Cosymplectic reduction for singular momentum maps

J. Phys. A: Math. Gen. 26 (1993) 5033–5043. Printed in the UK

Cosymplectic reduction for singular momentum maps†

Manuel de León‡ and Martín Saralegi§

Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain

Received 11 January 1993, in final form 8 June 1993

Abstract. In this paper we prove that the cosymplectic reduction of cosymplectic manifolds with symmetry due to C Albert may be obtained from the Marsden–Weinstein reduction theory. We also study the reduction of cosymplectic manifolds with singular momentum map by using the results of Sjamaar and Lerman for the symplectic case.

1. Introduction

As is well known, the existence of symmetries allows us to reduce Hamiltonian systems. More precisely, if G is a Lie group of symmetries of a Hamiltonian system (M, ω, H) with a momentum map $J : M \longrightarrow \mathfrak{g}^*$, then we obtain a reduced Hamiltonian system (M_0, ω_0, H_0), where $0 \in \mathfrak{g}^*$ is a regular value of J and $M_0 = J^{-1}(0)/G$. This is the statement of the Marsden and Weinstein theorem [1, 8, 9, 10]. But if 0 is not a regular value, then M_0 is not a symplectic manifold. In fact, it is not even a manifold. Sjamaar and Lerman [11] have proved that in this case M_0 is a stratified space supporting a natural Poisson structure whose restriction to the strata defines a symplectic structure.

On the other hand the reduction of time-dependent Hamiltonian systems has been recently developed by Albert [3, 4] in the framework of cosymplectic manifolds. The reduced Hamiltonian system is defined on a reduced cosymplectic manifold M_0. The purpose of this paper is to extend the Albert reduction procedure to the case of singular values. To do this we first reformulate the cosymplectic reduction theorem of Albert. In fact, we show that, by extending the phase space, the reduced Hamiltonian system can be obtained directly from the Marsden and Weinstein reduction procedure. Using this construction we prove that for a singular value $0 \in \mathfrak{g}^*$, M_0 is a cosymplectic stratified space.

The paper is structured as follows. In section 2, we give a brief background on cosymplectic manifolds. Section 3 is devoted to obtaining the cosymplectic reduction theorem from the symplectic one by extending the phase space. In section 4 we recall the main results of Sjamaar and Lerman. The cosymplectic reduction for singular values is developed in section 5. Finally, we study the dynamics on the reduced cosymplectic stratified space.

† Supported by DGICYT-SPAIN, Proyecto PB91-0142.

‡ E-mail: CEEML02 @ CC.CSIC.ES.

§ E-mail: SARALEGI @ CC.CSIC.ES.

2. Cosymplectic manifolds

A *cosymplectic manifold* is a triple (M, Ω, η) consisting of a smooth $(2n+1)$-dimensional manifold M with a closed 2-form Ω and a closed 1-form η, such that $\Omega^n \wedge \eta \neq 0$ (see [7, 6]). In particular, $\Omega^n \wedge \eta$ yields a volume form on M. Consider the bundle homomorphism

$$b : TM \longrightarrow T^*M \qquad X \in T_xM \rightarrow b(X) = i_X\Omega(x) + (i_X\eta(x))\eta(x).$$

Then b is a vector bundle isomorphism. We denote by R the *Reeb vector field*, defined by

$$i_R\Omega = 0 \qquad i_R\eta = 1$$

i.e. $R = b^{-1}(\eta)$.

There exist, in the neighbourhood of every point, canonical coordinates (t, q^i, p_i), $i = 1, \ldots, n$, such that

$$\Omega = \mathrm{d}q^i \wedge \mathrm{d}p_i, \eta = \mathrm{d}t$$

(t, q^i, p_i) will be called *Darboux coordinates*. Then we have $R = \partial/\partial t$.

To each function $f \in C^\infty(M)$ one can associate three vector fields on M:

(1) The *gradient vector field* grad f, which is defined by

$$\operatorname{grad} f = b^{-1}(\mathrm{d}f).$$

or, equivalently,

$$i_{\operatorname{grad} f}\Omega = \mathrm{d}f - R(f)\eta \qquad i_{\operatorname{grad} f}\eta = R(f).$$

(2) The *Hamiltonian vector field* X_f according to

$$X_f = b^{-1}(\mathrm{d}f - R(f)\eta)$$

or, equivalently,

$$i_{X_f}\Omega = \mathrm{d}f - R(f)\eta \qquad i_{X_f}\eta = 0.$$

(3) The *evolution vector field* $E_f = R + X_f$.

In Darboux coordinates we find

$$\begin{aligned} \operatorname{grad} f &= \frac{\partial f}{\partial t}\frac{\partial}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} \\ X_f &= \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} \\ E_f &= \frac{\partial}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}\,. \end{aligned}$$

Cosymplectic manifolds are a natural framework to develop the geometric formulation of time-dependent Hamiltonian systems (see [3, 6, 7]). The dynamics on a cosymplectic manifold (M, Ω, η) are introduced by giving a Hamiltonian function $h \in C^\infty(M)$. In fact, the integral curves of the evolution vector field E_h satisfy the *Hamilton* or *motion equations* corresponding to h:

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial h}{\partial p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial h}{\partial q^i}\,.$$

An alternative way to introduce the evolution vector field E_h is as follows: first we modify the cosymplectic structure (Ω, η) to obtain a new cosymplectic structure $(\Omega_h = \Omega + dh \wedge \eta, \eta)$. Then E_h is just the Reeb vector field of the modified cosymplectic structure (Ω_h, η).

