GYSIN SEQUENCES

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Associated with a principal action $\Phi\colon \mathbf{S}^1 \times M \to M$ of the unit circle on a manifold M there exists a long exact sequence, said *Gysin sequence*,

$$(1) \qquad \cdots \longrightarrow H^i(M) \longrightarrow H^{i-1}(M/\mathbf{S}^1) \xrightarrow{\wedge[e]} H^{i+1}(M/\mathbf{S}^1) \longrightarrow H^{i+1}(M) \longrightarrow \cdots,$$

where $H^*(M)$ and $H^*(M/\mathbf{S}^1)$ are the deRham cohomology of the manifolds M and $M/\mathbf{S}^1$ and $[e] \in H^2(M/\mathbf{S}^1)$ is the *Euler class* (see for example [4]). Notice that the Gysin sequence express the cohomology of M in terms of transversal data of the flow defined by the action Φ.

In this work we extend the scope of the Gysin sequence to any isometric action $\Phi\colon \mathbf{R}\times M \to M$ of the real numbers on a compact riemannian manifold M. Observe that Φ becomes a $\mathbf{S}^1$-principal action when it is periodic and free. The orbit space $M/\mathbf{R}$ is a manifold but $M/\mathbf{R}$ is wilder in the general case. So, the cohomology of the orbit space appearing in the Gysin sequence we construct is not necessarily the usual deRham cohomology. Three different cases have been already considered.

• *The action is almost free (i.e. without fixed points) and periodic.* The orbit space is an orbifold (or Sataké manifold) and the Gysin sequence of [9] is exactly (1), where the deRham cohomology $H^*(M/\mathbf{S}^1)$ must be understood in the category of orbifolds.

• *The action is almost free but not necessarily periodic.* The orbits of Φ define on M a foliation $\mathcal{F}$. We have shown that $H^*(M)$ can be computed from the basic cohomology $H^*(M/\mathcal{F})$ by means of the Gysin sequence

$$(2) \qquad \cdots \longrightarrow H^i(M) \longrightarrow H^{i-1}(M/\mathcal{F}) \xrightarrow{\wedge[e]} H^{i+1}(M/\mathcal{F}) \longrightarrow H^{i+1}(M) \longrightarrow \cdots,$$

where $[e] \in H^2(M/\mathcal{F})$ is the Euler class (cf. [6]).

• *The action Φ is periodic but fixed points are allowed.* The orbit space $M/\mathbf{S}^1$ is a stratified space. Thus, the deRham intersection cohomology $IH^*_{\overline{r}}(M/\mathbf{S}^1)$ (cf. [3]) is a natural candidate to replace the cohomology of the orbit space appearing in the Gysin sequence. We have shown in [10] that the cohomology of M can be calculated by means of the following Gysin sequence

$$(3) \qquad \cdots \longrightarrow H^i(M) \to IH^{i-1}_{\overline{r}-\overline{2}}(M/\mathbf{S}^1) \xrightarrow{\wedge[e]} IH^{i+1}_{\overline{r}}(M/\mathbf{S}^1) \longrightarrow H^{i+1}(M) \longrightarrow \cdots,$$

where $[e] \in IH^2_{\overline{2}}(M/\mathbf{S}^1)$ is the Euler class. The shifting of the perversity comes from the fact that the pervers degree of e is two.

The sequences (2) and (3) become (1) when the action is periodic and almost free.

The goal of this work is to treat the general picture where no restrictions about periodicity or freeness of Φ are considered. For a generic isometric action the geometrical situation involves at the same time a stratification and a foliation as follows:

- The action Φ defines a natural stratification $\mathcal{S}$ on M where the singular strata are the connected component of the fixed point set F.
- The orbits of Φ define a singular foliation $\mathcal{F}$ on M which restriction to the regular stratum $M - F$ is a foliation $\mathcal{F}^*$.

The main point to construct the Gysin sequence in this framework is gathering the cohomologies used in (2) and (3) in a new natural cohomology: the basic intersection cohomology.

