fibre bundle with P with standard fibre $\mathfrak{g}$ = Lie(G). Let $\underline{L}$, $\underline{Q}$, $\underline{T}$ denote the Γ-sheaves of germs of sections of the fibre bundles L, Q, T, respectively. Then the following sequence of sheaves is exact:

$$0 \to \underline{L} \to \underline{Q} \to \underline{T} \to 0. \tag{6}$$

Denote by Q^r the bundle TP^r/G^r over T^rN, by T^r the bundle TT^rN. Let $i:N \to T^rN$ be the zero section. We have the following commutative diagram of sheaves and their (Γ,Γ^r)-cohomomorphisms over i:

$$\begin{tikzcd} 0 \arrow[r] & \underline{L}^r \arrow[r] \arrow[d] & \underline{Q}^r \arrow[r] \arrow[d] & \underline{T}^r \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{L} \arrow[r] & \underline{Q} \arrow[r] & \underline{T} \arrow[r] & 0 \end{tikzcd} \tag{7}$$

In the proof of the fact that the vertical arrows, defined by the mapping π^r, are (Γ,Γ^r)-cohomomorphisms we use the existence and properties of liftings of vector fields (cf. [2], [6]).

From diagram (7) we get

$$\begin{tikzcd} 0 \arrow[r] & \underline{\mathrm{Hom}}(\underline{T}^r,\underline{L}^r) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\underline{T}^r,\underline{Q}^r) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\underline{T}^r,\underline{T}^r) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{\mathrm{Hom}}(\underline{T},\underline{L}) \arrow[r] & \underline{\mathrm{Hom}}(\underline{T},\underline{Q}) \arrow[r] & \underline{\mathrm{Hom}}(\underline{T},\underline{T}) \arrow[r] & 0 \end{tikzcd} \tag{8}$$

and therefore the following diagram of long exact sequences is commutative:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\Gamma^r}(T^r,L^r) \arrow[r] \arrow[d] & \mathrm{Hom}_{\Gamma^r}(T^r,Q^r) \arrow[r] \arrow[d] & \mathrm{Hom}_{\Gamma^r}(T^r,T^r) \arrow[r, "\delta"] \arrow[d] & H^1(B\Gamma^r,\underline{\mathrm{Hom}}(\underline{T}^r,\underline{L}^r)) \arrow[d] \\ 0 \arrow[r] & \mathrm{Hom}_{\Gamma}(T,L) \arrow[r] & \mathrm{Hom}_{\Gamma}(T,Q) \arrow[r] & \mathrm{Hom}_{\Gamma}(T,T) \arrow[r, "\delta"] & H^1(B\Gamma,\underline{\mathrm{Hom}}(\underline{T},\underline{L})). \end{tikzcd} \tag{9}$$

Let us take as the sheaf $\underline{F}_1$ the sheaf $\underline{\mathrm{Hom}}(\underline{T}^r,\underline{L}^r)$ and as $\underline{F}_2$ the sheaf $\underline{\mathrm{Hom}}(\underline{T},\underline{L})$. As the (Γ,Γ^r)-cohomomorphism F over i we take the corresponding vertical arrow. Then from diagrams (9) and (5) we get the following commutative diagram, taking into account that the sheaf $\mathcal{F}_1^*\underline{\mathrm{Hom}}(\underline{T}^r,\underline{L}^r)$ is equal to the sheaf $\underline{\mathrm{Hom}}(\underline{N(\mathcal{F}^r)},\underline{P(g^r)})$ and $\mathcal{F}_2^*\underline{\mathrm{Hom}}(\underline{T},\underline{L})$ to the sheaf $\underline{\mathrm{Hom}}(\underline{N(\mathcal{F})},\underline{P(g)})$, where $P(g^r)$ and $P(g)$ are the associated fibre bundles with the standard fibre g^r and g, respectively:

$$\begin{tikzcd} \mathrm{Hom}_{\Gamma^r}(T^r,T^r) \arrow[r] \arrow[d, "a"] & H^1(B\Gamma^r,\underline{\mathrm{Hom}}(\underline{T}^r,\underline{L}^r)) \arrow[r, "\hat{\mathcal{F}}_1^*"] \arrow[d] & H^1(N^r(M,\mathcal{F}),\underline{\mathrm{Hom}}(\underline{N(\mathcal{F}^r)},\underline{P(g^r)})) \arrow[d, "b"] \\ \mathrm{Hom}_{\Gamma}(T,T) \arrow[r] & H^1(B\Gamma,\underline{\mathrm{Hom}}(\underline{T},\underline{L})) \arrow[r, "\hat{\mathcal{F}}_2^*"] & H^1(M,\underline{\mathrm{Hom}}(\underline{N(\mathcal{F})},P(g))). \end{tikzcd} \tag{10}$$

According to [3], the Atiyah-Molino class $AM[\mathcal{F}^r]$ of the lifted foliation $\mathcal{F}^r$ is equal to

$$AM[\mathcal{F}^r] = \hat{\mathcal{F}}_1^* \delta(Id_T r).$$

Then

$$b(AM[\mathcal{F}^r]) = b(\hat{\mathcal{F}}_1^*\delta(Id_T r)) = \hat{\mathcal{F}}_2^*\delta a(Id_T r).$$

One can easily check, directly from the definition, that $a(Id_T r) = id_T$. Thus $b(AM[\mathcal{F}^r]) = \hat{\mathcal{F}}_2^*\delta(Id_T) = AM(\mathcal{F})$.

Up till now we have considered the foliation $\mathcal{F}^r$ as a Γ^r-foliation. Let Γ_r be the groupoid of germs of automorphisms of the r-prolongation $P^r(N,G)$ of the G-structure $P(N,G)$. This groupoid contains the groupoid Γ^r as an open subgroupoid. We have also to consider the foliation $\mathcal{F}^r$ as a Γ_r-foliation and look at the relations between the Atiyah-Molino classes. It is not difficult to check that they are equal. This last remark effectively ends the sketch of the proof of Theorem B.

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Fibrilations and group actions

1. FIBRED MANIFOLDS

The aim of this paper is to present some results on fibred structures which may be viewed as generalisations of fibre bundles, and to report some joint work with D. Canarutto concerning the stability of frame bundle incompleteness which has significance for quantization in general relativity.

The context is that of fibred manifolds (or surmersions) the geometry of which, following Ehresmann, has been studied in particular by Libermann, Kolař, Mangiarotti and Modugno, Ferraris and Francaviglia. The geometry is quite rich because a fibred manifold may be viewed as the least structure needed to supp[illegible] connection.

A *fibred man*[illegible] smooth surjective submersion $E \xrightarrow{p} B$. Then p has maximum rank [illegible] F has an open neighbourhood V and a manifold F_V with a diffeomorphism $\Phi_V : V \cong pV \times F_V$ over p. We shall call F_V a *fibril*. A *morphism* [illegible] fibred manifolds is a commuting diagram of smooth fibre-preserving [illegible]:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "\psi"] & B' \end{tikzcd}$$

There is a natural composition of such diagrams, so yielding a category FM. One reason for studying this category is that it admits pullbacks; in fact, that is a consequence of the following result, which says that every finite diagram in FM has a left limit.

Theorem The category FM is finitely left complete.

Proof It is sufficient to show that FM admits finite products and equalizers. The former is clear enough. Consider the equalizer diagram of commuting squares:

$$\begin{tikzcd} E_2 \arrow[d, "p_2"'] & E_1 \arrow[l, "f_2"'] \arrow[r, "f_1"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ B_2 & B_1 \arrow[l, "\check{f}_2"'] \arrow[r, "\check{f}_1"] & B_2 \end{tikzcd} \quad \equiv \quad F_1 \overset{f_1}{\underset{f_2}{\rightrightarrows}} F_2 .$$

Our candidate left limit object is $E \overset{p}{\twoheadrightarrow} B$ ($\equiv F$, say) with

$$\left.\begin{aligned} E &= \{y \in E_1 \mid f_1(y) = f_2(y)\} \\ &= \{x \in p_1E \mid \check{f}_1(x) = \check{f}_2(x)\} \end{aligned}\right\} \qquad p = p_{1|E} .$$

Now, $p_1E = B$, E is a smooth submanifold of E_1 and p is evidently a smooth surjection.

Take any $y \in E$, then by commutativity in the tangent diagram,

$$Tp(T_yE) = Tp_1(T_yE) = T_{p(y)}B$$

so p inherits the submersion property from p_1.

The morphism required for the universal property can be obtained from inclusions and compositions.

2. SHEAF STRUCTURES

A smooth map $p:E \to B$ is a *sheaf manifold over* B if p is an open, local diffeomorphism.

The categories of sheaves on smooth manifolds and sheaf manifolds are related via a functor which carries sheaf manifolds to sheaves of their smooth local sections.

Proposition A fibred manifold $E \overset{p}{\twoheadrightarrow} B$ is a sheaf manifold over B if p is open and its fibres are discrete spaces.

In the topological case, sheaves and sheaf spaces are actually equivalent as categories over a given base space, for there the functor, carrying sheaf spaces to sheaves of their continuous local sections has an inverse which carries sheaves to sheaf spaces of germs of local sections. However, this inverse is not available to us even for fibred topological spaces, because the latter need not have discrete fibres. Of course, the fibrils of a fibred manifold constitute a sheaf and this may allow any common algebraic structure to be exploited.

<u>Proposition</u> Every fibred manifold $E \overset{p}{\twoheadrightarrow} B$ determines a sheaf SE of smooth local sections of p.

<u>Proof</u> First we obtain a presheaf cofunctor

$$
S : T(B) \to \mathrm{Set} : \begin{array}{c} U \\ i \big\downarrow \\ V \end{array} \longrightarrow \begin{array}{c} S(U,E) = \{\sigma \in E^V \mid p^\sigma = 1_V\} \\ \big\uparrow \rho_V^U \\ S(V,E) \end{array} : \sigma \to \sigma|_U = \sigma i
$$

Since $E \overset{p}{\twoheadrightarrow} B$ is a fibred manifold, every point of E has a sectionable neighbourhood over a sufficiently small base set; so SE is not empty.

Now take any open cover $\{U_\alpha \mid \alpha \in A\}$ of any $U \in T(B)$. Suppose that we have a collection $\{\sigma_\alpha \in S(U_\alpha,E) \mid \alpha \in A\}$ with

$$
(\forall \alpha,\beta \in A) \quad \rho_{U_\alpha}^{U_\alpha \cap U_\beta} \sigma_\alpha = \rho_{U_\beta}^{U_\alpha \cap U_\beta} \sigma_\beta .
$$

Hence $\sigma_\alpha i_\alpha = \sigma_\beta i_\beta$, $j_\alpha i_\alpha = j_\beta i_\beta$. Then by functoriality of S

$$
\rho_U^{U_\alpha \cap U_\beta} = \rho_{U_\alpha}^{U_\alpha \cap U_\beta} \rho_U^{U_\alpha} = \rho_{U_\beta}^{U_\alpha \cap U_\beta} \rho_U^{U_\beta} .
$$

$$
\begin{tikzcd}
& U_\alpha \cap U_\beta \arrow[dl, hook, "i_\alpha"'] \arrow[dr, hook, "i_\beta"] & \\
U_\alpha \arrow[r, "\sigma_\alpha"] \arrow[dr, hook, "j_\alpha"'] & E & U_\beta \arrow[l, "\sigma_\beta"'] \arrow[dl, hook, "j_\beta"] \\
& U \arrow[u, "\sigma"] &
\end{tikzcd}
$$

Define $\sigma : U \to E : x \to \sigma_\alpha(x)$ if $x \in U_\alpha$.
It follows that $\sigma \in S(U,E)$ and, as required, it satisfies

$$\rho_U^{U_\alpha}\sigma = \sigma j_\alpha = \sigma_\alpha .$$

<u>Corollary</u> S determines a functor from the category of fibred manifolds over B to the category of sheaves over B.

3. <u>G-FIBRILATIONS</u>

A *group-fibrilation* is a fibred manifold $E \overset{p}{\twoheadrightarrow} B$ with a left action α over B by a Lie group G acting as a group of fibred manifold automorphisms of E. We shall denote this by $G \times E \xrightarrow{\alpha} E \overset{p}{\twoheadrightarrow} B$ and call it a *G-fibrilation*. Logically we might call a fibred manifold a *fibrilation*.

A *morphism* of group-fibrilations is a commuting diagram of smooth maps with $\dot{f}$ a Lie group homomorphism:

$$\begin{tikzcd}
G \times E \arrow[r, "\alpha"] \arrow[d, "(\dot{f}, f)"'] & E \arrow[r, two heads, "p"] \arrow[d, "f"'] & B \arrow[d, "\check{f}"] \\
G' \times E' \arrow[r, "\alpha'"] & E' \arrow[r, two heads, "p'"] & B'
\end{tikzcd}$$

Again we have a category, $GF\ell$, under diagram composition. It is closed under finite products and we can use the FM equalizer but in general we cannot obtain a suitable G-fibrilation equalizer except by taking it with trivial G, and likewise for pullbacks.

<u>Proposition</u> Let $G \times E \xrightarrow{\alpha} E \overset{p}{\twoheadrightarrow} B$ be a G-fibrilation.

Given a smooth curve $c : (-\varepsilon, \varepsilon) \to B$ and any $y_0 \in p^{\leftarrow}(c(0))$, there is some positive $\delta < \varepsilon$ and a smooth curve $\bar{c} : [-\delta, \delta] \to E$ with $p\bar{c} = c_{|[-\delta,\delta]}$. Moreover $\alpha_\gamma \bar{c}$ is another such curve for all smooth curves $\gamma : [-\delta, \delta] \to G$ with $\gamma(0) = 1$.