The *Poisson bracket* of two functions $f, f' \in C^\infty(M)$ is defined by

$$\{f, f'\} = \Omega(\operatorname{grad} f, \operatorname{grad} f') = \Omega(X_f, X_{f'}) = \Omega(E_f, E_{f'}) .$$

Then M turns out to be a Poisson manifold, the symplectic leaves of which are precisely the leaves of the integrable distribution $\ker \eta$.

If $h \in C^\infty(M)$ is a Hamiltonian function on M, then we have

$$E_h f = X_h f + R(f) \tag{2.1}$$

for any function $f \in C^\infty(M)$. In terms of Poisson brackets (2.1) becomes

$$E_h f = \{f, h\} + R(f) . \tag{2.2}$$

From (2.2) we deduce that the flow $\gamma_h(t, x)$ of E_h is characterized by

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_h(t, x)) = \{f, h\}(\gamma_h(t, x)) + R(f)(\gamma_h(t, x)) . \tag{2.3}$$

An *automorphism* of the cosymplectic structure (M, Ω, η) is a diffeomorphism $\phi : M \longrightarrow M$ such that

$$\phi^*\Omega = \Omega \qquad \phi^*\eta = \eta.$$

3. Cosymplectic reduction of cosymplectic manifolds with symmetry

Suppose that there exists a left action $\Phi : G \times M \longrightarrow M$ of a Lie group G on a cosymplectic manifold (M, Ω, η). We always assume that both G and M are connected. The Lie algebra of G will be denoted by $\mathfrak{g}$ and its dual by $\mathfrak{g}^*$. For each $g \in G$ we let $\Phi_g \equiv \Phi(g, \cdot)$, the induced transformation on M. The fundamental vector field, or infinitesimal generator, associated with $\xi \in \mathfrak{g}$ is the vector field ξ_M on M defined by $\xi_M(x) = \frac{d}{dt}\Phi(\exp t\xi, x)\big|_{t=0}$.

An action Φ of a Lie group G on a cosymplectic manifold (M, Ω, η) is called *cosymplectic*, if for each $g \in G$ the corresponding Φ_g is an automorphism of the cosymplectic structure, i.e. $\Phi_g^*\Omega = \Omega$, $\Phi_g^*\eta = \eta$.

A momentum map is a function $J : M \longrightarrow \mathfrak{g}^*$ such that if we define

$$J_\xi(x) = \langle \xi, J(x) \rangle$$

for all $\xi \in \mathfrak{g}$, then $R(J_\xi) = 0$ and the Hamiltonian vector field X_{J_ξ} is just ξ_M. The momentum map J is said to be *Ad^*-equivariant* if

$$J \circ \Phi_g = Ad^*_{g^{-1}} \circ J$$

for each $g \in G$, where Ad^* is the co-adjoint representation of G on $\mathfrak{g}^*$.

For given $\mu \in \mathfrak{g}^*$ we denote by G_μ the isotropy group of μ. By the Ad^*-equivariance of J it follows that the level subset $Z_\mu = J^{-1}(\mu)$ is an invariant subset for the restriction of Φ to G_μ. Moreover, if μ is a regular value of J, then Z_μ is a submanifold of M and Φ induces a smooth action of G_μ on Z_μ. Following Libermann and Marle [7] we will say that this action is *simple* if the orbit space Z_μ/G_μ admits a manifold structure such that the canonical projection $\pi_\mu : Z_\mu \longrightarrow Z_\mu/G_\mu$ is a surjective submersion. This will for instance be the case if the action is free and proper. In the following it will always be assumed that G_μ is connected, so the fibers of π_μ are also connected.

Albert [3, 4] has established the following cosymplectic reduction theorem.

Theorem 3.1. There exists a unique cosymplectic structure (Ω_μ, η_μ) on the quotient space $M_\mu = Z_\mu/G_\mu$ such that

$$j_\mu^*\Omega = \pi_\mu^*\Omega_\mu \quad \text{and} \quad j_\mu^*\eta = \pi_\mu^*\eta_\mu,$$

with $j_\mu : Z_\mu \longrightarrow M$ the inclusion map and $\pi_\mu : Z_\mu \longrightarrow M_\mu$ the canonical projection. Further, the restriction of the Reeb vector field R to Z_μ projects onto M_μ and its projection R_μ is just the Reeb vector field for the reduced cosymplectic structure (Ω_μ, η_μ).

Now suppose that h is a Hamiltonian function on M such that it is G-invariant, i.e. $h \circ \Phi_g = h$, for any $g \in G$. Then $h \circ j_\mu$ projects to a function h_μ defined on M_μ. Denote by E_h the evolution vector field determined by h. Then E_h is tangent to Z_μ and it projects to a vector field $(E_h)_\mu$ on M_μ which is precisely the evolution vector field E_{h_μ} determined by h_μ on the reduced cosymplectic manifold M_μ. Hence the dynamics on M are projected onto the dynamics on M_μ. Notice that

$$\dim M_\mu = \dim M - \dim G - \dim G_\mu$$

and thus we have reduced the number of motion equations. The main problem now is to reconstruct the dynamics on M from the dynamics on M_μ.