We define a *basic intersection differential form* as a differential form defined on $M - F$ which is basic, relatively to the foliation $\mathcal{F}^*$, and which is an intersection differential form, relatively to the stratification $\mathcal{S}$. The *basic intersection cohomology* of $\mathcal{F}$ is the cohomology of the complex of basic intersection differential forms and it is denoted by $IH^*_{\overline{r}}(M/\mathcal{F})$. This cohomology coincides with $H^*(M/\mathbf{S}^1)$ (resp. $H^*(M/\mathcal{F})$, resp $IH^*_{\overline{r}}(M/\mathbf{S}^1)$) if the action Φ is almost free and periodic (resp. almost free, resp. periodic).

The main result of this work is the construction of the Gysin sequence

$$\cdots \longrightarrow H^i(M) \longrightarrow IH^{i-1}_{\overline{r}-\overline{2}}(M/\mathcal{F}) \xrightarrow{\wedge[e]} IH^{i+1}_{\overline{r}}(M/\mathcal{F}) \longrightarrow H^{i+1}(M) \longrightarrow \cdots,$$

where $[e] \in IH^2_{\overline{2}}(M/\mathcal{F})$ is the Euler class. So, we can compute the deRham cohomology of M in terms of the basic intersection cohomology of $\mathcal{F}$. We end the work giving a geometrical interpretation of the vanishing of the Euler class in terms of the transversal triviality of $\mathcal{F}$.

For the sequel we fix an isometric action $\Phi\colon \mathbf{R} \times M \to M$ of the real numbers on an m-dimensional compact riemannian manifold (M, μ). We shall suppose that this action is not *trivial*, that is, the set F of fixed points is not the whole manifold M. In this work all the manifolds are connected and smooth.

1 Geometry of the action

The action Φ defines naturally on M a stratification and a foliation.

1.1 Stratification. Consider the homomorphism $\Phi_0\colon \mathbf{R} \to Iso(M,\mu)$ defined by $\Phi_0(g)(x) = \Phi(g,x)$, where $Iso(M,\mu)$ stands for the group of isometries of (M,μ). Notice that the map Φ_0 is not constant and that Φ is periodic if and only if $\Phi_0(\mathbf{R}) = \mathbf{S}^1$. Since M is a compact manifold a result of Myers-Steenrod [8] asserts that the group $Iso(M,\mu)$ is a compact Lie group which acts smoothly on M[1]. Write $\mathbf{T}$ the closure of $\Phi_0(\mathbf{R})$ on $Iso(M,\mu)$, which is necessarily a torus. The induced action $\widetilde{\Phi}\colon \mathbf{T} \times M \to M$ is smooth. We have the relation $\Phi_0(g)(x) = \widetilde{\Phi}(g,x)$ for $(g,x) \in \mathbf{T} \times M$.

Recall that the topology of $Iso(M,\mu)$ is stronger than the pointwise convergence topology (cf. [8]). So, a closed subset $N \subset M$ is $\mathbf{R}$-invariant iff it is $\mathbf{T}$-invariant (and we will just say that N is invariant). The same commentary applies to equivariant and invariant maps. On the other hand, the set F is also the set of fixed points of $\widetilde{\Phi}$ and therefore it is an invariant manifold (not necessarily connected). We shall write $\{F_\alpha\}_{\alpha\in\Delta}$ the family of its connected components. The action Φ is *almost free* is there is not any fixed point (i.e. $\Delta = \emptyset$).

Each F_α is called a *singular stratum.* The open dense subset $M - F$ is the *regular stratum.* The family $\{M - F,\ \{F_\alpha\}_{\alpha\in\Delta}\}$ define a stratification on M by invariant submanifolds. Notice that this stratification does not depend on the choice of the riemannian metric μ.

