12. Blown-up intersection cohomology of a cone. Theorem E.

In this section, X is an n-dimensional *compact* filtered space and we represent the open cone as the quotient $\mathring{c}X = X \times [0,\infty[/X \times \{0\}$, whose apex is denoted by $\mathtt{w}$. The formal dimension of $\mathring{c}X$ is $n+1$ relatively to the conical filtration, $(\mathring{c}X)_i = \mathring{c}X_{i-1}$ if $i \geqslant 1$ and $(\mathring{c}X)_0 = \{\mathtt{w}\}$. The purpose of this section is to prove the following proposition, cf. also [4, Corollary 1.47] and [24, Proposition 3.1.1].

Theorem E. *Let X be a compact filtered space. Consider the open cone, $\mathring{c}X = X \times [0,\infty[/X \times \{0\}$, equipped with the conical filtration and a perversity $\overline{p}$. We also denote by $\overline{p}$ the perversity induced on X. The following properties are verified for any commutative ring, R.*

(a) *The inclusion $\iota\colon X \to \mathring{c}X$, $x \mapsto [x,1]$, induces an isomorphism, $\mathscr{H}^k_{\overline{p}}(\mathring{c}X;R) \xrightarrow{\cong} \mathscr{H}^k_{\overline{p}}(X;R)$, for each $k \leqslant \overline{p}(\mathtt{w})$.*

(b) *For each $k > \overline{p}(\mathtt{w})$, we have $\mathscr{H}^k_{\overline{p}}(\mathring{c}X;R) = 0$.*

12.1 Simplices on a filtered space and its cone. First we link the complexes of X and $\mathring{c}X$. The formal dimension of the cone being different from that of the original space, we introduce some operations which increase or decrease the length of filtrations.

- If $\Delta = \Delta_0 * \cdots * \Delta_{n+1}$ is a regular simplex, of formal dimension $n+1$, we define a regular simplex, of formal dimension n, by $\widehat{\Delta} = \Delta_1 * \cdots * \Delta_{n+1}$. Its filtration is characterized by $\widehat{\Delta}_i = \Delta_{i+1}$, for each $i \in \{0,\ldots,n\}$.
- Let $\sigma\colon \Delta_\sigma = \Delta_0 * \cdots * \Delta_{n+1} \to \mathring{c}X$ be a regular simplex of $\mathring{c}X$. Since $\sigma(\widehat{\Delta}_\sigma) \subset X \times]0,\infty[$, we define the restriction
$$\hat{\sigma}\colon \Delta_{\hat{\sigma}} = \widehat{\Delta}_\sigma \xrightarrow{\sigma} X \times]0,\infty[.$$
- For each regular simplex of $\mathring{c}X$, $\sigma\colon \Delta_\sigma = \Delta_0 * \cdots * \Delta_{n+1} \to \mathring{c}X$, the image of a point $x \in \Delta_\sigma$ can be written as,
$$\sigma(x) = [\sigma_1(x), \sigma_2(x)] \in \mathring{c}X = X \times [0,\infty[/X \times \{0\}.$$
Associated to the simplex σ, there is the following regular simplex of $\mathring{c}X$,
$$\mathtt{c}\sigma\colon \Delta_{\mathtt{c}\sigma} = (\{\mathtt{p}\} * \Delta_0) * \cdots * \Delta_{n+1} \to \mathring{c}X,$$
defined by $\mathtt{c}\sigma((1-t)\mathtt{p}+tx) = [\sigma_1(x), t\sigma_2(x)]$. Moreover, if one considers $\hat{\sigma}\colon \widehat{\Delta}_\sigma \to X \times]0,\infty[\hookrightarrow \mathring{c}X$ as a filtered simplex of the cone, then $\mathtt{c}\hat{\sigma}$ is a face of $\mathtt{c}\sigma$.

The *truncation* of a cochain complex is defined for all positive integers s by

$$(\tau_{\leqslant s}C)^r = \begin{cases} C^r & \text{if } r < s, \\ \mathcal{Z}C^s & \text{if } r = s, \\ 0 & \text{if } r > s, \end{cases} \tag{38}$$

where $\mathcal{Z}C^s$ means the R-module of cocycles whose degree is s.

Construction of $f\colon \widetilde{N}^*(X \times]0,\infty[;R) \to \widetilde{N}^*(\mathring{c}X;R)$.

Let $\sigma\colon \Delta_\sigma = \Delta_0 * \cdots * \Delta_{n+1} \to \mathring{c}X$ and $\omega \in \widetilde{N}^*(X \times]0,\infty[;R)$. We denote by $\lambda_{\mathtt{c}\Delta_0}$ the cocycle $\mathbf{1}_{(\emptyset,1)} + \sum_{e \in \mathcal{V}(\Delta_0)} \mathbf{1}_{([e],0)} \in N^0(\mathtt{c}\Delta_0)$. We set

$$f(\omega)_\sigma = \lambda_{\mathtt{c}\Delta_0} \otimes \omega_{\hat{\sigma}}. \tag{39}$$

Proposition 12.2. *Let $(\mathring{\mathtt{c}}X, \overline{p})$ be a perverse space over the cone of the compact space X and let $(X\times]0,\infty[,\overline{p})$ be the induced perverse space. The correspondence defined above induces a cochain map,*

$$f\colon \tau_{\leqslant \overline{p}(\mathtt{w})} \widetilde{N}^*_{\overline{p}}(X\times]0,\infty[;R) \to \tau_{\leqslant \overline{p}(\mathtt{w})} \widetilde{N}^*_{\overline{p}}(\mathring{\mathtt{c}}X;R).$$