Next we shall prove that the reduction of cosymplectic manifolds may be obtained from the Marsden–Weinstein procedure by extending the phase space M.

Lemma 3.2. Let M be a manifold and Ω, η two differential forms on M with degrees 2 and 1 respectively. Consider on $\widetilde{M} = M \times \mathbb{R}$ the differential 2-form $\omega = \mathrm{pr}^*\Omega + \mathrm{pr}^*\eta \wedge ds$, where $s \in \mathbb{R}$ and $\mathrm{pr} : \widetilde{M} \longrightarrow M$ is the canonical projection. Then

(a) (M, Ω, η) is a cosymplectic manifold if and only if $(\widetilde{M}, \omega)$ is a symplectic manifold.

(b) In such a case, pr is a Poisson morphism.

Proof. We proceed in two steps. (a) Let $\dim \widetilde{M} = 2n$. A straightforward calculation gives the relation $\omega^n = n \cdot \mathrm{pr}^*(\Omega^{n-1} \wedge \eta) \wedge \mathrm{d}s$. Thus

$$\Omega^{n-1} \wedge \eta \text{ is a volume form } \Longleftrightarrow \omega^n \text{ is a volume form.}$$

(b) Let $f, f' : M \longrightarrow \mathbb{R}$ be two smooth functions. We need to prove

$$\mathrm{pr}^*\{f, f'\}_M = \{\mathrm{pr}^*f, \mathrm{pr}^*f'\}_{\widetilde{M}} \tag{3.1}$$

where

$$\{f, f'\}_M = \Omega(X_f, X_{f'}) \quad \text{and} \quad \{\mathrm{pr}^*f, \mathrm{pr}^*f'\}_{\widetilde{M}} = \omega(X_{\mathrm{pr}^*f}, X_{\mathrm{pr}^*f'}). \tag{3.2}$$

Here $\{\ ,\ \}_M$ (respectively, $\{\ ,\ \}_{\widetilde{M}_0}$) denotes the Poisson bracket on M (respectively, $\widetilde{M}_0$). Also, X_f (respectively, X_{pr^*f}) denotes the Hamiltonian vector field associated to f (respectively, pr^*f) relative to (Ω, η) (respectively, ω). So,

$$X_f \text{ is the solution of: } i_{X_f}\Omega = \mathrm{d}f - R(f)\eta, \quad i_{X_f}\eta = 0$$

and

$$X_{\mathrm{pr}^*f} \text{ is the solution of: } i_{X_{\mathrm{pr}^*f}}\omega = \mathrm{d}\,\mathrm{pr}^*f.$$

In a similar way, the same relations hold for f'. A straightforward calculation gives $X_{\mathrm{pr}^*f} = X_f - R(f)\dfrac{\partial}{\partial s}$ and therefore we get (3.1). Here R is the Reeb vector field determined by the cosymplectic structure (Ω, η). We observe that the Hamiltonian vector field $X_{\mathrm{pr}^* f}$ projects onto M and its projection is just the Hamiltonian vector field X_f. $\square$

Suppose again that M is a cosymplectic manifold with cosymplectic structure (Ω, η) and G is a Lie group acting on M such that there exists a momentum map $J : M \longrightarrow \mathfrak{g}^*$. Then we can extend the action of G on M to $\widetilde{M}$ as follows:

$$\widetilde{\Phi} : G \times \widetilde{M} \longrightarrow \widetilde{M} \qquad \widetilde{\Phi}_g(x, s) = (\Phi_g(x), s)$$

for any $s \in \mathbb{R}$, $x \in M$ and $g \in G$. Now we construct the 2-form $\widetilde{\omega} = \Omega + \eta \wedge \mathrm{d}s$ on $\widetilde{M}$. Thus ω is a symplectic form and the group G acts on $(\widetilde{M}, \omega)$ by symplectomorphisms. Also the momentum map J may be extended to a momentum map $\widetilde{J} : \widetilde{M} \longrightarrow \mathfrak{g}^*$ by $\widetilde{J}(x, s) = J(x)$. A direct computation shows that J is Ad^*-equivariant iff $\widetilde{J}$ is Ad^*-equivariant too. Further, since the level set $\widetilde{Z}_\mu = \widetilde{J}^{-1}(\mu) = Z_\mu \times \mathbb{R}$, then we also have that $\mu \in \mathfrak{g}^*$ is a regular value of J iff it is a regular value of $\widetilde{J}$. In such a case we can apply the Marsden–Weinstein procedure and obtain a reduced symplectic manifold $\widetilde{M}_\mu$ endowed with a reduced symplectic form ω_μ such that