Associated with this stratification there exists a *system of tubular neighborhoods* $\{\tau_\alpha\colon U_\alpha \to F_\alpha\}_{\alpha\in\Delta}$, that is:

a) $U_\alpha \subset M$ is an invariant neighborhood of F_α with $U_\alpha \cap U'_\alpha = \emptyset$ if $\alpha \neq \alpha'$,

[1] We refer the reader to [2] for the notions related with compact Lie group actions.

b) $\tau_\alpha : U_\alpha \to F_\alpha$ is a $\mathbf{T}$-vector bundle with structure group $O(n)$, and

c) the restriction of τ_α to F_α is the identity.

The torus $\mathbf{T}$ fixes each point of F and therefore acts on each fiber of τ_α's. So, for each $\alpha \in \Delta$, we can find an orthogonal action $\Phi_\alpha : \mathbf{T} \times \mathbf{S}^{n_\alpha - 1} \to \mathbf{S}^{n_\alpha - 1}$ and an atlas $\mathcal{A}_\alpha$ of τ_α such that, if $\varphi : \tau_\alpha^{-1}(W) \to W \times \mathbf{R}^{n_\alpha}$ is a chart of this atlas then

$$\Phi(g, \varphi^{-1}(w, \theta)) = \varphi^{-1}(w, |\theta| \cdot \Phi_\alpha(g, \theta/|\theta|))$$

for each $g \in \mathbf{T}$, $w \in W$ and $\theta \in \mathbf{R}^{n_\alpha - 1}$ (see for example [4, pag.139]). We have written $|\cdot|$ the euclidean norm on $\mathbf{R}^n$. We shall write $\rho_\alpha : U_\alpha \to [0, \infty[$ the *distance map* defined by $\rho_\alpha(\varphi^{-1}(w, \theta)) = |\theta|$.

The system of tubular neighborhoods is not unique but two of them are equivariantly diffeomorphic (cf. [2, pag 312]). The action Φ_α is unique up to conjugation.

1.2 Foliation. We denote by X the *fundamental vector field* of the action Φ. It is defined by $X(x) = (\Phi_x)_*(1)$ for $x \in M$, $\Phi_x : \mathbf{R} \to M$, given by $\Phi_x(g) = \Phi(g, x)$, and $(\Phi_x)_*$ the tangent map. The flow of X determines a singular foliation[2] $\mathcal{F}$ on M whose singular orbits are the points of F; the restriction of this flow to $M - F$ gives a one dimensional regular foliation.The foliation $\mathcal{F}$ does not depend on the choice of the riemannian metric μ.

2 Differential forms

Before introducing the basic intersection cohomology we remind some facts about basic cohomology and intersection cohomology. Both notions are not treated in the whole generality but in simpler framework covering the purposes of this work.

2.1 Basic cohomology. Consider on a manifold N a dimensional foliation $\mathcal{G}$ given by a vector field Y. A *basic differential form* is a differential form ω on N satisfying

$$i_Y \omega = 0 \qquad \text{and} \qquad i_Y d\omega = 0,$$

where i_Y stands for the interior product by Y. The differential complex of basic differential forms is denoted by $\Omega^*(N/\mathcal{G})$ and its cohomology $H^*(N/\mathcal{G})$ is the *basic cohomology* of $\mathcal{G}$. This denomination is justified by the following fact: if $\mathcal{G}$ is defined by the fibers of a smooth submersion $\kappa : N \to B$ then the pull back $\kappa^* : \Omega^*(B) \to \Omega^*(N/\mathcal{G})$ is an isomorphism and therefore the basic cohomology $H^*(N/\mathcal{G})$ is isomorphic to the deRham cohomology $H^*(B)$.

When we restrict ourselves to basic differential forms with compact support we obtain the basic cohomology with compact support $H^*_c(N/\mathcal{G})$.

2.2 Intersection cohomology. The intersection cohomology is defined for stratified spaces and involves the notion of pervers degree.

[2]We refer the reader to [11] for the notions related with singular foliations.