<u>Proof</u> Since $E \overset{p}{\twoheadrightarrow} B$ is a fibred manifold, we can find some open neighbourhood V of y_0, a fibril F_V and diffeomorphism

$$\phi : V \cong pV \times F_V .$$

Take $\delta > 0$ with $c[-\delta, \delta] \subset pV$, possible since E is a manifold. Define : $p_2 : pV \times F_V \to F_V \quad : (x, v) \to v$. Then,

$$\bar{c} : [-\delta, \delta] \to E : t \to \phi^{-1}(c(t), p_2\phi(y_0))$$

is smooth and projects onto the restriction of c to $[-\delta,\delta]$. Also, if $\gamma(t) \in G$ then the action automorphism $\alpha_{\gamma(t)}$ acts vertically so $\alpha_\gamma \bar{c}$ is well-defined for the same δ, though it may leave V.

The usefulness of this lifting is essentially measured by δ, on a scale from zero to ε; the more of the curve that can be lifted the better. It may be of value to take the supremum of δ over all fibril neighbourhoods of y. Each lifted curve $\bar{c}$ determines a transport process among fibrils over the base curve and this extends, via the action of G, also to their vertical translation.

Next we define suitable connections on G-fibrilations, essentially in the same way as Libermann [8].

A *connection* on a G-fibrilation $G \times E \xrightarrow{\alpha} E \twoheadrightarrow B$ is a smooth (dimB)-dimensional distribution on E

$$\Gamma : y \mapsto H_y \subseteq T_y E \text{ with } Tp_{|H_y} : H_y \cong T_{p(y)} B$$

that is *invariant* under α, namely:

$$H_{\alpha_g(y)} = T\alpha_g\, H_y \qquad (\forall y \in E,\ \forall g \in G).$$

We see that a G-fibrilation generalises the notion of a G-bundle. It has a local product structure that is not locally trivial nor even a fibration, since the fibres need not be homotopic. Moreover, the action of G is not necessarily transitive nor free. However, locally a G-fibrilation has a sufficiently simple structure to make differential analysis easy through adapted charts. Moreover, it can support the useful geometric notion of an invariant connection. The study of these was begun in [3] where some principal bundle theory was adapted to obtain induced and coinduced connections from group-fibrilation morphisms. Jet calculus and connection geometry on fibred manifolds (cf. [8], [9], [4], [7] for example, and references therein) is transferable to G-fibrilations which may prove a useful setting for variational problems with group symmetries.

We turn now to a result concerning a very particular type of G-fibrilation; namely, a principal G-bundle.

4. CONNECTION-STABILITY OF BASE SINGULARITIES

It is well known that the notion of geodesic completeness is inadequate for pseudo-Riemannian manifolds, such as spacetimes, where it has become the practice to lift the problem to a convenient bundle by a method due to Schmidt [10] (cf. also [2] for a detailed account and survey). General theorems suggest that any theory of gravity is likely to predict physical singularities in the classical geometry. Recently, we have proved the following result for manifolds with linear connections.

Theorem Bundle-incompleteness is stable under perturbations of the connection.

The geometric details of the proof are given in [1] and they depend on Modugno's structure of connections [9]. This is a fibred manifold $JP/G \twoheadrightarrow M$, where in our case $P \equiv P(G,M)$ is the frame bundle, sections of which are connections $\Gamma: P \to JP$ invariant under G (cf. Libermann [8]). Our trick is to use a canonical connection (cf. García [5]) on the fibred manifold $JP/G \times P \twoheadrightarrow JP/G$ to obtain a bilinear form on its total space. Now, this restricts to become a Riemannian metric on certain submanifolds which have diffeomorphisms to the frame bundle and these become isometries for each choice of connection. Then, if M is bundle-incomplete with respect to one connection, it is also bundle-incomplete with respect to a nearby connection.

This theorem has physical significance in that it lends weight to the belief that general relativistic singularities cannot be quantized away. It was already known from the work of Gotay and Isenberg [6] that geometric quantization of a massless Klein-Gordon scalar field on a positively curved spacetime could not prevent the collapse of the state vector. Our result is more general and not tied to any particular method of quantization. It may also be useful to extend it to the case of nonlinear connections when they can be made to induce suitable metric structures on some convenient fibred manifold total space.

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The theory of formal connections and fibred connections in fibred manifolds

1. INTRODUCTION

In the framework of higher order calculus of variations in a fibred manifold $\underline{Y} = (Y,X,\eta)$ one often encounters fields of objects which may be naturally identified with sections of vector bundles of the kind

$$V^p_q(Y) \otimes_Y T^r_s(X)$$

where V and T are standard functors and (p,q,r,s) are non-negative integers. Objects of this type are called in short "(fields of) fibred tensors", because of their transformation properties under changes of fibred coordinates in $\underline{Y}$. As an example, we can mention Lagrangians, their vertical differentials, momenta, etc.

The local structure of higher order calculus of variations is fairly well understood, both at the Lagrangian and at the Hamiltonian level. However, in many physically interesting situations one needs to deal also with global problems, which only recently have received serious consideration and have been given a reasonably satisfactory formulation. Among the global problems that have stimulated a lot of interest and a number of different interpretations we recall the problem of a correct global definition of the so-called "Poincaré-Cartan form" (which has long been known to exist uniquely for standard mechanical systems, later proved to exist uniquely also for higher order mechanics and first order field theory, but recently shown to be highly non-unique in the most general situation; [2], [8], [9], [10], [11], [12], [13], [17], [19]).

There are of course several techniques to handle global problems (direct methods of globalization from local results, or intrinsic methods based on sophisticated tools such as sheaf theory, cohomology, etc.). In the direct approach, one of the standard procedures consists in trying to patch together local expressions by showing that their transformation laws may be interpreted as transition functions of some suitable bundle. A major difficulty which arises in applications to higher order calculus of variations is hidden in the wide use of the so-called "formal derivative operator", which unfortunately does not transform fibred tensors into fibred tensors. More precisely, if

$$t^{i_1,i_2,\dots,i_p,\alpha_1,\alpha_2,\dots,\alpha_r}_{j_1,j_2,\dots,j_q,\beta_1,\beta_2,\dots,\beta_s}$$

are the local components of a fibred tensor t, the formal partial dervatives

$$d_\mu\, t^{i_1,i_2,\dots,i_p,\alpha_1,\alpha_2,\dots,\alpha_r}_{j_1,j_2,\dots,j_q,\beta_1,\beta_2,\dots,\beta_s}$$

are no longer components of a fibred tensor. Accordingly, it is convenient to replace higher-order (formal) derivatives of fibred tensors with suitably defined "formal covariant derivatives", constructed in such a way that they transform again as fibred tensors. For this purpose, one needs first to introduce suitable global objects which are called "formal connections" and then use a formal connection to define a "fibred connection" which allows calculation of formal covariant derivatives of any fibred tensor.

A preliminary short discussion of formal connections and fibred connections in fibred manifolds has already been given in [2] and [5] and the purpose of this paper is to provide a more detailed exposition of this subject. Applications to higher order calculus of variations have already been discussed in [2], [3], [5], where the existence was shown, by an explicit construction, of an infinite family of Poincaré-Cartan forms parametrized by a family of "fibred connections".

In this paper we shall first define the relevant notions in the classical coordinate formalism and then we shall turn to more intrinsic definitions in terms of principal fibrations and exact sequences of vector bundles. Section 2 will be devoted to a short discussion of preliminaries and notation; in Section 3 we shall develop the theory of formal connections and formal (first-order) covariant derivatives; Sections 4 and 5 will contain the intrinsic description of these notions.

2. PRELIMINARIES AND NOTATION

We shall here recall some standard definitions and set the notation which will be used throughout this paper. We assume that the reader is already familiar with differential geometry in fibred manifolds and with the theory of jet-prolongations (details and references may be found in [13] and [18]). All manifold and fibred manifold structures considered here are assumed to be smooth in the category of (paracompact) topological manifolds over the reals.

Let X be a manifold and let $\underline{Y} = (Y,X,\eta)$ be a fibred manifold over the manifold X. The *vertical bundle* of $\underline{Y}$ is $\underline{V}[\eta] = (V_X(Y),X,\eta\circ\nu_Y)$, where $V_X(Y) = \mathrm{Ker}(T\eta) \subseteq T(Y)$ and ν_Y is the restriction to $V_X(Y)$ of the canonical projection $\tau_Y:T(Y) \to Y$. If $\underline{U} = (U,Y,\nu)$ is a fibred manifold having for basis the total space Y of $\underline{Y}$ (namely, we have a *double fibration* over X), then the composition $\eta\circ\nu$ defines a fibred manifold $(U,X,\eta\circ\nu)$. We recall that in this case $V_X(U)$ stands for $\mathrm{Ker}(T(\eta\circ\nu))$, while $V_Y(U)$ stands for $\mathrm{Ker}(T\nu)$. Whenever there is no need to specify the basis manifold of the fibration we shall omit the basis from the notation (writing, for example, $V(Y)$ instead of $V_X(Y)$).

For any quadruple of non-negative integers (p,q,r,s) we shall also set $V^p_q(Y) = [V(Y)]^{\otimes p} \otimes_Y[V^*(Y))]^{\otimes q}$ and we define the following family of vector bundles

$$FT^{(p,r)}_{(q,s)}(Y) = V^p_q(Y) \otimes_Y T^r_s(X)$$

where $T^r_s(X)$ denotes the standard tensor power of $T(X)$. The sections of $FT^{(p,r)}_{(q,s)}(Y)$ over Y will be called (*fields of*) *fibred tensors* over $\underline{Y}$.

For any point $y \in Y$ we consider the space $VF_y(Y)$ consisting of all bases of the vector space $V_y(Y) = (\nu_Y)^{-1}(y)$ and we form the union

$$VF(Y) = \bigcup_{y\in Y} VF_y(Y).$$

This space is endowed with a natural manifold structure and it is fibred over Y by the canonical projection $\phi_Y:VF(Y) \to Y$. Moreover, there is a canonical action of the linear group $GL(n;\mathbb{R})$ ($n = \dim(Y) - \dim(X)$) onto the fibres $VF_y(Y)$, which induces on $VF(Y)$ a natural structure of principal $GL(n;\mathbb{R})$-bundle over Y. The bundle $(VF(Y),Y,\phi_Y;GL(n;\mathbb{R}))$ is shortly denoted by $\underline{VF}[\eta]$ and it is called the *bundle of vertical frames* of $\underline{Y}$.

The *k-th order jet-prolongation* of $\underline{Y}$ (where k is any non-negative integer) is denoted by $\underline{J}^k[\eta] = (J^k_X(Y),X,\eta^k)$. Also in this case, whenever there is no danger of confusion we shall omit the indication of the basis manifold X. For any pair (r,s) of integers there is a canonical embedding $i^{r,s}:J^{r+s}(Y) \to J^r(J^s(Y))$. For any local section $\sigma:X \to Y$ we denote by

$j^k\sigma : X \to J^k(Y)$ the k-th order jet-prolongation of σ.

If $\underline{Z} = (Z, X, \zeta)$ is a further fibred manifold over X, a *fibred morphism* from the fibred manifold $\underline{Y}$ into the fibred manifold $\underline{Z}$ is a map $F: Y \to Z$ such that $\zeta \circ F = \eta$. For any integer $r \geq 1$ and any fibred morphism $F: J^k(Y) \to Z$, with $k > 0$, we define the *r-th order (holonomic) prolongation* $\rho^r(F)$ of the morphism F, by setting

$$\rho^r(F) = J^r(F) \circ i^{r,k} : J^{r+k}(Y) \to J^r(Z) \tag{2.1}$$

where $J^r(F) : J^r(J^k(Y)) \to J^r(Z)$ denotes the standard r-th order jet-prolongation of the fibred mornhism F.

Let $f: J^k(Y) \to \mathbb{R}$ be a smooth map. There exists a unique 1-form Df over the manifold $J^{k+1}(Y)$ such that the following holds:

$$(j^{k+1}\sigma)^*(Df) = d((j^k\sigma)^*(f)) \text{ and } i(\xi)(Df) = 0$$

for any (local) section $\sigma: X \to Y$ and any field of vertical vectors $\xi: J^{k+1}(Y) \to V(J^{k+1}(Y))$; here $i(\cdot)$ denotes the interior product between vectors and forms. The unique 1-form Df which satisfies the properties above is called the *formal differential* of the map f.

For our later purposes we now turn to list some coordinate notations which will be used throughout the paper. Consider a fibred manifold $\underline{Y}$, with $m = \dim(X)$ and $n = \dim(Y) - \dim(X)$. If $\bar{\psi} = (U; x^\lambda)$, with $1 \leq \lambda \leq m$, is a local chart of the manifold X, its domain is denoted by $\mathrm{Dom}(\bar{\psi})$; a *fibred chart* of $\underline{Y}$ over $\hat{\psi}$ is denoted by $\psi = (W; x^\lambda, y^i)$ (with $\lambda = 1, \dots, m$ and $i = 1, \dots, n$); the local coordinates associated to a fibred chart ψ will be called *fibred coordinates*. If $\psi' = (W'; x^{\lambda'}, y^{i'})$ is a further fibred chart whose domain has a non-empty intersection with $\mathrm{Dom}(\psi)$ and

$$x^{\lambda'} = \phi^{\lambda'}(x^\lambda), \quad y^{i'} = \phi^{i'}(x^\lambda, y^i)$$

are the corresponding transition functions, the following notation will be used to indicate their partial derivatives:

$$X^{\lambda'}_\lambda = X^{\lambda'}_\lambda(x^\mu) = \partial\phi^{\lambda'}/\partial x^\lambda, \quad Y^{i'}_i = Y^{i'}_i(x^\lambda, y^k) = \partial\phi^{i'}/\partial y^i$$

$$Y^{i'}_{jk} = Y^{i'}_{jk}(x^\lambda, y^m) = \partial^2\phi^{i'}/\partial y^j \partial y^k, \; Y^{i'}_{j\mu} = Y^{i'}_{j\mu}(x^\lambda, y^m) = \partial^2\phi^{i'}/\partial y^j \partial x^\mu$$

and so on.

A fibred chart ψ of $\underline{Y}$ induces canonically in $\underline{V}[\eta]$ a fibred chart ψ_V with coordinates (x^λ, y^i, v^i) and a fibred chart ψ_k in $\underline{J}^k[\eta]$, with coordinates $(x^\lambda, y^i_{\underline{\nu}})$ (here $\underline{\nu} = (\nu_1, \ldots, \nu_m) \in N^m$ is a multi-index with $0 \le |\underline{\nu}| \le k$). All charts induced canonically by fibred charts of $\underline{Y}$ will be called *natural fibred charts*.