Proof. First, we check that the application f, defined locally at the level of simplices, extends globally to $\widetilde{N}^*(X\times]0,\infty[;R)$. For this, we must establish $\delta_k^* f(\omega)_\sigma = f(\omega)_{\sigma\circ\delta_k}$, for each $\omega \in \widetilde{N}^*(X\times]0,\infty[;R)$, each regular simplex, $\sigma\colon \Delta_\sigma = \Delta_0 * \cdots * \Delta_{n+1} \to \mathring{\mathtt{c}}X$, and any regular face operator, $\delta_k\colon \nabla \to \Delta_\sigma$, with $k \in \{0,\ldots,\dim\Delta_\sigma\}$. Let j_0 denote the dimension of Δ_0. To determine the effect of δ_k on the operation $\sigma \mapsto \hat{\sigma}$, we must distinguish $k > j_0$ of $k \leqslant j_0$. For the sake of convenience, we set $\delta_s = \mathrm{id}$ if $s < 0$. From the construction of $\hat{\sigma}$, we have

$$\widehat{\sigma\circ\delta_k} = \left\{ \begin{array}{cll} \hat{\sigma}\circ\delta_{k-j_0-1} & \text{if} & k > j_0, \\ \hat{\sigma} & \text{if} & k \leqslant j_0, \end{array}\right.$$

which implies $\widehat{\sigma\circ\delta_k} = \hat{\sigma}\circ\delta_{k-j_0-1}$, with the previous convention. We conclude

$$\delta_k^* f(\omega)_\sigma = \delta_k^*\left(\lambda_{\mathtt{c}\Delta_0} \otimes \omega_{\hat{\sigma}}\right) = \left\{ \begin{array}{cll} \lambda_{\mathtt{c}\Delta_0} \otimes \delta^*_{k-j_0-1}\omega_{\hat{\sigma}} & \text{if} & k > j_0, \\ \lambda_{\mathtt{c}\nabla_0} \otimes \omega_{\hat{\sigma}} & \text{if} & k \leqslant j_0. \end{array}\right.$$

It follows $\delta_k^* f(\omega)_\sigma = \lambda_{\mathtt{c}\nabla_0} \otimes \omega_{\widehat{\sigma\circ\delta_k}} = f(\omega)_{\sigma\circ\delta_k}$.

Since the 0-cochain $\lambda_{\mathtt{c}\Delta_0}$ is a cocycle, the compatibility with the differentials is immediate from the equalities

$$\delta\left(f(\omega)_\sigma\right) = \delta\left(\lambda_{\mathtt{c}\Delta_0} \otimes \omega_{\hat{\sigma}}\right) = \lambda_{\mathtt{c}\Delta_0} \otimes \delta\omega_{\hat{\sigma}} = f(\delta\,\omega)_\sigma.$$

The map f being compatible with the differentials, it remains to show that the image by f of a $\overline{p}$-allowable cochain, $\omega \in \widetilde{N}^*(X\times]0,\infty[;R)$, is a $\overline{p}$-allowable cochain in $\widetilde{N}^*(\mathring{\mathtt{c}}X;R)$. We choose ω of degree less than or equal to $\overline{p}(\mathtt{w})$ and refer to Definition 3.4 for the property of $\overline{p}$-allowability. For the stratum reduced to $\mathtt{w}$, the allowability comes directly from $\|f(\omega)_\sigma\|_{n+1} \leqslant |\omega_{\hat{\sigma}}| \leqslant \overline{p}(\mathtt{w})$. Now consider a singular stratum S of X and a regular simplex $\sigma\colon \Delta_\sigma = \Delta_0 * \cdots * \Delta_{n+1} \to \mathring{\mathtt{c}}X$, such that $\sigma(\Delta_\sigma) \cap (S\times]0,\infty[) \neq \emptyset$. Let $\ell = \operatorname{codim}_{X\times]0,\infty[}(S\times]0,\infty[)$ and notice the equivalence of the conditions $\sigma(\Delta_\sigma) \cap (S\times]0,\infty[) \neq \emptyset$ and $\hat{\sigma}(\Delta_{\hat{\sigma}}) \cap (S\times]0,\infty[) \neq \emptyset$. For such stratum, we have $\ell \in \{1,\ldots,n\}$ and $\|f(\omega)_\sigma\|_\ell = \|\lambda_{\mathtt{c}\Delta_0} \otimes \omega_{\hat{\sigma}}\|_\ell = \|\omega_{\hat{\sigma}}\|_\ell$. The result is a consequence of the inequality $\|\omega_{\hat{\sigma}}\|_\ell \leqslant \|\omega\|_{S\times]0,\infty[} \leqslant \overline{p}(S\times]0,\infty[)$, arising from the $\overline{p}$-allowability of ω. □

Remark 12.3. Nous avons toujours

$$\omega \in \widetilde{N}^*(X\times]0,\infty[;R) \Longrightarrow f(\omega) \in \widetilde{N}^*(\mathring{\mathtt{c}}X;R)$$

Construction of $g\colon \widetilde{N}^*(\mathring{\mathtt{c}}X;R) \to \widetilde{N}^*(X\times]0,\infty[;R)$.