2.2.1 Simple stratified space. Consider E a stratified space[3]. The space E is decomposed in a locally finite family of manifolds called *strata*. We shall say that E is a *simple stratified space* if there exists a dense stratum E_0 and if the others strata $\{E_\alpha\}_{\alpha\in\nabla}$ are closed. Recall that that we also have a family $\{\sigma_\alpha\colon V_\alpha \to E_\alpha\}_{\alpha\in\nabla}$ of fiber bundles and a family of continuous functions $\{\delta_\alpha\colon V_\alpha \to [0,\infty[\}_{\alpha\in\nabla}$ satisfying:

- V_α is an open neighborhood of E_α, with $V_\alpha \cap V'_\alpha = \emptyset$ if $\alpha \neq \alpha'$,
- the fiber of σ_α is the *cone* $cL_\alpha = L_\alpha \times [0,1[/L_\alpha \times \{0\}$ over a compact manifold L_α, called the *link* of E_α, and
- there exists an atlas $\mathcal{B}_\alpha = \{\varphi\colon \sigma_\alpha^{-1}(W) \to W \times cL_\alpha\}$ of σ_α such that
  - the restriction $\varphi\colon W = \sigma_\alpha^{-1}(W) \cap E_\alpha \to W \times cL_\alpha$ is given by $\varphi(x) = (x, \text{vertex of } cL_\alpha)$,
  - the restriction $\varphi\colon \sigma_\alpha^{-1}(W) - E_\alpha \to W \times L_\alpha \times]0,\infty[$ is a diffeomorphism and
  - $\delta_\alpha\varphi^{-1}\colon W \times L_\alpha \times]0,\infty[\to]0,\infty[$ is the canonical projection and $\delta_\alpha(E_\alpha) = \{0\}$.

The family $\{\sigma_\alpha \times \delta_\alpha\}_{\alpha\in\nabla}$ will be called *control data*. We shall write $\check{E}_\alpha = \delta_\alpha^{-1}(]0,1/2[)$, which is a fiber bundle over E_α.

2.2.2 Remarks.

a) We have already seen that Φ induces a stratification on M. The manifold M is a simple stratified space relatively to this stratification. We shall consider the following control data: $\{\tau_\alpha \times \rho_\alpha\}_{\alpha\in\Delta}$. The control data are unique up to equivariant diffeomorphism [2].

b) Suppose that the action is periodic and semifree (free outside of F). Put $M/\mathbf{S}^1$ the orbit space and $\pi\colon M \to M/\mathbf{S}^1$ the canonical projection. The orbit space $M/\mathbf{S}^1$ is also a simple stratified space, the strata being $\pi(M-F)$ et $\{\pi(F_\alpha) \equiv F_\alpha\}_{\alpha\in\Delta}$. We shall consider the following control data: $\{\gamma_\alpha \times \xi_\alpha\}_{\alpha\in\nabla}$ with $(\gamma_\alpha \times \xi_\alpha)(\pi(x)) = (\tau_\alpha(x), \delta_\alpha(x))$ for $\pi(x) \in V_\alpha$. Notice that each action Φ_α is free and so the link of F_α is the projective space $\mathbf{CP}^{m_\alpha}$, with $m_\alpha = (n_\alpha - 1)/2$.

c) When the action is just periodic then the orbit space $M/\mathbf{S}^1$ is a stratified space but no simple (see for example [13]). The topology is here more complex because the strata are not separated but incident. The elements of the control data are asked to satisfy some extra compatibility conditions. The links of strata are not manifolds but also stratified spaces. For sake of simplicity we stress the simple case in this work.