For any fibred chart ψ of Y, the formal differential Df is defined over the domain of ψ_{k+1} and its local representation is $Df = (d_\lambda f) dx^\lambda$, where the coefficients $d_\lambda f$ are given by

$$d_\lambda f = \partial f/\partial x^\lambda + \sum_{|\underline{\nu}|=0}^{k} y^i_{\underline{\nu}+\underline{1}_\lambda} (\partial f/\partial y^i_{\underline{\nu}}). \tag{2.2}$$

Here standard multi-index notation has been used: $\underline{1}_\lambda$ denotes the multi-index $(0,\ldots,0,1,0,\ldots,0)$ (with 1 in the λ-th position) and summation of multi-indices is defined componentwise. The meaning of Df is clear from the local expressions above. We remark that the partial differential operator d_λ is often called the *formal partial derivative* with respect to the coordinate x^λ.

Let us now consider any field t of fibred tensors in $FT^{(p,r)}_{(q,s)}(Y)$, having local components

$$t^{i_1, i_2, \ldots, i_p, \alpha_1, \alpha_2, \ldots, \alpha_r}_{j_1, j_2, \ldots, j_q, \beta_1, \beta_2, \ldots, \beta_s}$$

in any natural fibred chart. Easy but tedious calculations show that the local functions defined by

$$d_\mu \, t^{i_1, i_2, \ldots, i_p, \alpha_1, \alpha_2, \ldots, \alpha_r}_{j_1, j_2, \ldots, j_q, \beta_1, \beta_2, \ldots, \beta_s}$$

do not transform as the local components of a field of fibred tensors in $FT^{(p,r)}_{(q,s+1)}(Y)$. For example, we have the following transformation law for the formal partial derivatives of the components v^i of a vertical vector field:

$$d_{\mu'} v^{i'} = Y^{i'}_i [d_\mu v^i + (Y^i_{jk} y^k_\mu + Y^i_{j\mu}) v^j] X^\mu_{\mu'} \; . \tag{2.3}$$

This local formula is our starting point toward the definition of formal connections, which will be discussed in the next section.

We finally recall the following well known alternative intrinsic definitions of connections in a principal G-bundle $\underline{P} = (P, M, \pi; G)$, over any manifold

M with structure group any Lie group G

(i) There exists a natural action $J^1(P) \times G \to J^1(P)$, which is induced by the first prolongation of the canonical (right) action of G onto P

This prolonged action is free and differentiable and it admits a quotient manifold $K_M(P) = J^1(P)/G$ Moreover, it can be shown that the canonical projection k_π $K_M(P) \to M$ is a (surjective) submersion. The fibred manifold $\underline{K}[\pi] = (K_M(P), M, k_\pi)$ is an affine bundle, because the affine structure of $J^1(P)$ over P can be shown to pass to the quotient Finally, there exists a canonical one-to-one correspondence between the space of global sections of the bundle $\underline{K}[\pi]$ and the space of all connections of the principal bundle $\underline{P}$ (see [7])

(ii) Given any G-bundle $\underline{P}$, there exists a short exact sequence of vector bundles and vector bundle morphisms over M,

$$0 \to V(P)/G \to T(P)/G \to T(M) \to 0,$$

and any splitting ω $T(M) \to T(P)/G$ of this sequence defines a connection of $\underline{P}$ (and vice versa).

3 FORMAL CONNECTIONS AND FIBRED CONNECTIONS

As we already announced in the Introduction, the purpose of this paper is to define a family of objects which allow replacement of the partial formal derivatives of the components of any fibred tensor by local functions which still have a "fibred-tensorial behaviour Objects of this kind will be called "fibred connections' and it turns out that their major ingredient is a section of a suitably defined affine bundle over $J^1(Y)$, which will be called a "formal connection" over Y. In order to define fibred connections and formal covariant derivatives of fibred tensors, it will be convenient to define first the notion of formal covariant derivative of a fibred morphism $F: J^k(Y) \to V(Y)$, with k any non-negative integer Derivation of fibred tensors will then be defined by standard tensorization procedures.

Let us consider a fibred manifold $\underline{Y} = (Y, X, \eta)$ together with a fibred morphism $F. J^k(Y) \to V(Y)$. For any fibred chart ψ of $\underline{Y}$, we introduce a set of local smooth functions $\hat{\Gamma}^i_{j\mu}$ $\mathrm{Dom}(\psi_1) \to \mathbb{R}$ and we set by definitions

$$\nabla_\mu \xi^i(F) = d_\mu \xi^i(F) + \hat{\Gamma}^i_{j\mu} \xi^j(F) \qquad (3\ 1)$$

where $\xi^i(F) = v^i \circ F$ are the components of the morphism F with respect to the given fibred chart ψ. Let us then require that the local expressions (3.1) above define the components of a fibred tensor in $FT^{(1,0)}_{(0,1)}(Y)$. Easy calculations based on relation (2.3) tell us that the local functions $\hat{\Gamma}^i_{j\mu}$ should obey the following transformation laws:

$$\hat{\Gamma}^{i'}_{j'\mu'} = (Y^{i'}_i \ \Gamma^i_{j\mu} - Y^{i'}_{jk} \ y^k_\mu - Y^{i'}_{j\mu})Y^j_{j'} \ X^\mu_{\mu'} \tag{3.2}$$

for any pair (ψ,ψ') of fibred charts of $\underline{Y}$ such that $\mathrm{Dom}(\psi) \cap \mathrm{Dom}(\psi') \neq \emptyset$.

It is easy to show that the relations (3.2) are invertible any satisfy the composition property of a cocycle over the manifold $J^1(Y)$ with values in the Lie group $GL(n^2.m;\mathbb{R})$. Accordingly, they define the transition functions of a bundle $\underline{\hat{C}}[\eta^1] = (\hat{C}(J^1(Y)),J^1(Y),\hat{c})$ over $J^1(Y)$ itself, which is unique up to isomorphisms. It is easily seen from (3.2) that this bundle can be given a canonical structure of affine bundle over $J^1(Y)$; moreover, whenever (Y,X,η) is affine, $\underline{\hat{C}}[\eta^1]$ is the pull-back over $J^1(Y)$ of an affine bundle over the basis X. The bundle $\underline{\hat{C}}[\eta^1]$ will be called the *bundle of formal connections* over the fibred manifold $\underline{Y}$; being an affine bundle, it admits global sections $\hat{\Gamma}:J^1(Y) \to \hat{C}(J^1(Y))$, which we shall call *formal connections* over $\underline{Y}$.

Turning to local coordinate expressions, let us first remark that any fibred chart ψ of $\underline{Y}$ induces in a canonical way a natural fibred chart of the affine bundle $\underline{\hat{C}}[\eta^1]$, with fibred coordinates $(x^\lambda,y^i,y^i_\mu,\hat{\Gamma}^i_{j\mu})$. In such a natural chart, the local representation of a formal connection $\hat{\Gamma}$ over $\underline{Y}$ has then the following expression:

$$\hat{\Gamma}:(x^\lambda,y^i,y^i_\sigma) \longrightarrow (x^\lambda,y^i,y^i_\sigma,\hat{\Gamma}^i_{j\mu}(x^\lambda,y^k,y^k_\sigma)) \tag{3.3}$$

where the functions $\hat{\Gamma}^i_{j\mu}(x^\lambda,y^k,y^k_\sigma)$ are defined in the domain $\mathrm{Dom}(\psi_1)$ of the given chart and transform according to (3.2).

We turn now to define the *formal covariant derivative* of a fibred morphism. We consider then any formal connection $\hat{\Gamma}$ over $\underline{Y}$ and we set, in any fibred chart of Y,

$$\nabla_\mu \ v^a = v^a_\mu + \hat{\Gamma}^a_{h\mu}(x^\lambda,y^i,y^i_\sigma)v^h \tag{3.4}$$

where $(x^\lambda,y^i,v^a,y^i_\sigma,v^a_\sigma)$ are the natural fibred coordinates in $J^1(V(Y))$ induced by the given chart of Y. It is easily checked that the relations (3.3) define in fact a global vector bundle morphism over η^1_0

$$\nabla : J^1(V(Y)) \longrightarrow V(Y) \otimes_Y T^*(X)$$

which will be called a *formal covariant derivation* (associated to the formal connection $\hat{\Gamma}$).

Let now $F : J^k(Y) \to V(Y)$ be any (global) fibred morphism over Y. We consider the holonomic prologation $\rho^1(F) = J^1(F) \circ i^{1,k}$ and we define fibred morphisms $\tilde{\nabla}(F)$ and $\nabla(F)$ by setting

$$\tilde{\nabla}(F) = \nabla \circ J^1(F) : J^1(J^k(Y)) \longrightarrow V(Y) \otimes_Y T^*(X) \tag{3.5}$$

and

$$\nabla(F) = \nabla \circ \rho^1(F) : J^{k+1}(Y) \longrightarrow V(Y) \otimes_Y T^*(X). \tag{3.6}$$

These fibred morphisms are respectively called the *anholonomic* and the *holonomic formal covariant derivative* of F with respect to $\hat{\Gamma}$; they are the unique (global) morphisms which fit into the commutative diagram, Figure 1.

$$\begin{tikzcd}
J^1(V(Y)) \arrow[rr, "\nabla"] \arrow[ddd] & & V(Y) \otimes_Y T^*(X) \\
& J^1(J^k(Y)) \arrow[ul, "J^1(F)"'] \arrow[ur, "\tilde{\nabla}(F)"] & \\
& J^{k+1}(Y) \arrow[u] \arrow[uul, "\rho^1(F)"] \arrow[uur, "\nabla(F)"'] \arrow[d] & \\
V(Y) & J^k(Y) \arrow[l, "F"'] \arrow[d] & \\
& Y &
\end{tikzcd}$$

Figure 1

Consider now a local section $v : X \to V(Y)$ and define its *formal covariant derivative* (with respect to $\hat{\Gamma}$), as the section of $V(Y) \otimes_Y T^*(X)$ obtained by setting

$$\nabla(v) = \nabla \circ j^1(v) : X \longrightarrow V(Y) \otimes_Y T^*(X). \tag{3.7}$$

For any local section $\sigma^k : X \to J^k(Y)$ the composition $F \circ \sigma^k$ is a local section of $V(Y)$ over X. Therefore we can calculate its formal covariant derivative, which is easily shown to satisfy the following property:

$$\nabla(F \circ \sigma^k) = \tilde{\nabla}(F) \circ j^1(\sigma^k). \tag{3.8}$$

In particular, if σ^k is the jet-prolongation $j^k(\sigma)$ of a local section $\sigma : X \to Y$, relation (3.8) and Figure 1 imply the following:

$$\nabla(F \circ j^k(\sigma)) = \nabla(F) \circ j^{k+1}(\sigma). \tag{3.9}$$

We are now in a position to define formal covariant derivatives of any field of fibred tensors over Y. In fact, let us first remark that standard tensorization procedures allow us to extend the notion of formal covariant derivative to morphisms from $J^k(Y)$ into any bundle $V^p_q(Y)$, for any pair (p,q). On the other hand, whenever a linear connection γ is given on X, one may calculate covariant derivatives of any tensor field over X. Accordingly, any pair $\Gamma = (\hat{\Gamma},\gamma)$, consisting of a formal connection $\hat{\Gamma}$ over $\underline{Y}$ and of a linear connection γ over the basis X, will naturally allow us to define formal covariant derivatives of morphisms from $J^k(Y)$ into any bundle of the form $V^p_q(Y) \otimes_Y T^r_s(X) = FT^{(p,r)}_{(q,s)}(Y)$. Any such pair Γ will be thence called a *fibred connection* over $\underline{Y}$.

A standard construction then provides uniquely, for any fibered connection Γ and any quadruple (p,q,r,s) of non-negative integers, a global vector bundle morphism

$$\nabla^{(p,r)}_{(q,s)} : J^1(FT^{(p,r)}_{(q,s)}(Y)) \longrightarrow FT^{(p,r)}_{(q,s+1)}(Y)$$

over Y, which will be called the *formal covariant derivation* (of fibred tensors of type (p,q,r,s)) associated to the fibred connection Γ. (In the sequel, whenever there is no danger of confusion, the indication of the type (p,q,r,s) will be omitted and we shall more simply write ∇). The local coordinate expressions for ∇ may be easily calculated from the definitions given above. As an example, the expression of the relevant part of ∇ in the case $p = q = r = s = 1$ is given by:

$$F^{a\alpha}_{b\beta\mu} \circ \nabla = \nabla_\mu f^{a\alpha}_{b\beta} = f^{a\alpha}_{b\beta\mu} + \hat{\Gamma}^a_{h\mu}(x^\lambda, y^i, y^i_\sigma) f^{h\alpha}_{b\beta} + \gamma^{\alpha}_{\rho\mu}(x^\lambda) f^{a\rho}_{b\beta}$$
$$- f^{a\alpha}_{k\beta} \; \hat{\Gamma}^k_{b\mu}(x^\lambda, y^i, y^i_\sigma) - f^{a\alpha}_{b\sigma} \; \gamma^{\sigma}_{\beta\mu}(x^\lambda) \tag{3.10}$$

where $(x^\lambda, y^i, f^{a\alpha}_{b\beta}\; y^i_\sigma\; f^{a\alpha}_{b\beta\sigma})$ and $(x^\lambda, y^i, F^{a\alpha}_{b\beta\mu})$ are the natural fibred coordinates in $J^1(V^1_1(Y) \oplus_Y T^1_1(X))$ and $V^1_1(Y) \oplus_Y T^1_2(X)$, respectively, induced by a fibred chart ψ of $\underline{Y}$. The generalization of formula (3.10) to arbitrary values of the four integers (p,q,r,s) is analogous to the standard one for covariant derivatives of arbitrary tensor fields over a manifold and to avoid complicated expressions, it will not be reported here.