Let $\omega \in \widetilde{N}^*(\mathring{\mathtt{c}}X;R)$ and $\tau\colon \Delta_\tau \to X\times]0,\infty[$ a regular simplex. We denote by $\mathtt{c}\tau\colon \Delta_{c\tau} = \{\mathtt{p}\} * \Delta_\tau \to \mathring{\mathtt{c}}X$ the cone over τ defined above. Notice $\widetilde{\Delta_{\mathtt{c}\tau}} = \mathtt{c}\{\mathtt{p}\} \times \widetilde{\Delta_\tau}$.

Let $\mathtt{v}_0$ *be the apex of the cone over the component of filtration degree 0 of a filtered simplex.* The cone $\mathtt{c}\{\mathtt{p}\}$ having two vertices $\mathtt{p}$ and $\mathtt{v}_0$, the cochain $\omega_{\mathtt{c}\tau}$ decomposes into

$$\omega_{\mathtt{c}\tau} = \mathbf{1}_{\mathtt{p}} \otimes \gamma_{\mathtt{p}} + \mathbf{1}_{\mathtt{v}_0} \otimes \gamma_{\mathtt{v}_0} + \mathbf{1}_{\mathtt{p}*\mathtt{v}_0} \otimes \gamma'_{\mathtt{v}_0}, \tag{40}$$

with $\gamma_{\mathtt{p}}, \gamma_{\mathtt{v}_0}, \gamma'_{\mathtt{v}_0} \in \widetilde{N}^*(\Delta_\tau)$. We set

$$g(\omega)_\tau = \gamma_{\mathtt{v}_0}.$$

Proposition 12.4. *Let* $(\mathring{\mathtt{c}}X, \overline{p})$ *be a perverse space with* X *compact and* $(X\times]0,\infty[, \overline{p})$ *the induced perverse space. The correspondence defined above induces a cochain map,*

$$g\colon \tau_{\leqslant \overline{p}(\mathtt{w})}\widetilde{N}^*_{\overline{p}}(\mathring{\mathtt{c}}X;R) \to \tau_{\leqslant \overline{p}(\mathtt{w})}\widetilde{N}^*_{\overline{p}}(X\times]0,\infty[;R).$$

The proof follows the pattern of that of Proposition 12.2; we leave it to the reader.

Remark 12.5. Nous avons toujours

$$\omega \in \widetilde{N}^*(\mathring{\mathtt{c}}X;R) \Longrightarrow g(\omega) \in \widetilde{N}^*(X\times]0,\infty[;R)$$

Specify the compositions $f\circ g$ *and* $g\circ f$*.*

(a) Let $\omega \in \widetilde{N}^*(X;R)$. Consider a regular simplex, $\tau\colon \Delta_\tau \to X\times]0,\infty[$ and its associated map $\mathtt{c}\tau\colon \{\mathtt{p}\} * \Delta_\tau \to \mathring{\mathtt{c}}X$. Following (39), one has

$$f(\omega)_{\mathtt{c}\tau} = \lambda_{\mathtt{c}\{\mathtt{p}\}} \otimes \omega_\tau = \mathbf{1}_{\mathtt{p}} \otimes \omega_\tau + \mathbf{1}_{\mathtt{v}_0} \otimes \omega_\tau.$$

It follows, according to (40), $g(f(\omega))_\tau = \omega_\tau$ and $g\circ f = \mathrm{id}$.

(b) Let $\omega \in \widetilde{N}^*(\mathring{\mathtt{c}}X;R)$. Consider a regular simplex $\sigma\colon \Delta_\sigma = \Delta_0 * \cdots * \Delta_{n+1} \to \mathring{\mathtt{c}}X$, and its associated map $\mathtt{c}\sigma\colon (\{\mathtt{p}\} * \Delta_0) * \cdots * \Delta_{n+1} \to \mathring{\mathtt{c}}X$. The cochain $\omega_{\mathtt{c}\sigma}$ decomposes into

$$\omega_{\mathtt{c}\sigma} = \underbrace{\sum_{F\lhd \mathtt{c}\Delta_0} \mathbf{1}_F \otimes \gamma_F}_{\omega_\sigma} + \sum_{F\lhd \mathtt{c}\Delta_0} \mathbf{1}_{\mathtt{p}*F} \otimes \gamma'_F + \mathbf{1}_{\mathtt{p}} \otimes \gamma_\emptyset. \tag{41}$$

Since the cochain ω is globally defined and the simplex $\mathtt{c}\hat{\sigma}$ is a face of $\mathtt{c}\sigma$, we deduce $\omega_{\mathtt{c}\hat{\sigma}} = \mathbf{1}_{\mathtt{v}_0} \otimes \gamma_{\mathtt{v}_0} + \mathbf{1}_{\mathtt{p}*\mathtt{v}_0} \otimes \gamma'_{\mathtt{v}_0} + \mathbf{1}_{\mathtt{p}} \otimes \gamma_\emptyset$. It follows:

$$f(g(\omega))_\sigma = \lambda_{\mathtt{c}\Delta_0} \otimes g(\omega)_{\hat{\sigma}} = \lambda_{\mathtt{c}\Delta_0} \otimes \gamma_{\mathtt{v}_0}. \tag{42}$$