$$\widetilde{j}_\mu^* \omega = \widetilde{\pi}_\mu^* \omega_\mu$$

with $\widetilde{j}_\mu : \widetilde{Z}_\mu \longrightarrow \widetilde{M}$ the inclusion map and $\widetilde{\pi}_\mu : \widetilde{Z}_\mu \longrightarrow \widetilde{M}_\mu = \widetilde{Z}_\mu / G_\mu$ the canonical projection. We remark that the action of G_μ on Z_μ is simple if and only if the action of G_μ on $\widetilde{Z}_\mu$ is simple. A direct computation shows that $\widetilde{M}_\mu = M_\mu \times \boldsymbol{R}$. Also, the reduced cosymplectic structure may be obtained from ω_μ as follows:

$$\Omega_\mu = (\iota_0)^* \omega_\mu \qquad \eta_\mu = -(\iota_0)^* (i_{\partial/\partial s}\, \omega_\mu)$$

where $\iota_0 : M_\mu \longrightarrow \widetilde{M}_\mu = M_\mu \times \mathbb{R}$ is given by $\iota_0(x) = (x, 0)$.

Now suppose that a Hamiltonian function h on M is given. We define an *extended Hamiltonian* $\widetilde{h}$ on $\widetilde{M}$ by

$$\widetilde{h}(x, s) = h(x) + s \qquad x \in M \qquad s \in \mathbb{R}\,.$$

Thus, the corresponding Hamiltonian vector field $X_{\widetilde{h}}$ is given by

$$i_{X_{\widetilde{h}}}\, \omega = \mathrm{d}\widetilde{h}$$

and then we deduce

$$X_{\widetilde{h}} = E_h - R(h) \frac{\partial}{\partial s} \tag{3.3}$$

from which we obtain $X_{\widetilde{h}}(s) = -R(h)$. Then if we know the reduced Hamiltonian system $X_{\widetilde{h}_\mu}$ we obtain that

$$E_{h_\mu} = X_{\widetilde{h}_\mu} - X_{\widetilde{h}_\mu}(s) \frac{\partial}{\partial s} \tag{3.4}$$

and, conversely, given the reduced evolution vector field E_{h_μ} then we have

$$X_{\widetilde{h}_\mu} = E_{h_\mu} - R_\mu(h_\mu) \frac{\partial}{\partial s}\,.$$

Observe that the vector field $X_{\widetilde{h}}$ is projectable onto M and its projection is just E_h. Hence the Hamiltonian flow of $X_{\widetilde{h}}$ projects onto the flow of E_h. The same is true for the reduced vector fields $X_{\widetilde{h}_\mu}$ and E_{h_μ}. Moreover, a direct computation shows that

$$\{\widetilde{f}, \widetilde{f}'\}_{\widetilde{M}} = \mathrm{pr}^*(\{f, f'\}_M + R(f) - R(f'))\,.$$

Remark. We notice that M may be extended by multiplying by S^1 and then we substitute $\mathrm{d}s$ by θ, where θ is the length form on S^1. The above procedure also works in this case.

4. Symplectic reduction on singular values

Consider (M, ω) a symplectic manifold, $\Phi : G \times M \longrightarrow M$ a left action by symplectomorphisms and $J : M \longrightarrow \mathfrak{g}^*$ a momentum map. For the sake of simplicity we shall suppose that G is compact (although most of the results hold for proper actions of Lie groups) and that $0 \in \mathfrak{g}^*$ is a singular value, that is, the group G does not act freely on the zero level set $Z = J^{-1}(0)$ (the 'shifting trick' allows one to talk exclusively about reduction at 0 [11]).

Note that the quotient $M_0 = Z/G$ cannot be a symplectic manifold: it is not even a manifold! But the symplectic structure remains. Sjamaar and Lerman [11] proved that this quotient is a singular manifold whose strata are symplectic manifolds, these symplectic structures fit together nicely and define a structure of Poisson algebra on M_0. This gives rise to the notion of stratified symplectic space.

4.1. Symplectic stratified space

A *stratified space* X is a singular manifold which is the union of a locally finite family $\mathcal{S}_X$ of smooth manifolds such that the local structure of X is conical (see for example [5, p 12] for the exact definition).

A *smooth structure* on X is a subalgebra $C^\infty(X)$ of the algebra of continuous functions having the property that, for any $f \in C^\infty(X)$ and for any stratum S, the restriction $f|_S$ is smooth.

A stratified space X endowed with a smooth structure is said to be a *smooth stratified space*.

Example 1. The zero level set Z possesses a natural structure of smooth stratified space, where

$$\mathcal{S}_Z = \{\text{connected components of } Z_{(K)} \text{ with } K \text{ subgroup of } G\},$$

and

$$C^\infty(Z) = \{f : Z \to \mathbb{R} \ / \ \text{there exists a } G\text{-invariant } F \in C^\infty(M) \text{ with } f = j_0^* F\}.$$

Here, K denotes a subgroup of G, $Z_{(K)} = \{x \in Z \ / \ \text{the isotropy subgroup } G_x \text{ is conjugate to } K\}$ and $j_0 : Z \to M$ the natural inclusion.