It is now easy to define also the *anholonomic* and *holonomic formal covariant derivative* of any morphism $F: J^k(Y) \to V^p_q(Y) \oplus_Y T^r_s(X)$. In fact, we may construct a commutative diagram by the obvious replacements in Figure 1, which yield the following:

$$\tilde{\nabla}(F) = \nabla \circ J^1(F) : J^1(J^k(Y)) \longrightarrow V^p_q(Y) \oplus_Y T^r_{s+1}(X) \tag{3.11}$$

and

$$\nabla(F) = \nabla \circ \rho^1(F) : J^{k+1}(Y) \longrightarrow V^p_q(Y) \oplus_Y T^r_{s+1}(X). \tag{3.12}$$

In terms of these notions, we have the following intrinsic characterization of the operator ∇. Let us first remark that the set of all fibred morphisms $F: J^k(Y) \to FT^{(p,r)}_{(q,s)}(Y)$, for all integers (k,p,q,r,s), forms a graded algebra $\mathcal{FT}(Y)$ over the reals. (This algebra is in fact the pullback over $J^\infty(Y)$ of the graded algebra of all fibred tensors over $\underline{Y}$.) Then ∇ is uniquely characterized by the following property:

<u>Theorem 1</u> Given any fibred connection $(\hat{\Gamma}, \gamma)$, the differential operator ∇ defined by (3.12) is the unique derivation of the graded algebra $\mathcal{FT}(Y)$ which satisfies the following properties:

(i) ∇ restricted to functions coincides with the formal derivative D;

(ii) ∇ restricted to vertical vector fields coincides with the operator ∇ defined by (3.6);

(iii) ∇ restricted to "horizontal" tensor fields coincides with the covariant derivation with respect to γ;

(iv) ∇ commutes with contractions.

Proof The proof of this theorem is straightforward, by recalling that ∇ is local by definition and applying a classical theorem of Willmore concerning the extension of differential operators on tensor bundles (see, e.g., [1], p. 50).

4. FORMAL CONNECTIONS THROUGH VERTICAL FRAMES

In this section we shall provide a first intrinsic definition of the bundle $\hat{\underline{C}}[\eta^1]$ of formal connections, discussing an equivalent construction through suitable quotients of the first-order jet-prolongations of the bundle $\underline{VF}[\eta]$ of vertical frames in $\underline{Y}$.

Let us then consider the principal bundle $\underline{VF}[\eta] = (VF(Y),Y,\phi_Y;G)$ of *vertical frames* of the fibred manifold (Y,X,η), where $G = GL(n;\mathbb{R})$ with $n = \dim(Y) - \dim(X)$. We denote by $A : VF(Y) \times G \to VF(Y)$ the canonical (right) action of G onto VF(Y). If we prolong this action with respect to the projection ϕ_Y, we obtain a natural right action

$$A_Y^1 : J_Y^1(VF(Y)) \times G \longrightarrow J_Y^1(VF(Y))$$

whose quotient manifold defines the bundle $\underline{K}[\phi_Y] = (K_Y(VF(Y)),Y,k_{\phi Y})$. The sections of $\underline{K}[\phi_Y]$ are in one-to-one correspondence with the linear connections of the vector bundle $\underline{V}[\eta]$, which will be called the *vertical connections* of $\underline{Y}$.

Composing ϕ_Y with η, we obtain a further fibred manifold $(VF(Y),X,\eta\circ\phi_Y)$. Although this is not a principal bundle over X, we may adapt to it the above construction. In fact, there exists a natural right action

$$A_X^1 : J_X^1(VF(Y)) \times G \longrightarrow J_X^1(VF(Y))$$

which is obtained by prolonging A with respect to the projection $\eta\circ\phi_Y$. This action A_X^1 is free and differentiable and it admits a quotient manifold $K_X(VF(Y)) = J_X^1(VF(Y))/G$, having a natural projection over $J^1(Y)$ which makes it an affine bundle over the manifold $J^1(Y)$ itself. Turning to local calculations in natural fibred coordinates, one can easily show that the bundle $\hat{\underline{C}}[\eta^1]$ and the bundle $\underline{K}[\eta\circ\phi_Y]$ constructed above admit the same transition functions, so that they are canonically isomorphic as affine bundles over $J^1(Y)$.

Since the natural composition of functions induces a (natural) epimorphism between first jets of functions, there exists a natural epimorphism

$$J^1(VF(Y);Y) \times {}_Y J^1(Y;X) \to J^1(VF(Y);X)$$

which by restriction defines an epimorphism α from $J^1_Y(VF(Y)) \times {}_Y J^1(Y)$ onto $J^1_X(VF(Y))$. It is not hard to show that this epimorphism is equivariant under the prolonged actions A^1_Y and A^1_X so that it passes to the quotients and defines uniquely a natural epimorphism $\tilde{\alpha}$ which fits into the commutative diagram, Figure 2

$$\begin{tikzcd}
J^1_Y(VF(Y)) \times {}_Y J^1(Y) \arrow[r, "\alpha"] \arrow[d, "K_{\varphi_Y} \times id"'] & J^1_X(VF(Y)) \arrow[d, "K_{\eta \cdot \varphi_Y}"] \\
K_Y(VF(Y)) \times {}_Y J^1(Y) \arrow[r, "\tilde{\alpha}"'] & K_X(VF(Y)) \cong \hat{C}(J^1(Y))
\end{tikzcd}$$

Figure 2

A local coordinate description of the projection $\tilde{\alpha}$ may be given as follows. Let us fix a fibred chart $\psi = (W;x^\lambda,y^i)$ of $\underline{Y}$ and let us denote by $(x^\lambda,y^i,y^i_\sigma,\Gamma^i_{j\mu},\Gamma^i_{jk})$ and $(x^\lambda,y^i,y^i_\sigma,\hat{\Gamma}^i_{j\mu})$ respectively the induced fibred coordinates in $K_Y(VF(Y)) \times {}_Y J^1(Y)$ and $K_X(VF(Y))$. Then the epimorphism $\tilde{\alpha}$ reads as follows:

$$\hat{\Gamma}^i_{j\mu} = \Gamma^i_{j\mu} + y^k_\mu \, \Gamma^i_{jk} \tag{4.1}$$

from which it is immediately seen that $\tilde{\alpha}$ is in fact an affine morphism of affine bundles over $J^1(Y)$.

5. FORMAL CONNECTIONS AS SPLITTINGS OF EXACT SEQUENCES

We give here a further description of formal connections, in terms of splitting of exact sequences of bundles. Let us then consider the exact diagram of vector bundles and vector bundle morphisms over the manifold Y shown in Figure 3, where G is the group $GL(n;\mathbb{R})$, which acts naturally on $VF(Y)$; i_1 and i_2 are natural embeddings; π_i (with $i = 1,2,3,4$) are natural projections.

$$\begin{tikzcd}
 & & ① & ② & & \\
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & V_Y(VF(Y))/G \arrow[d] \arrow[r, equal] & V_Y(VF(Y))/G \arrow[d] & & \\
③ & 0 \arrow[r] & V_X(VF(Y))/G \arrow[r, "i_1"'] \arrow[d, "\pi_1"] & T(VF(Y))/G \arrow[r, "\pi_4"'] \arrow[d, "\pi_2"] & \eta^*(T(X)) \arrow[r] \arrow[d, equal] & 0 \\
④ & 0 \arrow[r] & V_X(Y) \arrow[r, "i_2"'] \arrow[d] & T(Y) \arrow[r, "\pi_3"'] \arrow[d] & \eta^*(T(X)) \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

Figure 3

We then define affine bundles $\underline{C}_i[\eta] = (C_i(Y),Y,c_i)$ $(i = 1,2,3,4)$ by setting

$$C_1(Y) = \{\Gamma_1 \in (V_X(VF(Y))/G) \otimes_Y (V_X(Y))^* \mid \pi_1 \circ \Gamma_1 = id_1\}$$

$$C_2(Y) = \{\Gamma_2 \in (T(VF(Y))/G) \otimes_Y (T(Y))^* \mid \pi_2 \circ \Gamma_2 = id_2\}$$

$$C_3(Y) = \{\Gamma_3 \in T(Y) \otimes_Y (T(X))^* \mid \pi_3 \circ \Gamma_3 = id_3\}$$

$$C_4(Y) = \{\Gamma_4 \in (T(VF(Y))/G) \otimes_Y (T(X))^* \mid \pi_4 \circ \Gamma_4 = id_4\}$$

and taking for c_i the natural projections onto Y (here id_i are abbreviations for the appropriate identity mappings). From these definitions it follows directly that the spaces $\Gamma(c_i)$ of all global sections $\Gamma_i : Y \to C_i(Y)$ coincide with the spaces $S_i(\eta)$ of all splittings of the four exact lines of Figure 3.

We remark the following:

(i) The splittings $\Gamma_1 : V(Y) \to V_X(VG(Y))/G$ of the first short exact column (i.e., the elements of $\Gamma(c_1)$) allow definition of covariant derivatives of vertical tensor fields along "vertical directions". For this reason they might be called *very vertical connections*. Since they have no direct relevance to our present purposes they will not be discussed here.

(ii) The splittings $\Gamma_2: T(Y) \to T(VF(Y))/G$ (i.e., the elements of $\Gamma(c_2)$) coincide with the vertical connections over $\underline{Y}$ which have already been defined in Section 4 above.

(iii) The splittings $\Gamma_3: \eta^*(T(X)) \to T(Y)$ (i.e., the elements of $\Gamma(c_3)$) may be called *nonlinear connections* (or "generalized connections") over the fibred manifold $\underline{Y}$. They have been considered by several authors, also in view of their possible application to physical field theories (see, e.g., [15]).

(iv) The splittings $\Gamma_4: \eta^*(T(X)) \to T(VF(Y))/G$ (i.e., the elements of $\Gamma(c_4)$) will be called here *formal preconnections* over $\underline{Y}$. In fact, as we shall see below, although they do not correspond directly to formal connections, it is exactly this row of the diagram which allows us to define formal connections over $\underline{Y}$. The rest of this section will be devoted to a discussion of this claim.

We have the following:

Proposition 1 There are canonical projections $\pi_1^2: C_2(Y) \to C_1(Y)$ and $\pi_3^4: C_4(Y) \to C_3(Y)$ which define affine bundle structures.

Proof From the exactness of Figure 3 we have

$$\begin{aligned} im(i_1) &= ker(\pi_4) = ker(\pi_3 \circ \pi_2) \\ &= (\pi_2)^{-1}(ker(\pi_3)) = (\pi_2)^{-1}(im(i_2)) \end{aligned}$$

so that a canonical projection $\pi_1^2: C_2(Y) \to C_1(Y)$, may be defined by setting

$$\pi_1^2(\Gamma_2) = (i_1)^{-1} \circ \Gamma_2 \circ i_2. \tag{5.1}$$

All the bundles and mappings involved are affine and easy calculations show that also the fibration $(C_2(Y), C_1(Y), \pi_1^2)$ defines an affine bundle over the manifold $C_1(Y)$.

Let us then define a mapping $\pi_3^4 : C_4(Y) \to C_3(Y)$ by setting

$$\pi_3^4(\Gamma_4) = \pi_2 \circ \Gamma_4. \tag{5.2}$$

From the commutativity of Figure 3 and our definitions above, we see that π_3^4 is well defined and turns out to be an affine surjective submersion, so that $(C_4(Y),C_3(Y),\pi_3^4)$ is an affine bundle over $C_3(Y)$.

We have also the following result:

<u>Proposition 2</u> For any fibred manifold $\underline{Y}$ there exists a canonical epimorphism $j:C_2(Y) \times_Y C_3(Y) \to C_4(Y)$, defined by

$$j(\Gamma_2,\Gamma_3) = \Gamma_2 \circ \Gamma_3.$$

<u>Proof</u> Since Γ_2 is a splitting of (2) and Γ_3 is a splitting of (3), the composition is well defined and provides us with an injective mapping from $\eta^*(T(X))$ into $T(VF(Y))/G$. Owing to the commutativity of Figure 1 we have also $\pi_3 \circ \pi_2 = \pi_4$. This implies that $\Gamma_2 \circ \Gamma_3$ is a splitting of (4). Surjectivity of j is easily shown in local coordinates (see [6]).

Finally, we state the following:

<u>Theorem 2</u> There exist canonical isomorphisms $\lambda:C_3(Y) \to J^1(Y)$ and $\Lambda:C_4(Y) \to \hat{C}(J^1(Y))$ of affine bundles over Y, such that Figure 4 is commutative.

$$\begin{tikzcd}
C_2(Y) \arrow[d, "\pi_1^2"'] & C_2(Y)\times_Y C_3(Y) \arrow[l, "pr_1"'] \arrow[r, "j"] \arrow[dl] \arrow[dr, "pr_2"] & C_4(Y) \arrow[r, "\Lambda"] \arrow[d, "\pi_3^4"] & \hat{C}(J^1(Y)) \arrow[d, "\hat{c}"] \\
C_1(Y) \arrow[d] & & C_3(Y) \arrow[r, "\lambda"] \arrow[d] & J^1(Y) \arrow[d] \\
Y \arrow[rr, equal] & & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Figure 4

<u>Proof</u> Let us first recall that an equivalent definition of first-order jets of a fibred manifold $\underline{Z} = (Z,X,\zeta)$ assures the existence of a canonical one-to-one correspondence between global sections $\sigma:Z \to J^1(Z)$ and splittings $\bar{\sigma}:\zeta^*(T(X)) \to T(Z)$ of the canonical exact sequence

$$0 \longrightarrow V(Z) \longrightarrow T(Z) \longrightarrow T(X) \longrightarrow 0.$$

Accordingly, there exists a canonical one-to-one correspondence between splittings $\Gamma_3:\eta^*(T(X)) \to T(Y)$ and global sections $\sigma:Y \to J^1(Y)$, which defines uniquely a canonical isomorphism $\lambda:C_3(Y) \to J^1(Y)$ of affine bundles over Y. By analogy, from the above definition of $C_4(Y)$ one can see immediately that there exists a canonical one-to-one correspondence between splittings $\Gamma_4:\eta^*(T^*(X)) \to T(VF(Y))/G$ and global sections $\sigma:Y \to J^1_X(VF(Y))/G$, which in turn provides a canonical isomorphism $\Lambda:C_4(Y) \to \hat{C}(J^1(Y))$.