Construction of a homotopy $H\colon \widetilde{N}^*(\mathring{\mathtt{c}}X;R) \to \widetilde{N}^{*-1}(\mathring{\mathtt{c}}X;R)$**.**

If $\sigma\colon \Delta_\sigma = \Delta_0 * \cdots * \Delta_{n+1} \to \mathring{\mathtt{c}}X$ is a regular simplex, we define a map $H\colon \widetilde{N}^*(\Delta_{\mathtt{c}\sigma}) \to \widetilde{N}^{*-1}(\Delta_\sigma)$, i.e.,

$$H\colon N^*(\mathtt{c}(\mathtt{p} * \Delta_0)) \otimes N^*(\mathtt{c}\Delta_1) \otimes \cdots \otimes N^*(\Delta_{n+1}) \to N^{*-1}(\mathtt{c}\Delta_0) \otimes \cdots \otimes N^*(\Delta_{n+1}).$$

We decompose $\omega_{\mathtt{c}\sigma} \in \widetilde{N}^*(\Delta_{\mathtt{c}\sigma})$ as in the formula (41) and set:

$$(H(\omega))_\sigma = \sum_{\mathtt{v}_0 \neq F \lhd \mathtt{c}\Delta_0} (-1)^{|F|+1} \mathbf{1}_F \otimes \gamma'_F + \lambda_{\Delta_0} \otimes \gamma'_{\mathtt{v}_0}, \tag{43}$$

where $\mathtt{v}_0$ is the apex of the cone over the component filtration of degree 0 and λ_{Δ_0} the sum of 0-cochains on Δ_0.

Proposition 12.6. *Let* $(\mathring{\mathtt{c}}X, \overline{p})$ *be a perverse space with* X *compact.*

(a) *The equality (43) induces a linear map,* $H\colon \widetilde{N}^*(\mathring{\mathtt{c}}X;R)\to \widetilde{N}^{*-1}(\mathring{\mathtt{c}}X;R)$*, verifying*

$$\delta\circ H+H\circ\delta=\mathrm{id}-f\circ g.$$

(b) *Using the notation introduced in (38), the application* H *induces a map,*

$$H\colon \tau_{\leqslant\overline{p}(\mathtt{w})}\widetilde{N}^*_{\overline{p}}(\mathring{\mathtt{c}}X;R)\to \tau_{\leqslant\overline{p}(\mathtt{w})}\widetilde{N}^{*-1}_{\overline{p}}(\mathring{\mathtt{c}}X;R).$$

Proof. (a) We must establish the equality $\delta_k^*H(\omega)_\sigma=H(\omega)_{\sigma\circ\delta_k}$, for each cochain $\omega\in\widetilde{N}^*(\mathring{\mathtt{c}}X)$, each regular simplex $\sigma\colon\Delta_\sigma=\Delta_0*\cdots*\Delta_{n+1}\to\mathring{\mathtt{c}}X$ and each regular face operator $\delta_k\colon D=D_0*\cdots*D_{n+1}\to\Delta_\sigma$, with $k\in\{0,\ldots,\dim\Delta_\sigma\}$. Denoting by $\delta^{\mathtt{p}}_*$ the regular face operators of $\{\mathtt{p}\}*\Delta_\sigma$, we can write $\mathtt{c}(\sigma\circ\delta_k)=\mathtt{c}\sigma\circ\delta^{\mathtt{p}}_{k+1}$. If $k>\dim\Delta_0$, we set $k^\circ=k-\dim\Delta_0-1$ and $\delta^\circ_{k^\circ}\colon D_1*\cdots*D_{n+1}\to\Delta_1*\cdots*\Delta_{n+1}$ the induced face. Following (43), we have:

$$\delta_k^*H(\omega)_\sigma=\left\{\begin{array}{cl}\sum_{\mathtt{v}_0\neq F\lhd\mathtt{c}D_0}(-1)^{|F|+1}\mathbf{1}_F\otimes\gamma'_F+\lambda_{D_0}\otimes\gamma'_{\mathtt{v}_0} & \text{if } k\leqslant\dim\Delta_0,\\ \sum_{\mathtt{v}_0\neq F\lhd\mathtt{c}\Delta_0}(-1)^{|F|+1}\mathbf{1}_F\otimes\delta^{\circ,*}_{k^\circ}\gamma'_F+\lambda_{\Delta_0}\otimes\delta^{\circ,*}_{k^\circ}\gamma'_{\mathtt{v}_0} & \text{if } k>\dim\Delta_0.\end{array}\right.$$

Using the equality $\omega_{\mathtt{c}(\sigma\circ\delta_k)}=\omega_{\mathtt{c}\sigma\circ\delta^{\mathtt{p}}_{k+1}}=\delta^{\mathtt{p},*}_{k+1}\omega_{\mathtt{c}\sigma}$ and (41), we get:

$$\omega_{\mathtt{c}(\sigma\circ\delta_k)}=\left\{\begin{array}{cl}(\sum_{F\lhd\mathtt{c}D_0}\mathbf{1}_F\otimes\gamma_F+\mathbf{1}_{\mathtt{p}*F}\otimes\gamma'_F)+\mathbf{1}_{\mathtt{p}}\otimes\gamma_\varnothing & \text{if } k\leqslant\dim\Delta_0,\\ (\sum_{F\lhd\mathtt{c}D_0}\mathbf{1}_F\otimes\delta^{\circ,*}_{k^\circ}\gamma_F+\mathbf{1}_{\mathtt{p}*F}\otimes\delta^{\circ,*}_{k^\circ}\gamma'_F)+\mathbf{1}_{\mathtt{p}}\otimes\delta^{\circ,*}_{k^\circ}\gamma_\varnothing & \text{if } k>\dim\Delta_0,\end{array}\right.$$