Example 2. The above structure is invariant under the action of G, it induces on the reduced space M_0 a structure of smooth stratified space, where

$$\mathcal{S}_{M_0} = \{\pi_0(S) \ / \ S \in \mathcal{S}_Z\}$$

and

$$C^\infty(M_0) = \{f : M_0 \to \mathbb{R} \ / \ \pi_0^* f \in C^\infty(Z)\} .$$

Here, $\pi_0 : Z \to M_0$ is the canonical projection.

In the above two examples the smooth structure is taken from [2].

A smooth stratified space X is a *stratified symplectic space* if

- each stratum $S \in \mathcal{S}_X$ is a symplectic manifold,
- $C^\infty(X)$ is a Poisson algebra and
- the embedding $S \hookrightarrow X$ is Poisson.

In the same way, we define the notion of cosymplectic stratified space, exchanging symplectic by cosymplectic in the previous definition.

4.2. Vector fields on a smooth space

Although smooth stratified spaces are not manifolds, we still have the notion of vector field. A continuous map $f : X \to X'$ between two smooth stratified spaces is *smooth* if

$$g \circ f \in C^\infty(X) \quad \text{for any} \quad g \in C^\infty(X').$$

Example 3. The inclusion $j_0 : Z \hookrightarrow M$ is a smooth map.

Example 4. The projection $\pi_0 : Z \to M_0$ is a smooth map.

Example 5. The inclusion $S \hookrightarrow X$ is smooth, for any stratum $S \in \mathcal{S}_X$.

A continuous flow $\gamma : \mathbb{R} \times X \to X$ is a *smooth flow* if γ_t is smooth for any $t \in \mathbb{R}$. A linear map $V : C^\infty(X) \to C^\infty(X)$ is a *vector field* if there exists a smooth flow $\gamma : \mathbb{R} \times X \to X$ verifying

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t,x)) = V(f)(x) \quad \text{for any} \quad f \in C^\infty(X).$$

The set of vector fields on X will be denoted by $\mathfrak{X}(X)$.

Two vector fields $V \in \mathfrak{X}(X)$, $V' \in \mathfrak{X}(X')$ are *related* by a smooth function $f : X \to X'$ if

$$V(g \circ f)(x) = V'(g)(f(x))$$

for any $g \in C^\infty(X')$ and $x \in X$. We shall write $f_* V = V'$. If f is an onto map we shall say that f projects V onto V'. If f is the inclusion we shall say that $V = V'|_X$ is the restriction of V. The vector field V is *stratified* if for any stratum $S \in \mathcal{S}_X$ there exists $V_S \in \mathfrak{X}(S)$ with $V|_S = V_S$.

4.3. Symplectic reduction in a singular value

Sjamaar and Lerman proved that the reduced space M_0, endowed with the above smooth stratified structure, is in fact a stratified symplectic space [11].

The symplectic structure in each $Z_{(K)}/G$ is given by the differential 2-form $\omega_{(K)}$ determined by the equation:

$$\pi^*_{(K)} \omega_{(K)} = j^*_{(K)} \omega \tag{4.1}$$

where $j_{(K)} : Z_{(K)} \hookrightarrow M$ is the inclusion and $\pi_{(K)} : Z_{(K)} \to Z_{(K)}/G$ is the canonical projection.

The Poisson bracket $\{f, f'\}$ of two elements $f, f' \in C^\infty(M_0)$ is defined by the equation

$$\pi_0^*\{f, f'\} = j_0^*\{F, F'\} \tag{4.2}$$

where F, F' are G-invariant smooth functions on M with $j_0^* F = \pi_0^* f$, and similarly for f', F'.

4.4. Dynamics on stratified symplectic spaces

Dynamics on a stratified symplectic space are introduced by a Hamiltonian function $h \in C^\infty(M_0)$. Then the motion equations are given by:

$$X_h(f) = \{f, h\}_{M_0} \quad \text{for any} \quad f \in C^\infty(M_0) \tag{4.3}$$

which is written in terms of the Poisson bracket because M_0 is not a manifold. Sjamaar and Lerman proved that, for any Hamiltonian $h \in C^\infty(M_0)$, there exists a unique vector field $X_h \in \mathfrak{X}(M_0)$ satisfying the motion equations. The vector field X_h is called the *Hamiltonian vector field* associated to h.

The Hamiltonian vector field X_h is a stratified vector field. Moreover, since the natural inclusion $i_{(K)} : Z_{(K)} \to M_0$ is a Poisson map, then the restriction of X_h to $Z_{(K)}/G$ equals the Hamiltonian vector field $X_{i^*_{(K)}h}$, defined by the Hamiltonian function $i^*_{(K)}h$ on $(Z_{(K)}, \omega_{(K)})$.

5. Cosymplectic reduction on singular values

Following the ideas of section 3, we prove in this section that the cosymplectic reduction on singular values may be obtained from the Sjamaar–Lerman reduction theory by extending the phase space M. Here, the stratified cosymplectic space plays the role of the stratified symplectic space.

Consider the situation of section 3, that is, (M, Ω, η) is a cosymplectic manifold, $\Phi: G \times M \to M$ is a cosymplectic action and $J : M \to \mathfrak{g}^*$ is a momentum map. We shall suppose that G is compact and that $0 \in \mathfrak{g}^*$ is a singular value. The reduced space $M_0 = Z/G$ is also naturally endowed with a structure of smooth stratified space where S_{M_0} and $C^\infty(M_0)$ are defined as in the symplectic case.