2.2.3 Perversion. The *pervers degree* $||\omega||_\alpha$ of a differential form $\omega \in \Omega^*(E_0)$, relatively to $\alpha \in \Delta$, is the smallest integer k verifying:

$$i_{\zeta_0}\cdots i_{\zeta_k}\omega \equiv 0 \text{ for each family of vector fields } \{\zeta_i\}_{i=0}^k \text{ on } \check{E}_\alpha \text{ tangents to the fibers of } \sigma_\alpha.$$

We shall write $||\omega||_\alpha = -\infty$ if $\omega|_{\check{E}_\alpha} \cong 0$. For α, $\beta \in \Omega^*(\check{E}_\alpha)$we have the relations:

$$||\alpha+\beta||_\alpha \leq \max(||\alpha||_\alpha, ||\beta||_\alpha) \text{ and } ||\alpha\wedge\beta||_\alpha \leq ||\alpha||_\alpha + ||\beta||_\alpha. \tag{4}$$

A *perversity* is a map $\overline{r}\colon \Delta \to \mathbf{Z}$ (cf. [7]). For each integer $\ell \in \mathbf{Z}$ we shall write $\overline{\ell}$ the constant perversity defined by $\overline{\ell}(\alpha) = \ell$. Perversity is the parameter used to control the pervers degree of the differential forms employed to compute the intersection cohomology.

[3] We refer the reader to [12] for the notions related with stratified spaces.

2.2.4 Intersection differential forms. A differential form ω in E_0 is said to be a *intersection differential form* (or, $\overline{r}$-intersection differential form) if for each $\alpha \in \Delta$ the restriction of ω to E_α satisfies:

$$\max\left\{ ||\omega||_{\alpha}, ||d\omega||_{\alpha} \right\} \leq \overline{r}(\alpha).$$

The complex of intersection differential forms is denoted by $\Omega^*_{\overline{r}}(E)$. The cohomology of this complex, written $IH^*_{\overline{r}}(E)$, is the *intersection deRham cohomology*[4] of E.

When $\Delta = \emptyset$ the stratified space E is a manifold, the complex $\Omega^*_{\overline{r}}(E)$ is exactly the complex of differential forms $\Omega^*(E)$ of E and therefore $IH^*_{\overline{r}}(E)$ is the usual deRham cohomology $H^*(E)$.

2.2.5 The intersection cohomology enjoys the following properties (see [7] and [10]):

- $IH^*_{\overline{r}}(E)$ is isomorphic to the intersection homology $IH^{\overline{q}}_*(E)$, where $\overline{q}$ and $\overline{r}$ are complementary perversities.
- When each link is connected (that is, E is *normal*) we have $IH^*_{\overline{0}}(E) \cong H^*(E)$, singular cohomology of E with real coefficients.
- $IH^*_{\overline{r}}(E) \cong H^*(E)$ if E is a manifold and $\overline{0} \leq \overline{r} \leq \overline{t}$. We have written $\overline{t}$ the perversity defined by $\overline{r}(\alpha) = \dim L_\alpha - 1$.
- If $\overline{r} < \overline{0}$ then $IH^*_{\overline{r}}(E) \cong H^*_c(E - E_0)$, the deRham cohomology with compact supports.
- If $\overline{r} > \overline{t}$ then $IH^*_{\overline{r}}(E) \cong H^*(E - E_0)$.

Since the intersection homology does not depend on the control data these results show that intersection cohomology does either.

2.3 Basic intersection cohomology. This new cohomology is defined in the presence of an isometric action, where coexist a foliation and a stratification. It generalizes the previous basic cohomology and intersection cohomology.

Fix a perversity $\overline{r}\colon \Delta \to M$. A differential form ω on R is an *basic $\overline{r}$-intersection differential form*, or simply a *bif*, if ω is a basic differential form (relatively to the foliation $\mathcal{F}$ presented in 1.2) and a $\overline{r}$-intersection differential form (relatively to the stratified structure described in 2.2.2 a)). We shall write $\Omega^*_{\overline{r}}(M/\mathcal{F})$ the differential complex of bifs and $IH^*_{\overline{r}}(M/\mathcal{F})$ its cohomology, the *basic intersection cohomology.* Notice that this cohomology does not depend on the metric μ. It does not depend either on the control data because uniqueness.