It is immediate to see that Λ projects onto λ, i.e., the following holds:

$$\hat{C} \circ \Lambda = \lambda \circ \pi^4_3.$$

Therefore these affine isomorphisms fit into Figure 4 and make its right-hand square commutative. The rest of Figure 4 is commutative by virtue of Propositions 1 and 2 above.

We are now in a position to explain the terminology "formal preconnections" we used above to denote the splittings Γ_4 of (4), by showing how they allow one to construct an important sub-family of formal connections over $\underline{Y}$.

For this purpose, let us first consider the exact commutative diagram (Figure 5) of vector bundles and vector bundle morphisms over $J^1(Y)$, which is obtained by η^1_0-pull-back over $J^1(Y)$ of the commutative Figure 3. Define

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & (\eta^1_0)^*(V_Y(VF(Y))/G) \arrow[r, equal] \arrow[d] & (\eta^1_0)^*(V_Y(VF(Y))/G) \arrow[d] & & \\
0 \arrow[r] & (\eta^1_0)^*(V_X(VF(Y))/G) \arrow[r] \arrow[d] & (\eta^1_0)^*(T(VF(Y))/G) \arrow[r] \arrow[d] & (\eta^1)^*(T(X)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\eta^1)^*(V_X(Y)) \arrow[r] \arrow[d] & (\eta^1)^*(T(Y)) \arrow[r] \arrow[d] & (\eta^1)^*(T(X)) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Figure 5

then affine bundles over $J^1(Y)$ by setting

$$\underline{C}_i[\eta^1] = (\eta_0^1)^*(\underline{C}_i[\eta])$$

$(i = 1,2,3,4)$, so that their global sections $\tilde{\Gamma}_i : J^1(Y) \to \hat{C}(J^1(Y))$ can be canonically identified to the splittings of the four exact lines of Figure 5 (numbered as in Figure 3). From the definition of pull-back bundles, it follows that any section $\tilde{\Gamma}_i : J^1(Y) \to C_i(J^1(Y))$ may be canonically and uniquely identified to a function $\bar{\Gamma}_i : J^1(Y) \to C_i(Y)$ which satisfies the relation

$$c_i \circ \bar{\Gamma}_i = \eta_0^1$$

i.e., such that Figure 6 is commutative.

$$\begin{tikzcd} J^1(Y) \arrow[r, "\bar{\Gamma}_i"] \arrow[d, "\eta_0^1"'] & C_i(Y) \arrow[d, "c_i"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

Figure 6

We remark that all pull-backs $(\eta_0^1)^*(\Gamma_i)$, where Γ_i is any section of $\underline{C}_i[\eta]$, are sections of the bundles $\underline{C}_i[\eta^1]$, although the converse is not true (i.e., not all sections $\tilde{\Gamma}_i$ of $\underline{C}_i[\eta^1]$ are pull-backs). In particular, the short exact sequence

$$0 \to (\eta_0^1)^*(V_X(VF(Y))/G) \to (\eta_0^1)^*(T(VF(Y))/G) \to (\eta^1)^*(T(X)) \to 0$$

admits infinitely many splittings

$$\tilde{\Gamma}_4 : (\eta^1)^*(T(X)) \to (\eta_0^1)^*(T(VF(Y))/G)$$

which form a space, say $S_4(\eta^1)$, much larger than the space of pull-backs $(\eta_0^1)^*(\Gamma_4)$ of all splittings $\Gamma_4 : \eta^*(T(X)) \to T(VF(Y))/G$.

Let us now recall that there exists a canonical embedding

$$i : (\eta^1)^*(T(X)) \to (\eta_0^1)^*(T(Y))$$

which is in fact a splitting of the short exact sequence

$$0 \to (\eta_0^1)^*(V(Y)) \to (\eta_0^1)^*(T(Y)) \to (\eta^1)^*(T(X)) \to 0.$$

This implies that the affine bundle $C_3(J^1(Y)) \to J^1(Y)$ admits a canonical section

$$\tilde{K}_3 : J^1(Y) \to C_3(J^1(Y)).$$

Furthermore, the mapping $\bar{K}_3 : J^1(Y) \to C_3(Y)$ associated to $\tilde{K}_3$ satisfies the relation

$$\bar{K}_3 \circ \lambda = \mathrm{id} \tag{5.3}$$

where $\lambda : C_3(Y) \to J^1(Y)$ is the canonical isomorphism described in Theorem 2 above. As a consequence of (5.3), it follows that there is no section $\Gamma_3 : Y \to C_3(Y)$ whose η_0^1-pull-back coincides with $\tilde{K}_3$.

We claim the following:

<u>Theorem 3</u> There are infinitely many splittings $\tilde{\Gamma}_4 \in S_4(\eta^1)$ which are not pull-backs and which satisfy the following relation

$$\tilde{\pi}_3^4 \circ \tilde{\Gamma}_4 = \tilde{K}_3, \tag{5.4}$$

namely, they are projected onto the canonical section $\tilde{K}_3$. Moreover, the space of all these splittings $\tilde{\Gamma}_4 \in S_4(\eta^1)$ is in one-to-one correspondence with the space $\Gamma(\hat{C})$ of all formal connections over $\underline{Y}$.

<u>Proof</u> Let us first recall that the formal connections $\hat{\Gamma}$ over $\underline{Y}$ are by definition the sections of the affine bundle $\underline{\hat{C}}[\eta^1]$, so that they are the only functions $\hat{\Gamma} : J^1(Y) \to \hat{C}(J^1(Y))$ which fit into the commutative diagram, Figure 7. Let us also recall that there exists a canonical isomorphism of affine bundles over Y, $\Lambda : C_4(Y) \to \hat{C}(J^1(Y))$, so that all sections $\tilde{\Gamma}_4 : J^1(Y) \to C_4(J^1(Y))$ may be uniquely and canonically identified (through Λ) to all functions $\bar{\Gamma}_4 : J^1(Y) \to \hat{C}(J^1(Y))$ which fit into the commutative diagram, Figure 8.

$$
\begin{tikzcd}
J^1(Y) \arrow[r, "\hat{\Gamma}"] \arrow[d, "\eta_0^1"'] \arrow[dr, equal] & \hat{C}(J^1(Y)) \arrow[d, "\hat{c}"] \\
Y & J^1(Y) \arrow[l, "\eta_0^1"]
\end{tikzcd}
$$

Figure 7

$$
\begin{tikzcd}
J^1(Y) \arrow[r, "\bar{\Gamma}_4"] \arrow[d, "\eta_0^1"'] & \hat{C}(J^1(Y)) \arrow[d, "\hat{c}"] \\
Y & J^1(Y) \arrow[l, "\eta_0^1"]
\end{tikzcd}
$$

Figure 8

However, there are infinitely many functions $\bar{\Gamma}_4 : J^1(Y) \to \hat{C}(J^1(Y))$ which fit into Figure 8 but do not make also Figure 7 commutative (i.e., which are not formal connections). For example, if $\Gamma_4 : Y \to C_4(Y)$ is a splitting of $S_4(\eta)$, the mapping $\bar{\Gamma}_4$ uniquely associated to its pull-back $(\eta_0^1)^*(\Gamma_4) \in S_4(\eta^1)$ cannot make Figure 7 commutative.

Using relation (5.3), recalling the definition of $C_1(\eta^1)$ and the above identification, it is easy to see that a splitting $\tilde{\Gamma}_4$ is projected onto the canonical section $\tilde{K}_3$ (i.e., its satisfies (5.4)) if and only if its associated function $\bar{\Gamma}_4$ makes Figure 7 commutative. Accordingly, to generate the whole family of splittings $\tilde{\Gamma}_4$ satisfying relation (5.4) amounts to constructing them out of all formal connections, which are infinitely many. Finally, the fact that all splittings satisfying (5.4) are not pull-backs follows trivially from our remark above that the canonical section $\tilde{K}_3$ is not a pull-back as well.

Remark A formal construction which allows us to generate the whole set of splittings satisfying relation (5.4) through the existence of a surjective mapping from $S_2(\eta^1)$ onto $\Gamma(\hat{c})$ will be discussed elsewhere (see [6]), where we shall also give local coordinate descriptions of all the notions introduced in this paper.

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Horizontal lift of connections to a natural vector bundle

0. INTRODUCTION

Let $\pi:E \to M$ be a vector field and D be a connection in E, that is,

$$D : \mathfrak{X}(M) \times \underline{E} \ni (X,s) \dashrightarrow D_X s \in \underline{E}$$

is a mapping (where $\mathfrak{X}(M)$ denotes the module of vector fields of class C^∞ on M and $\underline{E}$ denotes the module of sections of class C^∞ of E) which satisfies the following conditions:

$$D_{fX + gY}s = f\, D_X s + g\, D_Y s,$$

$$D_X(s + s') = D_X s + D_X s',$$

$$D_X(fs) = X(f)\, s + f\, D_X s$$

for all vector fields X, Y on M, all functions f, g on M and all sections s, s' of E.

In Section 1 we recall the basic properties of a connection in a vector bundle E. In particular, we define the horizontal lift of vector fields from M to E.

In Section 2 we study vector fields on E.

At first, for each section σ of the dual vector bundle E^* we define a function $\tilde{\sigma}$ on E. This family of functions $\tilde{\sigma}$ is very important in the study of vector fields on E because two vector fields $\tilde{X}$ and $\tilde{Y}$ on E such that $\tilde{X}(\tilde{\sigma}) = \tilde{Y}(\tilde{\sigma})$ for all σ coincide on E (see Proposition 2.1). We prove (Proposition 2.2) that the horizontal lift X^D of a vector field X from M to E verifies the formula

$$X^D(\tilde{\sigma}) = \widetilde{D_X \sigma}\ .$$

Secondly, we define a vertical lift of sections of E. If s is a section of E then we define a vertical vector field s^V on E called the vertical lift of s. This vertical lift generalizes the previous definitions due to K.Yano,

S. Kobayashi, S. Ishihara [8], [9], [11] in the case of tangent bundles and due to K. Yano and E.M. Patterson [12], [13] in the case of cotangent bundles. Our definition generalizes also the horizontal lift of tensor fields to tensor bundles (see [4]). The vector field s^V verifies the condition (Proposition 2.8)

$$s^V(\tilde{\sigma}) = (\sigma.s)^V$$

for every section σ of E^*, where $f^V = f \circ \pi$ is the vertical lift of a function f from M to E.

In Section 2 we define also a vertical vector field $(R(X,Y))^{\square}$ on E, where

$$R(X,Y) = D_X \circ D_Y - D_Y \circ D_X - D_{[X,Y]}$$

is the curvature transformation of D. This vector field satisfies the following condition (Proposition 2.10)

$$(R(X,Y))^{\square}(\tilde{\sigma}) = \widetilde{R(X,Y)\sigma}$$

for each section σ of E^*. This vector field generalizes the constructions due to K. Yano, S. Kobayashi, S. Ishihara and E.M. Patterson [8], [9], [10], [11], [12], [13] in the case of tangent and cotangent bundles.

Next we study properties of these vector fields on E. We have the following formulas (Propositions 2.9, 2.13, 2.14):

$$[X^D,Y^D] = [X,Y]^D + (R(X,Y))^{\square}$$

$$[X^D,s^V] = (D_X s)^V$$

$$[s^V,s'^V] = 0$$

for all vector fields X, Y on M and all sections s, s' of E.

In Section 3 we define the horizontal lift of connections of order r to a natural vector bundle and we study its properties. Let $\pi : E \to M$ be a natural vector bundle. According to the theorem of R.S. Palais and C.-L. Terng [7], E is an associated vector bundle to F^rM for some number r, where F^rM denotes the principal fibre bundle of frames of order r. Let Γ be a connection of order r on M, that is, Γ is a connection in F^rM. For a such connection Γ we define a linear connection $\tilde{\nabla}$ on a manifold E called the horizontal lift of Γ

to E. This connection $\tilde{\nabla}$ satisfies the conditions (Theorem 3.1)

$$\tilde{\nabla}_{X^D} Y^D = (\nabla_X Y)^D$$

$$\tilde{\nabla}_{X^D} s^V = (D_X s)^V$$

$$\tilde{\nabla}_{s^V} X^D = \tilde{\nabla}_{s^V} s'^V = 0$$

for all vector fields X, Y on M and all sections s, s' of E, where ∇ is the linear part of Γ and D is the covariant derivation of sections of E determined by the connection Γ (see R. Crittenden [1]).

This construction generalizes the horizontal lifts of linear connections to tangent and cotangent bundles (see K. Yano, S. Ishihara and E.M. Patterson [9], [10], [13]) and also the horizontal lifts of linear connections to vector bundles associated with the principal fibre bundle of linear frames [3].

Next we study properties of the horizontal lift of connections of order r. Our results generalize the results obtained by K. Yano, S. Ishihara and E.M. Patterson [9], [10], [13] in the case of tangent and cotangent bundles.

The results of this paper can be generalized for an arbitrary natural bundle (no vector bundle). In this case we need another characterization of vertical vector fields on a natural bundle (in the construction of s^V the fact that E is a vector bundle is important). This generalization will be published separately.

1. PRELIMINARIES: CONNECTIONS IN A VECTOR BUNDLE

Let $\pi: E \to M$ be a vector bundle. We denote by $\underline{E}$ the module of all sections of class C^∞ of E and by $\mathfrak{X}(M)$ (resp. $\mathfrak{X}(E)$) the module of all vector fields of class C^∞ on M (resp. on E). A connection in E is a mapping

$$D: \mathfrak{X}(M) \times \underline{E} \ni (X,s) \longrightarrow D_X s \in \underline{E}$$

satisfying the following conditions:

$$D_{fX + gY} s = f\, D_X s + g\, D_Y s, \tag{1.1}$$

$$D_X(s + s') = D_X s + D_X s', \tag{1.2}$$

$$D_X(fs) = X(f)\, s + f\, D_X s \tag{1.3}$$

for all vectof fields X, Y on M, all functions f, g of class C^∞ on M and all sections s, s' of E.