We consider the notations $\widetilde{M}$, $\widetilde{\Phi}$, $\widetilde{J}$ and ω of section 3. Now, $(\widetilde{M}, \omega)$ is a symplectic manifold where G acts by symplectomorphisms and $\widetilde{J} : \widetilde{M} \to \mathfrak{g}^*$ is a momentum map with $0 \in \mathfrak{g}^*$ a singular value. We shall write $\widetilde{Z} = Z \times \mathbb{R}$ the zero level set of $\widetilde{J}$ and $\widetilde{M}_0 = \widetilde{Z}/G$ the reduced space, which is a symplectic singular space.

The bracket on $C^\infty(M_0)$ is given by the next lemma.

Lemma 5.1. Given f, $f' \in C^\infty(M_0)$ there exits a unique smooth function $\{f, f'\}_{M_0} \in C^\infty(M_0)$ satisfying

$$\{\mathrm{pr}_0^* f, \mathrm{pr}_0^* f'\}_{\widetilde{M}_0} = \mathrm{pr}_0^*\{f, f'\}_{M_0}. \tag{5.1}$$

Moreover, the algebra $C^\infty(M_0)$ endowed with this bracket is a Poisson algebra in a way such that the natural projection $\mathrm{pr}_0 : \widetilde{M}_0 \to M_0$ is a Poisson morphism.

Proof. Notice that, if (5.1) is satisfied, then $(C^\infty(M_0), \{-,-\})$ is a Poisson algebra and pr_0 is a Poisson morphism. Consider F and F' to be two G-invariant functions on M, with $j_0^* F = \pi_0^* f$ and $j_0^* F' = \pi_0^* f'$. The bracket $\{F, F'\}_{M_0}$ is a G-invariant smooth map on M. Then, there exists $g \in C^\infty(M_0)$ with $\pi_0^* g = \{F, F'\}_{M_0}$. Let $g = \{f, f'\}_{M_0}$. If the equation (5.1) holds, then the bracket is well defined.

Consider the canonical projections $\widetilde{\pi}_0 : \widetilde{Z} \to \widetilde{M}_0$ and $\mathrm{Pr} : \widetilde{Z} \to Z$. Since $\pi_0 \circ \mathrm{Pr} = \mathrm{pr}_0 \circ \widetilde{\pi}_0$ and the map $\widetilde{\pi}_0$ is onto then (5.1) is equivalent to

$$\widetilde{\pi}_0^*\{\mathrm{pr}_0^* f, \mathrm{pr}_0^* f'\}_{\widetilde{M}_0} = \mathrm{Pr}^* \pi_0^* \{f, f'\}_{M_0}.$$

From the definition of these brackets (cf (4.2)) we get

$$\widetilde{j}_0^*\{\mathrm{pr}^* F, \mathrm{pr}^* F'\}_{\widetilde{M}} = \mathrm{Pr}^* j_0^* \{F, F'\}_M$$

where $\mathrm{pr} : \widetilde{M} \to M$ is the canonical projection. Finally, the equality (5.1) comes from the relation $\mathrm{pr} \circ \widetilde{j}_0 = j_0 \circ \mathrm{Pr}$ and the fact that pr is a Poisson morphism (cf lemma 3.2). □

Theorem 5.1. The reduced space is a cosymplectic stratified space. For any subgroup K of G, the manifold $Z_{(K)}/G$ admits a cosymplectic structure $(\Omega_{(K)}, \eta_{(K)})$ such that

$$\pi_{(K)}^* \Omega_{(K)} = j_{(K)}^* \Omega \text{ and } \pi_{(K)}^* \eta = j_{(K)}^* \eta \tag{5.2}$$

where $j_{(K)} : Z_{(K)} \hookrightarrow M$ is the inclusion map and $\pi_{(K)} : Z_{(K)} \to Z_{(K)}/G$ the canonical projection.

Further, the restriction of the Reeb vector field R of (Ω, η) to Z projects onto a stratified vector field R_0 on M_0 and its restriction to any $Z_{(K)}/G$ is just the Reeb vector field $R_{(K)}$ for the reduced cosymplectic structure $(\Omega_{(K)}, \eta_{(K)})$.

Proof. We proceed in several steps.

• *Cosymplectic structure on the strata*. Let K be a subgroup of G. Notice the relation $\widetilde{M}_{(K)} = M_{(K)} \times \mathbb{R}$. The connected components of the manifold

$$\widetilde{Z}_{(K)}/G = (Z_{(K)}/G) \times \mathbb{R} \tag{5.3}$$

are the strata of the reduced space $\widetilde{M}_0$. The symplectic structure of the manifold $\widetilde{Z}_{(K)}/G$ is given by a differential form $\omega_{(K)}$ satisfying