Proposition 2.3.1 *The basic intersection cohomology verifies:*

a) If $\overline{r} < \overline{0}$ *then* $IH^*_{\overline{r}}(M/\mathcal{F}) \cong H^*_c((M - F)/\mathcal{F})$.

b) If $\overline{r} > \overline{t}$ *then* $IH^*_{\overline{r}}(M/\mathcal{F}) \cong H^*((M - F)/\mathcal{F})$.

c) $IH^*_{\overline{0}}(M/\mathcal{F}) \cong H^*(M/\mathcal{F})$.

[4] We refer the reader to [5] and [3] for the notions related with intersection homology and cohomology.

Proof.

a) Since $\Omega^*_{\overline{r}}(M/\mathcal{F}) = \{\omega \in \Omega^*((M-F)/\mathcal{F}) \ / \ \omega \equiv 0 \text{ on } \check{U}_\alpha \text{ for any } \alpha \in \Delta\}$.

b) Since $\Omega^*_{\overline{r}}(M/\mathcal{F}) = \Omega^*((M-F)/\mathcal{F})$.

c) Define the auxiliary complex $\mathcal{C}^*(M/\mathcal{F}) = \{\omega \in \Omega^*(M/\mathcal{F}) \ / \ \omega|_{M-F} \in \Omega^*_{\overline{0}}(M/\mathcal{F})\}$. We have:

$$\Omega^*_{\overline{0}}(M/\mathcal{F}) \longrightarrow \mathcal{C}^*(M/\mathcal{F}) \longleftarrow \Omega^*(M/\mathcal{F}),$$

where the left map is defined by restriction and the right one is the inclusion. We want to prove that these maps are quasi-isomorphisms (i.e., they induced an isomorphism in cohomology). We proceed in several stages

• Consider a partition of the unity subordinated to the covering $\{M-F, \{F_\alpha\}_{\alpha\in\Delta}\}$ made up of elements of $\Omega^0_{\overline{0}}(M/\mathcal{F})$. A such partition always exits because $\mathbf{T}$ is compact. Since the three complex concerned are $\Omega^0_{\overline{0}}(M/\mathcal{F})$-modules then we can apply the Mayer -Vietoris argument (see for example [1]). We reduce the problem to:

$$\Omega^*_{\overline{0}}(U/\mathcal{F}) \xrightarrow{\sim} \mathcal{C}^*(U/\mathcal{F}) \xleftarrow{\sim} \Omega^*(U/\mathcal{F}) \tag{5}$$

for any invariant open subset $U \subset \left(M - \bigcup_{\alpha\in\Delta} \check{U}_\alpha\right)$ and

$$\Omega^*_{\overline{0}}(U_\alpha/\mathcal{F}) \xrightarrow{\sim} \mathcal{C}^*(U_\alpha/\mathcal{F}) \xleftarrow{\sim} \Omega^*(U_\alpha/\mathcal{F}) \tag{6}$$

for any $\alpha \in \Delta$, where $\sim$ stands for quasi-isomorphism.

• The statement (5) it is true because $\Omega^*_{\overline{0}}(U/\mathcal{F}) = \mathcal{C}^*(U/\mathcal{F}) = \Omega^*(U/\mathcal{F})$. For (6) we consider a good covering $\{W\}$ of F_α and $\{f_W\}$ a subordinated partition of unity. The family $\{\tau_\alpha^{-1}(W)\}$ is a covering of U_α and $\{f_W \circ \tau_\alpha\}$ is a subordinated partition of unity made up of elements of $\Omega^0_{\overline{0}}(U_\alpha/\mathcal{F})$. Using the Mayer-Vietoris argument we transform (6) in:

$$\Omega^*_{\overline{0}}(\mathbf{R}^m/\mathcal{F}^\alpha) \xrightarrow{\sim} \mathcal{C}^*(\mathbf{R}^m/\mathcal{F}^\alpha) \xleftarrow{\sim} \Omega^*(\mathbf{R}^m)/\mathcal{F}^\alpha. \tag{7}$$