which gives

$$\begin{array}{rcl}H(\omega)_{\sigma\circ\delta_k}&=&\left\{\begin{array}{cl}(\sum_{\mathtt{v}_0\neq F\lhd\mathtt{c}D_0}(-1)^{|F|+1}\mathbf{1}_F\otimes\gamma'_F)+\lambda_{D_0}\otimes\gamma'_{\mathtt{v}_0}, & \text{if } k\leqslant\dim\Delta_0,\\ (\sum_{\mathtt{v}_0\neq F\lhd\mathtt{c}D_0}(-1)^{|F|+1}\mathbf{1}_F\otimes\delta^{\circ,*}_{k^\circ}\gamma'_F)+\lambda_{D_0}\otimes\delta^{\circ,*}_{k^\circ}\gamma'_{\mathtt{v}_0} & \text{if } k>\dim\Delta_0,\end{array}\right.\\ &=&\delta_k^*H(\omega)_\sigma.\end{array}$$

Let us study the behavior of H towards the differentials. Let $\sigma\colon\Delta_\sigma=\Delta_0*\cdots*\Delta_{n+1}\to\mathring{\mathtt{c}}X$ be a regular simplex and $\omega\in\widetilde{N}^*(\mathring{\mathtt{c}}X;R)$. Let us start from the equality (41) and calculate the differential in eliminating terms having a zero image by H.

$$(H(\delta\omega))_\sigma = H\left(\sum_{e\in\mathcal{V}(\mathtt{c}\Delta_0)}\mathbf{1}_{p*e}\otimes\gamma_\varnothing+\sum_{F\lhd\mathtt{c}\Delta_0}\mathbf{1}_{F*\mathtt{p}}\otimes\gamma_F+\sum_{\substack{F\lhd\mathtt{c}\Delta_0\\ e\in\mathcal{V}(\mathtt{c}\Delta_0)}}\mathbf{1}_{\mathtt{p}*F*e}\otimes\gamma'_F+\sum_{F\lhd\mathtt{c}\Delta_0}(-1)^{|F|+1}\mathbf{1}_{\mathtt{p}*F}\otimes\delta\gamma'_F\right).\tag{44}$$

Notice

$$\sum_{e\in\mathcal{V}(\mathtt{c}\Delta_0)}\mathbf{1}_{p*e}\otimes\gamma_\varnothing+\sum_{F\lhd\mathtt{c}\Delta_0}\mathbf{1}_{F*\mathtt{p}}\otimes\gamma_F=$$
$$\sum_{e\in\mathcal{V}(\Delta_0)}\mathbf{1}_{p*e}\otimes\gamma_\varnothing+\sum_{\mathtt{v}_0\neq F\lhd\mathtt{c}\Delta_0}\mathbf{1}_{F*\mathtt{p}}\otimes\gamma_F+\mathbf{1}_{\mathtt{p}*\mathtt{v}_0}\otimes(\gamma_\varnothing-\gamma_{\mathtt{v}_0}).$$

Replacing in (44) and developing the definition of H, we get:

$$\begin{aligned}(H(\delta\omega))_\sigma &= -\sum_{e\in\mathcal{V}(\Delta_0)} \mathbf{1}_e\otimes\gamma_\varnothing + \sum_{\mathtt{v}_0\neq F\lhd \mathtt{c}\Delta_0} \mathbf{1}_F\otimes\gamma_F + \lambda_{\Delta_0}\otimes(\gamma_\varnothing-\gamma_{\mathtt{v}_0}-\delta\gamma'_{\mathtt{v}_0})\\ &\quad + \sum_{\substack{F\lhd \mathtt{c}\Delta_0\\ e\in\mathcal{V}(\mathtt{c}\Delta_0)}} (-1)^{|F|}\mathbf{1}_{F*e}\otimes\gamma'_F + \sum_{\mathtt{v}_0\neq F\lhd\mathtt{c}\Delta_0}\mathbf{1}_F\otimes\delta\gamma'_F.\end{aligned}$$

On the other hand, the quantity $\delta\circ H$ can be written

$$\begin{aligned}\delta(H(\omega))_\sigma &= \sum_{\substack{\mathtt{v}_0\neq F\lhd\mathtt{c}\Delta_0\\ e\in\mathcal{V}(\mathtt{c}\Delta_0)}} (-1)^{|F|+1}\mathbf{1}_{F*e}\otimes\gamma'_F - \sum_{\mathtt{v}_0\neq F\lhd\mathtt{c}\Delta_0}\mathbf{1}_F\otimes\delta\gamma'_F\\ &\quad + \sum_{e\in\mathcal{V}(\Delta_0)}\mathbf{1}_{e*\mathtt{v}_0}\otimes\gamma'_{\mathtt{v}_0} + \lambda_{\Delta_0}\otimes\delta\gamma'_{\mathtt{v}_0}.\end{aligned}$$