$$\widetilde{\pi}^*_{(K)}\omega_{(K)} = \widetilde{j}^*_{(K)}\omega$$

where $\widetilde{\pi}_{(K)} : \widetilde{Z}_{(K)} \longrightarrow \widetilde{Z}_{(K)}/G$ is the canonical projection and $\widetilde{j}_{(K)} : \widetilde{Z}_{(K)} \hookrightarrow \widetilde{M}$ is the inclusion map. Since $\omega = \Omega + \eta \wedge ds$, a straightforward calculation shows the existence of two differential forms $\Omega_{(K)}$ and $\eta_{(K)}$ on $Z_{(K)}$ verifying

$$\omega_{(K)} = \Omega_{(K)} + \eta_{(K)} \wedge \mathrm{d}s\ .$$

Since $\omega_{(K)}$ is a symplectic form on $(Z_{(K)}/G) \times \mathbb{R}$ then $(\Omega_{(K)}, \eta_{(K)})$ defines a cosymplectic structure on $Z_{(K)}/G$ (cf lemma 3.2). By construction these forms satisfy (5.2).

• *The embedding $Z_{(K)}/G \hookrightarrow M_0$ is Poisson.* Consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{Z}_{(K)}/G & \xrightarrow{\ \widetilde{\imath}_{(K)}\ } & \widetilde{M}_0 \\ {\scriptstyle \mathrm{pr}_{(K)}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_0} \\ Z_{(K)}/G & \xrightarrow{\ i_{(K)}\ } & M_0 \end{array}$$

where the horizontal arrows are the natural inclusions and the vertical arrows are the canonical projections. From the lemmas 3.2 and 5.1 we know that $\mathrm{pr}_{(K)}$ and pr_0 are Poisson morphisms. Since $\widetilde{M}_0$ is a symplectic stratified space we know that $\widetilde{\imath}_{(K)}$ is a Poisson morphism. Since $\mathrm{pr}_{(K)}$ is an onto map, we get that $i_{(K)}$ is a Poisson morphism.

• *Reeb vector field*. Let $\gamma : \mathbb{R} \times M \to M$ be the smooth flow of the Reeb vector field R. Since R is invariant and satisfies $R(J) = 0$ then it induces the following smooth flows:

$$\gamma|_Z : \mathbb{R} \times Z \to Z \quad \text{and} \quad \gamma_0 : \mathbb{R} \times M_0 \to M_0.$$

Let $R_0 \in \mathfrak{X}(M_0)$ be the vector field defined by γ_0. By construction R_0 is the projection by π_0 of the restriction $R|_Z$. The invariance of γ implies that R_0 is a stratified vector field. Formula (5.2) implies that the restriction of R_0 to $Z_{(K)}/G$ is the Reeb vector field $R_{(K)}$. □

6. Dynamics on stratified cosymplectic spaces

Dynamics on a stratified cosymplectic space are introduced by giving a Hamiltonian function $h \in C^\infty(M_0)$. Then the motion equations are:

$$E_h(f) = \{f, h\}_{M_0} + R_0(f) \qquad \text{for any} \qquad f \in C^\infty(M_0) \tag{6.1}$$

which are written in terms of the Poisson bracket because M_0 is not a manifold. The vector field E_h is called the evolution vector field of the system.

6.1. Extended Hamiltonian

Given a Hamiltonian $h \in C^\infty(M_0)$, the map $\widetilde{h} : \widetilde{M}_0 \to \mathbb{R}$ defined by $\widetilde{h}(x,s) = h(x) + s$ is called the *extended Hamiltonian* of h. Consider $H : M \to \mathbb{R}$ a smooth function satisfying $j_0^* H = \pi_0^* h$. Its extended Hamiltonian $\widetilde{H} : \widetilde{M} \to \mathbb{R}$ is clearly a smooth function and therefore $\widetilde{h} \in C^\infty(\widetilde{M}_0)$ because $\widetilde{j}_0^* \widetilde{H} = \widetilde{\pi}_0^* \widetilde{h}$. We have

$$(\widetilde{\pi}_0)_* \left(X_{\widetilde{H}}|_{\widetilde{Z}}\right) = X_{\widetilde{h}} \,. \tag{6.2}$$

Write $\mathcal{R} \in \mathfrak{X}(\widetilde{M})$ (resp. $\mathcal{R}_0 \in \mathfrak{X}(\widetilde{M}_0)$) the natural lifting of $R \in \mathfrak{X}(M)$ to $\widetilde{M}$ (resp. $R_0 \in \mathfrak{X}(M_0)$ to $\widetilde{M}_0$). We have

$$(\widetilde{\pi}_0)_* \left(\mathcal{R}|_{\widetilde{Z}}\right) = \mathcal{R}_0 \quad \text{and} \quad (\mathrm{pr}_0)_* \mathcal{R}_0 = R_0 \,.$$

Consider $h = 0$ and $H = 0$. We have $i_{X_{\widetilde{H}}}\omega = ds = i_{\mathcal{R}}\omega$ and therefore $X_{\widetilde{H}} = \mathcal{R}$. Applying the above equations we get

$$X_{\widetilde{h}} = \mathcal{R}_0 \quad \text{and} \quad (\mathrm{pr}_0)_* X_{\widetilde{h}} = R_0 \,. \tag{6.3}$$

Theorem 6.2. For any Hamiltonian $h \in C^\infty(M_0)$ there exists a unique evolution vector field E_h. This vector field is a stratified vector field. For any $K < G$, its restriction to $Z_{(K)}/G$ is the evolution vector field $E_{I^*_{(K)}h}$ defined by $i^*_{(K)}h$ with respect to the cosymplectic structure $(\Omega_{(K)}, \eta_{(K)})$.