Let $\phi: E|U \to U \times R^N$ be a trivialization of E and let $E_1,\dots,E_N$ be the canonical base of R^N. We consider sections $\rho_1,\dots,\rho_N$ of $E|U$ defined by

$$\rho_a(x) = \phi^{-1}(x, E_a), \tag{1.4}$$

$a = 1,\dots,N$. $\rho_1,\dots,\rho_N$ are called the adapted sections to the trivialization ϕ. If $(U, x^1,\dots,x^n)$ is a chart on M, then there are (uniquely determined) functions Γ^a_{ib} on U such that

$$D_{\partial_i}\, \rho_a = \Gamma^b_{ia}\, \rho_b, \tag{1.5}$$

where $\partial_1,\dots,\partial_n$ is the canonical frame associated to $(U, x^1,\dots,x^n)$. †

The mapping D can be prolonged to a connection in the bundle

$$T^p_q E = \otimes^p E \otimes \otimes^q E^*$$

denoted also by D. This prolongation satisfies the following conditions:

$$D_X(t \otimes t') = D_X t \otimes t' + t \otimes D_X t', \tag{1.6}$$

$$D_X(f) = X(f), \tag{1.7}$$

$$D_X(C^i_j t) = C^i_j(D_X t) \tag{1.8}$$

for all $X \in \mathfrak{X}(M)$, $t \in \underline{T^p_q(E)}$, $t' \in \underline{T^{p'}_{q'}(E)}$ and $f \in C^\infty(M) = \underline{T^0_0(E)}$, where C^i_j is the operator of contraction.

Let $\rho^1,\dots,\rho^N$ be sections of $E^*|U$ such that $\rho^1(x),\dots,\rho^N(x)$ form the dual base to $\rho_1(x),\dots,\rho_N(x)$ for every point x of U, where $\rho_1,\dots,\rho_N$ are the adapted sections to a trivialization of $E|U$. From the conditions (1.6) - (1.8) we obtain

---

† We use the following convention: the indexes $i, j, k,\dots$ run from 1 to n, and the indexes $a, b, c,\dots$ run from 1 to N.

$$D_{\partial_i}\, \rho^a = - \Gamma^a_{ib}\, \rho^b \tag{1.9}$$

for any chart $(U, x^1, \dots, x^n)$ on M.

Let $\gamma:(a,b) \to M$ be a curve of class C^∞ and let $J_\gamma(E)$ be the set of all sections of E defined along γ, that is, an element of $J_\gamma(E)$ is a mapping $s:(a,b) \to E$ (of class C^∞) such that $\pi \circ s = \gamma$. For every curve γ, a connection D in E defines a mapping

$$D_\gamma : J_\gamma(E) \to J_\gamma(E)$$

called the covariant derivation along γ. If $s = s^a\, (\rho_a \circ \gamma)$ is an element of $J_\gamma(E)$, then for a chart $(U, x^1, \dots, x^n)$ on M we have

$$D_\gamma s = \{\frac{d}{dt} s^a + (\Gamma^a_{ib} \circ \gamma) \frac{d}{dt} \gamma^i\, s^b\}\, \rho_a, \tag{1.10}$$

where $\gamma^i = x^i \circ \gamma$, $i = 1, \dots, n$. From (1.10) we have:

<u>Proposition 1.1</u> If $\gamma:(a,b) \to M$ is a curve and y is an element of $E_{\gamma(t_0)} = \pi^{-1}(\gamma(t_0))$, $t_0 \in (a,b)$, then there is one and only one section $s \in J_\gamma(E)$ such that

$$s(t_0) = y, \tag{1.11}$$

$$D_\gamma s = 0. \tag{1.12}$$

Let y be a fixed element of E and $x = \pi(y)$. We denote by Γ_y the set of all velocity vectors $\dot{s}(0)$, where $s:(-\varepsilon, +\varepsilon) \to E$ is a section along $\gamma = \pi \circ s$ satisfying the conditions (1.11) and (1.12) with $t_0 = 0$.

Let $\phi: E|U \to U \times R^N$ be a trivialization and let $(U, x^1, \dots, x^n)$ be a chart on M. Now we can define a chart $(\pi^{-1}(U), x^i, y^a)$ on E called an induced chart, where

$$\begin{aligned} x^i(y) &= x^i(\pi(y)), \\ y &= y^a(y)\, \rho_a \end{aligned} \tag{1.13}$$

for all $y \in \pi^{-1}(U)$. Let $\partial_1, \dots, \partial_n, \delta_1, \dots, \delta_N$ be the canonical frame associated to the induced chart. If $X = \dot{\gamma}(0)$ is a velocity vector of γ and s is the unique section defined along γ satisfying the conditions (1.11) and

(1.12), then

$$\dot{s}(0) = X^i \partial_i - X^i \Gamma^b_{ia} y^a \delta_b. \tag{1.14}$$

This implies that:

Proposition 1.2 Γ_y is a vector subspace of T_yE and

$$T_yE = V_yE \oplus \Gamma_y,$$

where $V_yE = \ker d_y\pi = T_y(E_{\pi(y)})$ is the subspace of vertical vectors. In particular, $d_y\pi|\Gamma_y:\Gamma_y \to T_{\pi(y)}M$ is an isomorphism.

Let X be a vector field on M. Using Proposition 1.2 we can define its horizontal lift X^D by the formula

$$X^D(y) = (d_y\pi|\Gamma_y)^{-1}(X_{\pi(y)}). \tag{1.15}$$

It is easy to verify:

Proposition 1.3 If X,Y are vector fields on M, and f, g are functions on M, then

$$(fX + gY)^D = f^V X^D + g^V Y^D,$$

where $f^V = f \circ \pi$ and $g^V = g\circ\pi$ are vertical lifts of f and g.

From (1.15) and (1.14) we have

$$X^D(y) = X^i(\pi(y))\, \partial_i - X^i(\pi(y))\, \Gamma^b_{ia}(\pi(y))\, y^a \delta_b \tag{1.16}$$

for any induced chart on E.

2. VECTOR FIELDS ON E

Let $\sigma:M \to E^*$ be a section of the dual vector bundle E^*. σ defines a function $\tilde{\sigma}$ on E by the formula

$$\tilde{\sigma}(y) = \sigma_{\pi(y)}(y) \tag{2.1}$$

for every point y of E. (We observe that $\sigma_{\pi(y)}$ is an element of $E^*_{\pi(y)}$, that is, $\sigma_{\pi(y)}$ is a linear mapping $E_{\pi(y)} \to R$.) Using an induced chart it is easy to verify

$$\tilde{\sigma}(y) = \sigma_a(\pi(y))\, y^a, \tag{2.2}$$

where $\sigma = \sigma_a\, \rho^a$. Thus $\tilde{\sigma}$ is a function of class C^∞ on E. We have the following proposition.

<u>Proposition 2.1</u> If σ, σ' are sections of E^* and f, g are functions on M, then

$$\widetilde{f\sigma + g\sigma'} = f^V \tilde{\sigma} + g^V \tilde{\sigma}',$$

where $f^V = f \circ \pi$ is the vertical lift of f.

The proof is trivial. This family of functions $\tilde{\sigma}$ is very important to the study of vector fields on E because we have:

<u>Proposition 2.2</u> Let $\tilde{X}$ and $\tilde{Y}$ be vector fields of class C^∞ on E. If $\tilde{X}(\tilde{\sigma}) = \tilde{Y}(\tilde{\sigma})$ for every section σ of E^*, then $\tilde{X} = \tilde{Y}$.

<u>Proof</u> It is sufficient to show that the equality $\tilde{X}(\tilde{\sigma}) = 0$ for every section σ of E^* implies $\tilde{X} = 0$. Let

$$\tilde{X} = \tilde{X}^i\, \partial_i + \tilde{X}^a\, \partial_a$$

be the coordinates of $\tilde{X}$ with respect to an induced chart on E. From (2.2) we obtain

$$\tilde{X}^i\, (\partial_i \sigma_a)\, y^a + \tilde{X}^a\, \sigma_a = 0$$

for all functions σ_a, $a = 1,\dots,N$, on U. This implies that $\tilde{X}^i = 0$ and $\tilde{X}^a = 0$ for $i = 1,\dots,n$ and $a = 1,\dots,N$, that is, $\tilde{X} = 0$.

This proposition signifies that vector fields on E are uniquely determined by their actions on the functions of type $\tilde{\sigma}$, where σ is a section of E^*. We have:

<u>Proposition 2.3</u> If X is a vector field on M and σ is a section of E^*, then $X^D(\tilde{\sigma}) = \widetilde{D_X \sigma}$.

<u>Proof</u> Let $\sigma = \sigma_a\, \rho^a$. From (1.16), (2.2) and (1.9) we have

$$X^D(\tilde{\sigma}) = X^i \, \partial_i\sigma_a \, y^a - X^i \, \Gamma^b_{ia} \, y^a \, \sigma_b$$

$$= (D_X\sigma)_a \, y^a$$

$$= \widetilde{D_X\sigma}.$$

A vector field $\tilde{X}$ on E is called projectable on M if there is a vector field X on M such that

$$d\pi \circ \tilde{X} = X \circ \pi.$$

X is called projection of $\tilde{X}$ and X is uniquely determined by $\tilde{X}$. The set of all projectable vector fields on M is a Lie algebra and the projection mapping is a Lie algebra homomorphism. We have the following proposition [3].

Proposition 2.4 Let X and $\tilde{X}$ be vector fields on M and E respectively. $\tilde{X}$ is projectable on M and X is its projection if and only if, for each function f on M, we have

$$\tilde{X}(f^V) = (Xf)^V,$$

where $f^V = f \circ \pi$ is the vertical lift of f.

A vector field $\tilde{X}$ on E is called vertical if, for each point y of E, $\tilde{X}(y)$ is a vertical vector, that is $\tilde{X}(y)$ belongs to V_yE. A vertical vector field on E is projectable on M and its projection is zero. Thus, by Proposition 2.4, we have (see [3]):

Corollary 2.5 Let $\tilde{X}$ be a vector field on E. $\tilde{X}$ is vertical if and only if $\tilde{X}(f^V) = 0$ for each function f on M.

Corollary 2.6 If X is a vector field on M and f is a function on M, then

$$X^D(f^V) = (Xf)^V.$$

Since $E_{\pi(y)} = \pi^{-1}(\pi(y))$ is a vector space, there is for each point y of E a natural isomorphism

$$\psi_y : V_yE = T_y(E_{\pi(y)}) \longrightarrow E_{\pi(y)}. \tag{2.3}$$

If $s: M \to E$ is a section of E, then we can define a vector field s^V on E, called the vertical lift of s to E, by the formula

$$[illegible] \tag{2.4}$$

[illegible] induced cha[illegible]

$$[illegible] \tag{2.5}$$

[illegible] $\ldots, s_N$ are the adapted sections.

[illegible] generalizes the definitions of vertical lifts of vector fields to the tangent bundle (K. Yano, S. Kobayashi and S. Ishihara [8], [9], [11]) and vertical lifts of 1-forms to the cotangent bundle (K. Yano and E.M. Patterson [12], [13]). Our definition generalizes also the definition of vertical lifts of tensor introduced by J. Gancarzewicz and N. Rahmani [2]. We have

Proposition 2.7 If s, s' are sections of E and f, g are functions on M, then

$$(fs + gs')^V = f^V s^V + g^V s'^V.$$

Proposition 2.8 If s is a section of E, σ is a section of E* and f is a function on M, then we have

$$s^V(\tilde{\sigma}) = (\sigma \cdot s)^V,$$

$$s^V(f^V) = 0,$$

where $\sigma \cdot s$ is the function on M defined by the formula $(\sigma \cdot s)(x) = \sigma_x(s_x)$.

Proof From (2.5) and (2.2) we have

$$s^V(\tilde{\sigma}) = s^a \partial_a(\sigma_b y^b) = s^a \sigma_a = (\sigma \cdot s)^V.$$

The second formula is a consequence of Corollary 2.5.

Proposition 2.9 If s, s' are sections of E and X is a vector field on M, then

$$[s^V, s'^V] = 0,$$

$$[X^D, s^V] = (D_X s)^V.$$

Proof Let σ be a section of E^*. According to Proposition 2.8 we have

$$[s^V, s'^V](\tilde{\sigma}) = s^V(s'^V(\tilde{\sigma})) - s'^V(s^V(\tilde{\sigma}))$$

$$= 0.$$

Thus, by Proposition 2.2, $[s^V, s'^V] = 0$.

According to Propositions 2.8, 2.3 and Corollary 2.6 we have

$$[X^D, s^V](\tilde{\sigma}) = X^D(s^V(\tilde{\sigma})) - s^V(X^D(\tilde{\sigma}))$$

$$= D_X(\sigma \cdot s) - D_X\sigma \cdot s.$$

Using the formula $D_X(\sigma\cdot s) = D_X\sigma\cdot s + \sigma D_X s$ we obtain

$$[X^D, s^V](\tilde{\sigma}) = \sigma\cdot D_X s$$

$$= (D_X s)^V(\tilde{\sigma})$$

and hence, by Proposition 2.2, $[X^D, s^V] = (D_X s)^V$.

Remark Propositions 2.7, 2.8 and 2.9 generalize the analogical proposition shown in [2], [3], [8], [9], [10], [11], [12], [13].

We will introduce a new vertical vector field on E using the following proposition.