Using (42), the sum of the two expressions can be reduced to:

$$H(\delta\omega)_\sigma + \delta H(\omega)_\sigma = \sum_{F\lhd\mathtt{c}\Delta_0}\mathbf{1}_F\otimes\gamma_F - \lambda_{\mathtt{c}\Delta_0}\otimes\gamma_{\mathtt{v}_0} = \omega_\sigma - (f\circ g)(\omega)_\sigma. \tag{45}$$

(b) As in the proof of Proposition 12.2, we are reduced to consider a singular stratum T of $\mathring{\mathtt{c}}X$, a $\overline{p}$-allowable cochain $\omega\in\widetilde{N}^*(\mathring{\mathtt{c}}X)$, of degree less than or equal to $\overline{p}(\mathtt{w})$, and a regular simplex, $\sigma\colon\Delta_\sigma\to\mathring{\mathtt{c}}X$, with $\sigma(\Delta_\sigma)\cap T\neq\varnothing$. Let $\ell=\operatorname{codim}_{\mathring{\mathtt{c}}X}T\in\{1,\dots,n+1\}$. Notice that $\mathtt{c}\sigma(\Delta_{\mathtt{c}\sigma})\cap T\neq\varnothing$. Thus, according to the definition of perverse degree (cf. Definition 2.7), we have

$$\begin{aligned}\overline{p}(T) &\geqslant \|\omega\|_T \geqslant \|\omega_{\mathtt{c}\sigma}\|_\ell = \max\{\|\mathbf{1}_F\otimes\gamma_F\|_\ell,\ \|\mathbf{1}_{\mathtt{p}*F}\otimes\gamma'_F\|_\ell,\ \|\mathbf{1}_{\mathtt{p}}\otimes\gamma_\varnothing\|_\ell \mid F\lhd\mathtt{c}\Delta_0\}\\ &\geqslant \max\{\|\mathbf{1}_{\mathtt{p}*F}\otimes\gamma'_F\|_\ell \mid F\lhd\mathtt{c}\Delta_0\},\end{aligned}$$

where the equality uses (5) and (41). We develop this expression by distinguishing two cases.

- Let $\ell\neq n+1$. By definition, for any face $F\lhd\mathring{\mathtt{c}}\Delta_0$, we have the equality $\|\mathbf{1}_{\mathtt{p}*F}\otimes\gamma'_F\|_\ell = \|\mathbf{1}_F\otimes\gamma'_F\|_\ell$, if $F\neq\mathtt{v}_0$, and $\|\mathbf{1}_{\mathtt{p}*\mathtt{v}_0}\otimes\gamma'_{\mathtt{v}_0}\|_\ell = \|\lambda_{\Delta_0}\otimes\gamma'_{\mathtt{v}_0}\|_\ell$. It follows:
$$\overline{p}(T)\geqslant\max\{\|\mathbf{1}_F\otimes\gamma'_F\|_\ell,\ \|\lambda_{\Delta_0}\otimes\gamma'_{\mathtt{v}_0}\|_\ell \mid \mathtt{v}_0\neq F\lhd\mathtt{c}\Delta_0\} = \|H(\omega)_\sigma\|_\ell,$$
where the last equality uses (5) and (43). We deduce $\overline{p}(T)\geqslant\|H(\omega)\|_T$.
- Let $\ell=n+1$. In this case we have
$$\begin{aligned}\|H(\omega)_\sigma\|_{n+1} &= \max\{\|\mathbf{1}_{\mathtt{p}*F}\otimes\gamma'_F\|_{n+1},\ \|\lambda_{\Delta_0}\otimes\gamma'_{\mathtt{v}_0}\|_{n+1} \mid \mathtt{v}_0\neq F\lhd\mathtt{c}\Delta_0\}\\ &= \max\{|\gamma'_F|,\ |\gamma'_{\mathtt{v}_0}| \mid F\lhd\Delta_0\}\leqslant|\omega|-1\leqslant\overline{p}(\mathtt{w}),\end{aligned}$$
where the first equality uses (5) and (43). It follows $\overline{p}(\mathtt{w})\geqslant\|H(\omega)\|_{\mathtt{w}}$.

It has been shown $\|H(\omega)\|\leqslant\overline{p}$. The property $\|\delta H(\omega)\|\leqslant\overline{p}$ is deduced using (45). □

Remark 12.7. Nous avons toujours

$$\omega\in\widetilde{N}^*(\mathring{\mathtt{c}}X;R)\Longrightarrow H(\omega)\in\widetilde{N}^*(\mathring{\mathtt{c}}X;R)$$

et

$$\delta\circ H + H\circ\delta = \mathrm{id} - f\circ g.$$

The determination of the cohomology of a cone follows from the properties of f, g and H.

Proof of Theorem E. (a) From Propositions 12.2, 12.4 and 12.6, we deduce that the map $g\colon \tau_{\leqslant\overline{p}(\mathtt{w})}\widetilde{N}^*_{\overline{p}}(\mathring{\mathtt{c}}X;R) \to \tau_{\leqslant\overline{p}(\mathtt{w})}\widetilde{N}^*_{\overline{p}}(X\times]0,\infty[;R)$ is a quasi-isomorphism. We know from Theorem D, that the inclusion $I_1\colon X \to X\times]0,\infty[$ induces a quasi-isomorphism. It remains to prove $I_1^*\circ g = \iota^*$. The involved stratifications on $X\times]0,\infty[$ are different in these two quasi-isomorphisms but the cohomologies are the same (see 3.5).