Proof. We proceed in several steps.

• *Existence*. Consider $\widetilde{h} \in C^\infty(\widetilde{M}_0)$ the extended Hamiltonian $\widetilde{h}$ of h. Write $\gamma_{\widetilde{h}} : \mathbb{R} \times \widetilde{M}_0 \to \widetilde{M}_0$ the Hamiltonian flow of $X_{\widetilde{h}} \in \mathfrak{X}(\widetilde{M}_0)$. Then we have

$$\{\mathrm{pr}_0^* f, \widetilde{h}\}_{\widetilde{M}_0}(\gamma_{\widetilde{h}}(t,x,s)) = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{pr}_0(\gamma_{\widetilde{h}}(t,x,s))) \tag{6.4}$$

for any $f \in C^\infty(M_0)$, and $(x,s) \in \widetilde{M}_0$. Define the smooth flow $\gamma_h : \mathbb{R} \times M_0 \to M_0$ by

$$\gamma_h(t,x) = \mathrm{pr}_0(\gamma_{\mathrm{pr}_0^* h}(t,x,0)) \,. \tag{6.5}$$

Since pr_0 is a Poisson map, then (6.4) becomes

$$\{f,h\}_{M_0}(\gamma_h(t,x)) + \{\mathrm{pr}_0^* f, \widetilde{0}\}_{\widetilde{M}_0}(\gamma_{\widetilde{h}}(t,x,0)) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_h(t,x))$$

for any $f \in C^\infty(M_0)$, and $x \in M_0$. Applying (4.3) and (6.3) we get

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_h(t,x)) = \left(\{f,h\}_{M_0} + R_0\right)(\gamma_h(t,x))$$

that is, (6.1).

• *E_h is stratified*. The vector field $X_{\widetilde{h}}$ is stratified, then $\gamma_{\mathrm{pr}_0^* h}(\mathbb{R} \times \widetilde{Z}_{(K)}/G) \subset \widetilde{Z}_{(K)}/G$, and therefore $\gamma_h(\mathbb{R} \times Z_{(K)}/G) \subset Z_{(K)}/G$. Hence, the vector field E_h is stratified.

• *Restriction to a stratum*. The evolution vector field E_h is a solution of the equation

$$E_h = \{-,h\}_{M_0} + R_0 \,.$$

Restricting this equality to $Z_{(K)}/G$ we obtain

$$(i_{(K)})_* E_h = \{-, i_{(K)*}h\}_{Z_{(K)}/G} + R_{(K)} ,$$

where we have used the fact that $i_{(K)}$ is a Poisson map and the equality $i_{(K)*}R_0 = R_{(K)}$. So,

$$E_h|_{Z_{(K)}/G} = (i_{(K)})_* E_h = E_{i_{(K)*}h} .$$

• *Uniqueness*. Since $(Z_{(K)}/G, \Omega_{(K)}, \eta_{(K)})$ is a cosymplectic manifold, the restriction of γ_h to any stratum $Z_{(K)}/G$ is completely determined by the restriction $i^*_{(K)}h$. □

Acknowledgments

We wish to express our thanks to the referees for many helpful comments.

References

[1] Abraham R and Marsden J E 1978 *Foundations of Mechanics* 2nd ed (Reading, MA: Benjamin-Cummings)
[2] Arms J M, Cushman R H and Gotay M J 1991 A universal reduction procedure for Hamiltonian systems *The geometry of Hamiltonian systems* Meeting, Berkeley, CA, 1989 *Math. Sci. Res. Inst. Publ.* **22** (New York: Springer) pp 33–51
[3] Albert C 1989 Le théorème de réduction de Marsden–Weinstein en géométrie cosymplectique et de contact *J. Geom. Phys.* **6** (4) 627–49
[4] Cantrijn F, de León M and Lacomba E A 1992 *Gradient vector fields on cosymplectic manifolds*, *J. Phys. A: Math. Gen.* **25** 175-88
[5] Goresky M and MacPherson R 1988 *Stratified Morse theory* (New York: Springer)
[6] de León M and Rodrigues P R 1989 *Methods of Differential Geometry in Analytical Mechanics, North-Holland Mathematical Series* **158** (Amsterdam: North-Holland)
[7] Libermann P and Marle Ch 1987 *Symplectic Geometry and Analytical Mechanics* (Dordrecht: Reidel)
[8] Marsden J E 1992 *Lectures on Mechanics* London Math. Soc. Lecture Note Series (Cambridge: Cambridge University Press)
[9] Marsden J E, Montgomery R and Ratiu T 1990 Reduction, symmetry, and phases in mechanics *Memoirs of the AMS* **436**
[10] Marsden J and Weinstein A 1974 Reduction on Symplectic Manifolds with Symmetry *Rep. Math. Phys.* **5** 121–30
[11] Sjamaar R and Lerman E 1991 Stratified symplectic spaces and reduction *Ann. Math.* **134** 375–422