Proposition 2.10 Let $A : E^* \to E^*$ be a vector bundle homomorphism which covers the identity on M; that is, the diagram

$$\begin{tikzcd} E^* \arrow[rr, "A"] \arrow[dr] & & E^* \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

is commutative and the restrictions of A to fibres of E^* are linear. Then there is one and only one vector field $A^{\square}$ on E such that, for every section σ of E^*, we have

$$A^{\square}(\tilde{\sigma}) = \widetilde{A \circ \sigma}.$$

Proof The uniqueness of $A^{\square}$ is a consequence of Proposition 2.2. To prove the existence of $A^{\square}$ we consider a vector field .

$$\tilde{X} = \tilde{X}^i \, \partial_i + \tilde{X}^a \, \delta_a$$

on $E|U$ with coordinates $\tilde{X}^i$, $\tilde{X}^a$ with respect to an induced chart. For a section σ of E^*, by (2.2) we have

$$\tilde{X}(\tilde{\sigma}) = \tilde{X}^i(\partial_i \sigma_a)\, y^a + \tilde{X}^a \, \sigma_a. \tag{2.6}$$

If we denote

$$A(\rho^a) = A^a_b \, \rho^b, \tag{2.7}$$

then $A \circ \sigma = (\sigma_a \, A^a_b)\, \rho^b$, and hence, using (2.2), we have

$$\widetilde{A \circ \sigma} = A^a_b \, \sigma_a \, y^b. \tag{2.8}$$

Thus, if we set

$$\tilde{X}^i = 0, \quad \tilde{X}^a = A^a_b \, y^b \tag{2.9}$$

the equality $\tilde{X}(\tilde{\sigma}) = \widetilde{A \circ \sigma}$ is verified for every section σ of $E^*|U$. Thus we have constructed a vector field $\tilde{X}$ on $E|U$ such that $\tilde{X}(\tilde{\sigma}) = \widetilde{A \circ \sigma}$ for all σ.

Using two charts (U, x^i) and $(U', x^{i'})$ we can construct two vector fields $\tilde{X}$ and $\tilde{X}'$ on $E|U$ and $E|U'$ respectively. For any section σ of $E^*|(U \cap U') = (E^*|U) \cap (E^*|U')$ we have

$$\tilde{X}(\tilde{\sigma}) = \widetilde{A \circ \sigma} = \tilde{X}'(\tilde{\sigma})$$

and hence, according to Proposition 2.2, $\tilde{X}$ and $\tilde{X}'$ coincide on $E|(U \cap U')$. Thus, using an atlas on M, we can define a (global) vector field $A^\square$ on E such that $A^\square(\tilde{\sigma}) = \widetilde{A \circ \sigma}$.

This construction generalizes the operation γ defined by K. Yano, S. Kobayashi, S. Ishihara and E.M. Patterson [8], [9], [11], [12] in the case of tangent and cotangent bundles and also the lift $(\ )^\square$ defined by J. Gancarzewicz and N. Rahmani [2] in the case of tensor bundles.

According to (2.9) we have:

<u>Corollary 2.11</u> $A^\square$ is a vertical vector field on E. If (U, x^i) is a chart on M, then

$$A^\square = A^a_b \, y^b \, \delta_a$$

with respect to the induced chart, where A^a_b are defined by (2.7).

From Proposition 2.4 we obtain:

Corollary 2.12 If f is a function on M, then $A^{\square}(f^V) = 0$.

We have the following properties of $A^{\square}$.

Proposition 2.13 If A, B:E* → E* are vector bundle homomorphisms, s is a section of E and X is a vector field on M, then

$$[X^D, A^{\square}] = (D_X A)^{\square},$$

$$[s^V, A^{\square}] = (A^* \circ S)^V,$$

$$[A^{\square}, B^{\square}] = [A,B]^{\square},$$

where $A^*:E \to E$ is a homeomorphism of vector bundles such that $A^*|E_x$ is the transposed mapping of $A_x = A|E^*_x : E^*_x \to E^*_x$ and $[A,B] = A \circ B - B \circ A$.

Proof Let σ be a section of E*. Using Propositions 2.3 and 2.10 we have

$$[X^D, A^{\square}](\tilde{\sigma}) = X^D(A^{\square}(\tilde{\sigma})) - A^{\square}(X^D(\tilde{\sigma}))$$

$$= \widetilde{D_X(A \circ \sigma)} - \widetilde{A \circ D_X \sigma}.$$

We can interpret A as a section of E* ⊗ E. $A \circ \sigma$ is obtained from A and σ by the tensor product and contraction, thus using (1.6) - (1.8) we have

$$D_X(A \circ \sigma) = (D_X A) \circ \sigma + A \circ D_X \sigma$$

or

$$[X^D, A^{\square}](\tilde{\sigma}) = \widetilde{(D_X A) \circ \sigma}$$

$$= (D_X A)^{\square}(\tilde{\sigma}).$$

Hence, according to Proposition 2.2, we obtain the first formula.

Using Propositions 2.8, 2.10 and Corollary 2.12 we have

$$[s^V, A^{\square}](\tilde{\sigma}) = s^V(A^{\square}(\tilde{\sigma})) - A^{\square}(s^V(\tilde{\sigma}))$$

$$= s^V(\widetilde{A \circ \sigma}) - A^{\square}((\sigma \cdot s)^V)$$

$$= ((sA \circ \sigma) \cdot s)^V.$$

On the other hand, from (2.1) and (2.7), we obtain

$$
\begin{aligned}
(A\circ\sigma)\cdot s &= s^a (A\circ\sigma)_a \\
&= s^a A_a^b \sigma_b \\
&= \sigma_b (A^*\circ s)^b \\
&= \sigma\cdot(A^*\circ s) ,
\end{aligned}
$$

and hence

$$
\begin{aligned}
[s^V, A^\square](\tilde{\sigma}) &= (\sigma\cdot(A^*\circ s))^V \\
&= (A^*\circ s)^V(\tilde{\sigma}),
\end{aligned}
$$

that is, $[s^V, A^\square] = (A^*\circ s)^V$.

The verification of the last formula of our proposition is by analogy.

Let X and Y be two vector fields on M. We denote by

$$R(X,Y) = D_X \circ D_Y - D_Y \circ D_X - D_{[X,Y]} : \underline{E}^* \longrightarrow \underline{E}^*. \tag{2.10}$$

$R(X,Y)$ is called the curvature transformation of the connection D. From (1.1) - (1.3) (we have the same formulas for sections of E*) we obtain

$$R(X,Y)(\sigma + \sigma') = R(X,Y)\sigma + R(X,Y)\sigma'$$

$$R(X,Y)(f\sigma) = f\, R(X,Y)\sigma$$

for all sections σ, σ' of E* and any function f, and hence, $R(X,Y)$ can be considered as a vector bundle homeomorphism $R(X,Y):\underline{E}^* \longrightarrow \underline{E}^*$. The vector field $(R(X,Y))^\square$ is important for the characterization of the vertical component of $[X^D, Y^D]$. We have

<u>Proposition 2.14</u> If X and Y are two vector fields on M, then

$$[X^D, Y^D] = [X,Y]^D + (R(X,Y))^\square,$$

where $R(X,Y)$ is the curvature transformation of D defined by (2.10).

<u>Proof</u> Let σ be a section of E*. Using Propositions 2.3, 2.10 and formula (2.10) we have

$$
\begin{aligned}
[X^D, Y^D](\tilde{\sigma}) &= X^D(Y^D(\tilde{\sigma})) - Y^D(X^D(\tilde{\sigma})) \\
&= \widetilde{D_X(D_Y\sigma)} - \widetilde{D_Y(D_X\sigma)} \\
&= \widetilde{R(X,Y)\sigma} + \widetilde{D_{[X,Y]}\sigma} \\
&= [X,Y]^D(\tilde{\sigma}) + (R(X,Y))^{\square}(\tilde{\sigma}),
\end{aligned}
$$

and hence, using Proposition 2.2, we obtain our formula.

3. HORIZONTAL LIFTING OF CONNECTIONS TO A NATURAL VECTOR BUNDLE

Let $\phi: E \to M$ be a natural vector bundle. If $\phi: M \to M$ is a local diffeomorphism, then we denote by $\tilde{\phi}: E \to E$ the induced mapping. For each point x of M, $\tilde{\phi}(E_x) = E_{\phi(x)}$ and $\tilde{\phi}: E_x \to E_{\phi(x)}$ is an isomorphism, where $E_x = \pi^{-1}(x)$ is the fibre of E. By the theorem of R.S. Palais and C.-L. Terng [7] there exists a number r such that, for all local diffeomorphisms ϕ, $\psi: M \to M$ and every point x of M, the equality $j_x^r\phi = j_x^r\psi$ implies $\tilde{\phi}|E_x = \tilde{\psi}|E_x$. The smallest number r satisfying this property is called order of E.

Let r be the order of E. We suppose that $r \geq 1$. The vector bundle E is isomorphic to an associated fibre bundle with F^rM (see [7], [6]), where F^rM is the principal fibre bundle of frames of order r, that is,

$$
F^rM = \{j_0^r\phi : \phi \text{ is a diffeomorphism of a neighbourhood of } 0 \text{ in } R^n \\ \text{into some open subset of } M\}.
$$

Let F be the standard fibre of E. We denote by $\Phi: F^rM \times F \to E$ the canonical mapping for the associated fibre bundle E.

Let Γ be a connection in the principal fibre bundle F^rM (Γ is called connection of order r on M). Γ determines a horizontal distribution on E. If $y = \Phi(p,z)$ is a point of E, then

$$
H_y = d_p\Phi_z(\Gamma_p), \tag{3.1}
$$

where $\Phi_z: F^rM \to E$, $\Phi_z(p) = \Phi(p,z)$.

The connection Γ determines the covariant derivation of sections of associated fibre bundles with F^rM. In particular, we have the covariant derivation

$$
D: \mathfrak{X}(M) \times \underline{E} \ni (X,s) \longrightarrow D_Xs \in \underline{E}
$$

of sections of E. It is well known that D satisfies conditions (1.1) - (1.3), that is, using the terminology of Section 1, D is a connection in E (see [1]). It is easy to verify that the distribution H defined by (3.1) is the same as the distribution Γ defined in Section 1 for the connection D. Hence, the horizontal lift of vector fields with respect to D coincides with the usual horizontal lift of vector fields with respect to the connection Γ of order r on M.

Let $\pi^r_s : F^rM \to F^sM$, $s \le r$, be the natural projection, $\pi^r_s(j^r_0\phi) = j^s_0\phi$. Using this projection, for a given connection of order r on M we can induce a connection of order s, $s \le r$. In particular, the given connection Γ of order r on M induces a linear connection on M called linear part of Γ. We denote by ∇ the covariant derivation of vector fields with respect to the linear part of Γ.

The main theorem of this paper is the following one.

<u>Theorem 3.1</u> Let Γ be a connection of order r on M. If $\pi : E \to M$ is a natural vector bundle of order r, then there is one and only one linear connection $\tilde{\nabla}$ on the manifold E such that

$$\tilde{\nabla}_{X^D} Y^D = (\nabla_X Y)^D \tag{3.2}$$

$$\tilde{\nabla}_{X^D} s^V = (D_X s)^V \tag{3.3}$$

$$\tilde{\nabla}_{s^V} X^D = 0 \tag{3.4}$$

$$\tilde{\nabla}_{s^V} s'^V = 0 \tag{3.5}$$

for all vector fields X, Y on M and all sections s, s' of E, where ∇ is the covariant derivation of vector fields on M with respect to the linear part of Γ and D is the covariant derivation of sections of E with respect to Γ.

To prove this theorem we need the following lemma.

<u>Lemma 3.2</u> Let Γ be a connection of order r on M and $\tilde{\nabla}$ be a linear connection on E. For a chart (U, x^i) on M we denote by

$$\tilde{\nabla}_{\partial_i} \partial_j = \tilde{\Gamma}^k_{ij}\, \partial_k + \tilde{\Gamma}^a_{ij}\, \delta_a \tag{3.6}$$

$$\tilde{\nabla}_{\partial_i} \partial_a = \tilde{\Gamma}^j_{ia} \partial_j + \tilde{\Gamma}^b_{ia} \delta_b \tag{3.7}$$

$$\tilde{\nabla}_{\delta_a} \partial_i = \tilde{\Gamma}^j_{ai} \partial_j + \tilde{\Gamma}^b_{ai} \delta_b \tag{3.8}$$

$$\tilde{\nabla}_{\delta_a} \delta_b = \tilde{\Gamma}^i_{ab} \partial_i + \tilde{\Gamma}^c_{ab} \delta_c \tag{3.9}$$

the Christoffel symbols of $\tilde{\nabla}$ with respect to the induced chart on E. If conditions (3.2) - (3.5) are satisfied, then

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} \tag{3.10}$$

$$\tilde{\Gamma}^a_{ij} = (\partial_i \Gamma^a_{jb} + \Gamma^a_{ic} \Gamma^c_{jb} - \Gamma^k_{ij} \Gamma^b_{kb}) y_b \tag{3.11}$$

$$\tilde{\Gamma}^i_{ja} = 0 \tag{3.12}$$

$$\tilde{\Gamma}^b_{ia} = \Gamma^b_{ia} \tag{3.13}$$

$$\tilde{\Gamma}^j_{ai} = 0 \tag{3.14}$$

$$\tilde{\Gamma}^b_{ai} = \Gamma^b_{ia} \tag{3.15}$$

$$\tilde{\Gamma}^i_{ab} = 0 \tag{3.16}$$

$$\tilde{\Gamma}^c_{ab} = 0, \tag{3.17}$$

where Γ^i_{jk} are the Christoffel symbols of the linear part of Γ and Γ^a_{ib} are defined by (1.5).

<u>Proof</u> Let $\rho_1, \ldots, \rho_N$ be the adapted section of E to the induced chart. According to (2.5) we have

$$\rho^V_a = \delta_a. \tag{3.18}$$

Now formulas (3.18), (3.5) and (3.9) imply (3.16) and (3.17). Next from (1.16) we have

$$\partial^D_i = \partial_i - \Gamma^b_{ia} y^a \delta_b. \tag{3.19}$$

and hence, using (3.16), (3.17), (3.8) and (3.4), we obtain (3.14) and (3.15).