For this, consider $\omega \in \tau_{\overline{p}(\mathtt{w})}\widetilde{N}^*_{\overline{p}}(\mathring{\mathtt{c}}X;R)$ and $\tau\colon \Delta_\tau \to X$ a regular simplex. By definition, we have $(\iota^*\omega)_\tau = \omega_{\iota\circ\tau}$. Notice $\iota\circ\tau = \mathtt{c}(I_1\circ\tau)\circ\delta_0$, where $\delta_0(x) = 0\cdot\mathtt{p} + 1\cdot x$. It then follows

$$(\iota^*\omega)_\tau = \omega_{\mathtt{c}(I_1\circ\tau)\circ\delta_0} = \delta_0^*\,\omega_{\mathtt{c}(I_1\circ\tau)} =_{(40)} \gamma_{\mathtt{v}_0} = g(\omega)_{I_1\circ\tau} = I_1^*(g(\omega))_\tau.$$

To prove part (b), we consider a cocycle $\omega \in \widetilde{N}^k_{\overline{p}}(\mathring{\mathtt{c}}X;R)$ with $k > \overline{p}(\mathtt{w})$. Following Proposition 12.6, we have $\delta H(\omega) = \omega - f(g(\omega))$ and it suffices to establish the equality $f(g(\omega)) = 0$ and $H(\omega) \in \widetilde{N}^*_{\overline{p}}(X;R)$.

We prove $f(g(\omega)) = 0$ by contradiction, assuming that there is a regular simplex $\sigma\colon \Delta_\sigma \to \mathring{\mathtt{c}}X$, such that $f(g(\omega))_\sigma \neq 0$. Following (42), this implies $\gamma_{\mathtt{v}_0} \neq 0$. We get a contradiction,

$$\begin{aligned} k > \overline{p}(\mathtt{w}) &\geqslant \|f(g(\omega))\|_{\mathtt{w}} \geqslant \|f(g(\omega))_{\mathtt{c}\sigma}\|_{n+1} = \|\lambda_{\mathtt{c}(\mathtt{p}*\Delta_0)}\otimes\gamma_{\mathtt{v}_0}\|_{n+1} \\ &= \|\lambda_{\mathtt{p}*\Delta_0}\otimes\gamma_{\mathtt{v}_0}\|_{n+1} = |\gamma_{\mathtt{v}_0}| = k. \end{aligned}$$

Notice that $\delta H(\omega) = \omega$. But $H(\omega)$ does not belong to $\widetilde{N}^{k-1}_{\overline{p}}(\mathring{\mathtt{c}}X;R)$. We need to find another cochain integrating ω. In fact, we are going to construct $B(\omega) \in \widetilde{N}^{k-1}(\mathring{\mathtt{c}}X;R)$ and $C(\omega) \in \widetilde{N}^{k-1}_{\overline{p}}(\mathring{\mathtt{c}}X;R)$ verifying:

+ $H(\omega) - B(\omega) \in \widetilde{N}^{k-1}_{\overline{p}}(\mathring{\mathtt{c}}X;R)$
+ $\delta B(\omega) = \delta C(\omega)$.

This ends the proof since

$$\omega = \delta\,(H(\omega) - B(\omega)) + \delta C(\omega),$$

and $H(\omega) - B(\omega), C(\omega) \in \widetilde{N}^{k-1}_{\overline{p}}(\mathring{\mathtt{c}}X;R)$

In order to define these cochains, we consider a regular simplex $\sigma\colon \Delta_\sigma \to \mathring{\mathtt{c}}X$. Using (41) we define:

$$\begin{aligned} B(\omega) &= \lambda_{\Delta_0}\otimes\gamma'_{\mathtt{v}_0} \\ C(\omega) &= -\mathbf{1}_{\mathtt{v}_0}\otimes\gamma'_{\mathtt{v}_0} \end{aligned}$$

We proceed in three steps.

1 - $B(\omega), C(\omega) \in \widetilde{N}^{k-1}(\mathring{\mathtt{c}}X;R)$.

Vérifions la compatibilité aux faces. We prove $\delta_k^*B(\omega)_\sigma = B(\omega)_{\sigma\circ\delta_k}$ and $\delta_k^*C(\omega)_\sigma = C(\omega)_{\sigma\circ\delta_k}$ for each regular simplex, $\sigma\colon \Delta_\sigma = \Delta_0 * \cdots * \Delta_{n+1} \to \mathring{\mathtt{c}}X$, and any regular face operator, $\delta_k\colon \nabla = \nabla_0 * \cdots * \nabla_{n+1} \to \Delta_\sigma$, with $k \in \{0,\ldots,\dim\Delta_\sigma\}$.

We have

$$\delta_k^*B(\omega)_\sigma = \delta_k^*(\lambda_{\Delta_0}\otimes\gamma'_{\mathtt{v}_0}) = \lambda_{\nabla_0}\otimes\gamma'_{\mathtt{v}_0} = B(\omega)_{\sigma\circ\delta_k}.$$

and

$$\delta_k^* C(\omega)_\sigma = -\delta_k^*(\mathbf{1}_{\mathtt{v}_0} \otimes \gamma'_{\mathtt{v}_0}) = -\mathbf{1}_{\mathtt{v}_0} \otimes \gamma'_{\mathtt{v}_0} = C(\omega)_{\sigma\circ\delta_k}.$$

2 - $H(\omega) - B(\omega), C(\omega) \in \widetilde{N}^{k-1}_{\overline{p}}(\mathring{\mathtt{c}}X;R)$.

We need to prove $\max\{||H(\omega)_\sigma - B(\omega)_\sigma||_\ell, ||C(\omega)_\sigma||_\ell\} \leqslant \overline{p}(T)$ for each stratum T of $\mathring{\mathtt{c}}X$ meeting $\operatorname{Im}\sigma$, where $\ell = \operatorname{codim} T$. We distinguish two cases.

(a) $\ell \neq n+1$.
(b) $\ell = n+1$.

Let us see that.

(a) We notice that we have already prove that $||H(\omega)_\sigma||_\ell \leqslant \overline{p}(T)$. So, it suffices to prove that:

$$\max\{||B(\omega)_\sigma||_\ell, ||C(\omega)_\sigma||_\ell\} \leqslant \overline{p}(T).$$

We have

$$\begin{array}{rcl} \|H(\omega)_\sigma\|_{n+1} &=& \max\{\|\mathbf{1}_{\mathtt{p}*F} \otimes \gamma'_F\|_{n+1},\ \|\lambda_{\Delta_0} \otimes \gamma'_{\mathtt{v}_0}\|_{n+1} \mid \mathtt{v}_0 \neq F \lhd \mathtt{c}\Delta_0\} \\ &=& \max\{|\gamma'_F|,\ |\gamma'_{\mathtt{v}_0}| \mid F \lhd \Delta_0\} \leqslant |\omega| - 1 \leqslant \overline{p}(\mathtt{w}), \end{array}$$

(b) From (43) we have

$$H(\omega)_\sigma - B(\omega)_\sigma = \sum_{\mathtt{v}_0 \neq F \lhd \mathtt{c}\Delta_0} (-1)^{|F|+1} \mathbf{1}_F \otimes \gamma'_F$$

We have

$$\begin{array}{rcl} \max\{||H(\omega)_\sigma - B(\omega)_\sigma||_{n+1}, ||C(\omega)_\sigma||_{n+1}\} &=& \max\{||\mathbf{1}_F \otimes \gamma'_F||_{n+1}, ||\mathbf{1}_{\mathtt{v}_0} \otimes \gamma'_{\mathtt{v}_0}||_{n+1} \mid \mathtt{v}_0 \neq F \lhd \mathtt{c}\Delta_0\} \\ &=& \max\{||\mathbf{1}_F \otimes \gamma'_F||_{n+1} \mid F \lhd \Delta_0\} \\ &=& \max\{||\mathbf{1}_{\mathtt{p}*F} \otimes \gamma'_F||_{n+1} \mid F \lhd \Delta_0\} \overset{(41)}{\leqslant} ||\omega_{\mathring{\mathtt{c}}\sigma}||_{n+1} \leqslant ||\omega||_{\mathtt{w}} \leqslant \overline{p}(\mathtt{w}). \end{array}$$

3 - $\delta B(\omega) = \delta C(\omega)$.

It suffices tor prove that $\delta B(\omega)_\sigma = \delta C(\omega)_\sigma$. If $\delta\gamma'_{\mathtt{v}_0} = 0$ then we have

$$\delta B(\omega)_\sigma = \delta(\lambda_{\Delta_0} \otimes \gamma'_{\mathtt{v}_0}) = \sum_{e \in \mathcal{V}(\Delta_0)} \mathbf{1}_{e*\mathtt{v}_0} \otimes \gamma'_{\mathtt{v}_0} = -\delta(\mathbf{1}_{\mathtt{v}_0} \otimes \gamma'_{\mathtt{v}_0}) = \delta C(\omega_\sigma).$$

Let us see that $\delta\gamma'_{\mathtt{v}_0} = 0$. Following (41) we have

$$\omega_{\mathring{\mathtt{c}}\sigma} = \mathbf{1}_{\mathtt{v}_0} \otimes \gamma_{\mathtt{v}_0} + \mathbf{1}_{\mathtt{p}} \otimes \gamma_\varnothing + \mathbf{1}_{\mathtt{p}*\mathtt{v}_0} \otimes \gamma'_{\mathtt{v}_0} + \cdots.$$

Since $\delta\omega_{\mathtt{c}\sigma} = 0$, then the $\mathbf{1}_{\mathtt{p}*\mathtt{v}_0}$-term of $\delta\omega_{\mathtt{c}\sigma}$ is 0. We get

$$-\gamma_{\mathtt{v}_0} + \gamma_\varnothing - \delta\gamma'_{\mathtt{v}_0} = 0.$$

We have already seen that $\gamma_{\mathtt{v}_0} = 0$. On the other hand, si $\gamma_\varnothing \neq 0$ then we have

$$\overline{p}(\mathtt{w}) \geqslant ||\omega_\sigma||_{n+1} \geqslant ||\mathbf{1}_{\mathtt{p}} \otimes \gamma_\varnothing||_{n+1} = |\gamma_\varnothing| = |\omega_\sigma| = |\omega| = k > \overline{p}(\mathtt{w}).$$

This implies $\gamma_\varnothing = 0$ and therefore $\delta\gamma'_{\mathtt{v}_0} = 0$. $\square$