Using (3.7), (3.19) and (3.18) we can calculate

$$\tilde{\nabla}_{\partial_i^D} \rho_a^V = \tilde{\nabla}_{\partial_i} \delta_a - \Gamma^b_{ic}\, y^c\, \tilde{\nabla}_{\delta_b} \delta_a$$

$$= \tilde{\nabla}_{\partial_i} \delta_a.$$

On the other hand, using (1.5) and (2.5) we obtain

$$(D_{\partial_i} \rho_a)^V = \Gamma^b_{ia}\, \delta_b.$$

Thus the equality (3.3) implies (3.12) and (3.13). Finally, using (3.19) and (3.12) - (3.17) we calculate

$$\tilde{\nabla}_{\partial_i^D} \partial_j^D = \tilde{\nabla}_{\partial_i - \Gamma^a_{ib} y^b \delta_a} \partial_j - \Gamma^c_{jd}\, y^d\, \delta_c$$

$$= \tilde{\nabla}_{\partial_i} \partial_j - \{\partial_i \Gamma^b_{jc} - \Gamma^b_{ia} \Gamma^a_{jc}\}\, y^c\, \delta_b$$

$$(\tilde{\nabla}_{\partial_i} \partial_j)^D = \Gamma^s_{ij}\, \partial^D_s$$

$$= \Gamma^s_{ij}\, \partial_s - \Gamma^s_{ij}\, \Gamma^b_{sa}\, y^a\, \delta_b.$$

Hence, formulas (3.7) and (3.2) imply (3.10) and (3.11). The proof of our lemma is finished.

Proof of Theorem 3.1 The uniqueness of a linear connection $\tilde{\nabla}$ on E satisfying conditions (3.2) - (3.5) is clear because, according to Lemma 3.2, the Christoffel symbols of $\tilde{\nabla}$ are uniquely determined by the given connection Γ of order r. Thus we need to prove only the existence of $\tilde{\nabla}$.

Let (U, x^i) be a chart on M. We can define a linear connection $\tilde{\nabla}$ on $E|U$ such that its Christoffel symbols with respect to the induced chart are given by formulas (3.10) - (3.17). This linear connection $\tilde{\nabla}$ on $E|U$ verifies the conditions

$$\left.\begin{aligned}
&\tilde{\nabla}_{\partial_i^D} \partial_j^D = (\nabla_{\partial_i} \partial_j)^D \\
&\tilde{\nabla}_{\partial_i^D} \rho_a^V = (D_{\partial_i} \rho_a)^V \\
&\tilde{\nabla}_{\rho_a^V} \partial_i^D = 0 \\
&\tilde{\nabla}_{\rho_a^V} \rho_b^V = 0
\end{aligned}\right\} \qquad (3.20)$$

for i,j = 1,...,n and a,b = 1,...,N. Using the propositions of Sections 1 and 2 it is easy to prove that

$$\left.\begin{aligned}
&\tilde{\nabla}_{X^D} Y^D = (\nabla_X Y)^D \\
&\tilde{\nabla}_{X^D} s^V = (D_X s)^V \\
&\tilde{\nabla}_{s^V} X^D = \tilde{\nabla}_{s^V} s'^V = 0
\end{aligned}\right\} \qquad (3.21)$$

for all vector fields X, Y on U and all sections s, s' of $E|U$. We show only the first formula of (3.21). Let X and Y be vector fields on U. If we denote by

$$X = X^i \partial_i, \quad Y = Y^i \partial_i$$

the coordinates of X and Y with respect to the chart (U, x^i), then according to Proposition 1.3 we have

$$X^D = (X^i)^V \partial_i^D, \quad Y^D = (Y^i)^V \partial_i^D$$

and hence, using Propositions 1.3, Corollary 2.6 and the first equality of (3.20), we obtain

$$\begin{aligned}
\tilde{\nabla}_{X^D} Y^D &= (X^i)^V \{\partial_i^D((Y^j)^V)\, \partial_j^D + (Y^j)^V \tilde{\nabla}_{\partial_i^D} \partial_j^D\} \\
&= (X^i)^V \{(\partial_i Y^j)^V\, \partial_j^D + (Y^j)^V (\nabla_{\partial_i} \partial_j)^D\} \\
&= \{X^i [(\partial_i Y^j)\partial_j + Y^j \nabla_{\partial_i} \partial_j]\}^D \\
&= (\nabla_X Y)^D.
\end{aligned}$$

The other formulas of (3.21) are verified by analogy.

If (U, x^i) and $(U', x^{i'})$ are two charts on M, then we can define two linear connections $\tilde{\nabla}$ and $\tilde{\nabla}'$ respectively on $E|U$ and $E|U'$. From (3.21) we have

$$\tilde{\nabla}_{X^D} Y^D = (\nabla_X Y)^D = \tilde{\nabla}'_{X^D} Y^D$$

$$\tilde{\nabla}_{X^D} s^V = (D_X s)^V = \tilde{\nabla}'_{X^D} s^V$$

$$\tilde{\nabla}_{s^V} X^D = \tilde{\nabla}'_{s^V} X^D = 0$$

$$\tilde{\nabla}_{s^V} s'^V = \tilde{\nabla}'_{s^V} s'^V = 0$$

for all vector fields X, Y on $U \cap U'$ and all sections s, s' of $E|(U \cap U') = (E|U) \cap (E|U')$. Hence, by Lemma 3.1, the linear connections $\tilde{\nabla}$ and $\tilde{\nabla}'$ coincide on $E|(U \cap U')$.

Using an atlas on M we can define a linear connection $\tilde{\nabla}$ on E. This connection $\tilde{\nabla}$ verifies the conditions (3.2) - (3.5) and the proof is complete.

The linear connection $\tilde{\nabla}$ on E verifying conditions (3.2) - (3.5) is called the horizontal lift of Γ from M to E. The following three corollaries are immediate consequences of Theorem 3.1.

Corollary 3.3 (K. Yano, S. Ishihara [10], [9]). If ∇ is a linear connection on M, then there is one and only one linear connection $\tilde{\nabla}$ on TM such that

$$\tilde{\nabla}_{X^H} Y^H = (\nabla_X Y)^H, \quad \tilde{\nabla}_{X^H} Y^V = (\nabla_X Y)^V$$

$$\tilde{\nabla}_{X^V} Y^H = \tilde{\nabla}_{X^V} Y^V = 0$$

for all vector fields X and Y on M, where X^H is the horizontal lift of X to TM with respect to ∇.

Corollary 3.4 (K. Yano, E.M. Patterson [13], [9]). If ∇ is a linear connection on M, then there is one and only one linear connection $\tilde{\nabla}$ on T*M such that

$$\tilde{\nabla}_{X^H} Y^H = (\nabla_X Y)^H, \quad \tilde{\nabla}_{X^H} \phi^V = (\nabla_X \phi)^V$$

$$\tilde{\nabla}_{\phi^V} X^H = \tilde{\nabla}_{\phi^V} \omega^V = 0$$

for all vector fields X, Y on M and all 1-forms ϕ, ω on M, where X^H is the horizontal lift of X to T^*M with respect to ∇.

Corollary 3.5 (J. Gancarzewicz, N. Rahmani [3]). If E is a vector bundle associated to the principal fibre bundle LM of linear frames and ∇ is a linear connection on M, then there is one and only one linear connection $\tilde{\nabla}$ on E such that

$$\tilde{\nabla}_{X^H} Y^H = (\nabla_X Y)^H , \quad \tilde{\nabla}_{X^H} s^V = (\nabla_X s)^V$$

$$\tilde{\nabla}_{s^V} X^H = \tilde{\nabla}_{s^V} s'^V = 0$$

for all vector fields X, Y on M and all sections s, s' of E, where X^H is the horizontal lift of X to E with respect to ∇.

Next we will study the torsion tensor and the curvature tensor of the horizontal lift of a connection of order r to any natural vector bundle of order r (r is arbitrary). We have the following properties of these tensors.

Proposition 3.6 Let E be a vector bundle associated to F^rM and let Γ be a connection of order r on M. If $\tilde{\nabla}$ is the horizontal lift of Γ to E and $\tilde{T}$ is the torsion tensor of $\tilde{\nabla}$, then we have

$$\tilde{T}(X^D,Y^D) = (T(X,Y))^D - (R(X,Y))^{\square}$$

$$\tilde{T}(X^D,s^V) = \tilde{T}(s^V,s'^V) = 0$$

for all vector fields X, Y on M and all sections s, s' of E, where T is the torsion tensor of the linear part of ∇ and $R(X,Y)$ is the curvature transformation of Γ defined by (2.10).

Proof Using Theorem 3.1 and Proposition 2.10 we have

$$\begin{aligned}\tilde{T}(X^D,Y^D) &= \tilde{\nabla}_{X^D} Y^D - \tilde{\nabla}_{Y^D} X^D - [X^D,Y^D] \\ &= (\nabla_X Y)^D - (\nabla_Y X)^D - [X,Y]^D - (R(X,Y))^{\square} \\ &= (T(X,Y))^D - (R(X,Y))^{\square}.\end{aligned}$$

Next, using Theorem 3.1 and Proposition 2.9 we obtain

$$\tilde{T}(X^D,s^V) = \tilde{\nabla}_{X^D} s^V - \tilde{\nabla}_{s^V} X^D - [X^D,s^V]$$

$$= (D_X s)^V - (D_X s)^V = 0$$

$$\tilde{T}(s^V,s'^V) = \tilde{\nabla}_{s^V} s'^V - \tilde{\nabla}_{s'^V} s^V - [s^V,s'^V]$$

$$= 0.$$

To calculate the curvature tensor of $\tilde{\nabla}$ we need the following lemma.

Lemma 3.7 If $\tilde{\nabla}$ is the horizontal lift of a connection of order r to E and $A:E^* \to E^*$ is a vector bundle homomorphism, then

$$\tilde{\nabla}_{A^\square} X^D = 0, \qquad \tilde{\nabla}_{A^\square} s^V = 0$$

for every vector field X on M and every section s of E.

Proof Using an induced chart, according to Corollary 2.11 and formula (2.5) we have

$$A^\square = A^a_b\, y^b\, \delta_a\ , \quad s^V = s^a\, \delta_a$$

$$X^D = X^i(\partial_i - \Gamma^b_{ia}\, y^a\, \delta_b)$$

and hence, by Lemma 3.2, we obtain

$$\tilde{\nabla}_{A^\square} s^V = A^a_b\, y^b\, s^c\, \tilde{\nabla}_{\delta_a} \delta_c$$

$$= 0$$

$$\tilde{\nabla}_{A^\square} X^D = A^a_b\, y^b\, (\tilde{\nabla}_{\delta_a} \partial_i - \Gamma^c_{id}\, y^d\, \tilde{\nabla}_{\delta_a} \delta_c - \Gamma^c_{ia}\, \delta_c)$$

$$= 0$$

Now we have

Proposition 3.8 If $\tilde{\nabla}$ is the horizontal lift of a connection Γ of order r on M to a vector bundle associated with F^rM and $\tilde{R}$ is the curvature tensor of $\tilde{\nabla}$, then

$$\tilde{R}(X^D, Y^D)Z^D = (r(X,Y)Z)^D$$

$$\tilde{R}(X^D, Y^D)s^V = (R(X,Y)s)^V$$

$$\tilde{R}(X^D, s^V) = \tilde{R}(s^V, s'^V) = 0$$

for all vector fields X, Y, Z on M and all sections s, s' of [illegible] is the curvature transformation of Γ defined by (2.10) and $r(X,Y)$ [illegible] curvature tensor of the linear part of Γ.

<u>Proof</u> Using Theorem 3.1, Proposition 2.14 and Lemma 3.7 we have

$$\begin{aligned}
\tilde{R}(X^D, Y^D)Z^D &= \tilde{\nabla}_{X^D}(\tilde{\nabla}_{Y^D} Z^D) - \tilde{\nabla}_{Y^D}(\tilde{\nabla}_{X^D} Z^D) - \tilde{\nabla}_{[X^D, Y^D]} Z^D \\
&= (\nabla_X(\nabla_Y Z))^D - (\nabla_Y(\nabla_X Z))^D - (\nabla_{[X,Y]} Z)^D - \\
&\qquad - \tilde{\nabla}_{(R,X,Y))^{\square}} Z^D \\
&= (r(X,Y)Z)^D,
\end{aligned}$$

$$\begin{aligned}
\tilde{R}(X^D, Y^D)s^V &= \tilde{\nabla}_{X^D}(\tilde{\nabla}_{Y^D} s^V) - \tilde{\nabla}_{Y^D}(\tilde{\nabla}_{X^D} s^V) - \tilde{\nabla}_{[X^D, Y^D]} s^V \\
&= (D_X(D_Y s))^V - (D_Y(D_X s))^V - (D_{[X,Y]} s)^V \\
&\qquad - \tilde{\nabla}_{(R(X,Y))^{\square}} s^V \\
&= (R(X,Y)s)^V.
\end{aligned}$$

Using Proposition 2.9 we can calculate

$$\tilde{R}(X^D, s^V)Y^D = 0 \quad , \quad \tilde{R}(X^D, s^V)s'^V = 0$$

$$\tilde{R}(s^V, s'^V)X^D = 0 \quad , \quad \tilde{R}(s^V, s'^V)s''^V = 0$$

for all vector fields X, Y, Z on M and all sections s, s', s" of E. This remark finishes the proof of our proposition.

From Proposition 3.6 we have:

Proposition 3.9 Let $\tilde{\nabla}$ be the horizontal lift of a connection Γ of order r to a vector bundle E associated with F^rM. If the linear part of Γ is without torsion, then $\tilde{\nabla}$ is without torsion if and only if the curvature transformation R(X,Y) is zero for all vector fields X, Y on M.

From Proposition 3.8 we have:

Proposition 3.10 Let $\tilde{\nabla}$ be the horizontal lift of a connection Γ of order r on M to a vector bundle E associated with F^rM. Then the linear connection $\tilde{\nabla}$ is without curvature (that is, $\tilde{R} = 0$) if and only if the curvature transformation R(X,Y) of Γ defined by (2.10) is zero for all vector fields X and Y on M.

To prove this proposition it is sufficient to observe that if the curvature transformation R(X,Y) of Γ is zero then the linear part of Γ is without curvature.

Propositions 3.9 and 3.10 generalize the analogic propositions proved by K. Yano, S. Ishihara in the case of tangent bundles [10], [9], by K. Yano, E.M. Patterson in the case of cotangent bundles [13], [9] and by J. Gancarzewicz, N. Rahmani [3] in the case of a vector bundle associated with the principal fibre bundle of linear frames.

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