Here $\mathcal{F}^\alpha$ denotes the foliation determined by the orthogonal action $\Phi^\alpha\colon \mathbf{R}\times\mathbf{R}^m \to \mathbf{R}^m$ defined by $\Phi^\alpha(g,(u_1,\ldots,u_{m-n_\alpha},v_1,\ldots,v_{n_\alpha})) = (u_1,\ldots,u_{m-n_\alpha},\Phi_\alpha(g,v_1,\ldots,v_{n_\alpha})))$. The fixed point set is $F_\alpha = \mathbf{R}^{m-n_\alpha}\times\{0\} \equiv \mathbf{R}^{m-n_\alpha}$ and the control data we use are: $\{\tau_\alpha \times \rho_\alpha \colon \mathbf{R}^m \to F_\alpha \times [0,\infty[\}$ defined by $(\tau_\alpha\times\rho_\alpha)(u_1,\ldots,u_{m-n_\alpha},v_1,\ldots,v_{n_\alpha}) = (u_1,\ldots,u_{m-n_\alpha},|(v_1,\ldots,v_{n_\alpha})|)$.

• *Proving* $H^*(\mathbf{R}^m/\mathcal{F}^\alpha) \cong \mathbf{R}$. Consider the retraction $f\colon \mathbf{R}^m\times[0,1]\to\mathbf{R}^m$ defined by $f(w,t) = t\cdot w$. Put $F\colon \Omega^*(\mathbf{R}^m)\to\Omega^{*-1}(\mathbf{R}^m)$ the induced homotopy operator:

$$F\omega = \int_{[0,1]} i_{\partial/\partial t} f^*\omega \wedge dt,$$

where $\int_{[0,1]}$ is the integration along the fibers of the canonical projection $pr\colon \mathbf{R}^n\times[0,1]\to\mathbf{R}^n$. This operator satisfies $dF\omega + Fd\omega = \omega - \omega(0)$ for any $\omega \in \Omega^i(\mathbf{R}^m)$ (see for example [4, pag 178]). Here, $\omega(0)$ denotes the restriction of ω to $\{0\}$ ($\omega(0)\equiv 0$ if $i \neq 0$). Suppose we have proved $i_X F\omega = F i_X\omega$ for $\omega \in \Omega^*(\mathbf{R}^m/\mathcal{F})$. Then $F\colon \Omega^*(\mathbf{R}^m/\mathcal{F}) \to \Omega^{*-1}(\mathbf{R}^m/\mathcal{F})$ is a well defined homotopy operator which shows that the basic cohomology $H^*(\mathbf{R}^m/\mathcal{F}^\alpha)$ can be calculated with the complex $\{\omega(0) \ / \ \omega \in \Omega^0(\mathbf{R}^m/\mathcal{F}^\alpha)\}$, that is, $H^*(\mathbf{R}^m/\mathcal{F}^\alpha) \cong \mathbf{R}$.

It remains to prove $i_X F\omega = F i_X\omega$ for $\omega \in \Omega^*(\mathbf{R}^m/\mathcal{F})$. As $pr_*(X,0) = X$ and $[(X,0),\partial/\partial t] = 0$ then $i_X F\omega = \displaystyle\int_{[0,1]} i_{\partial/\partial t} i_{(X,0)} f^*\omega \wedge dt$. On the other hand, since any homothety on $\mathbf{R}^n$ is equivariant then $X = f_*(X,0)$ and therefore $i_X F\omega = F i_X\omega$.

• *Proving* $H^*(\mathcal{C}^\cdot(\mathbf{R}^m/\mathcal{F}^\alpha)) \cong \mathbf{R}$ We check that the homotopy operator $F\colon \Omega^*(\mathbf{R}^m/\mathcal{F}^\alpha) \to \Omega^{*-1}(\mathbf{R}^m/\mathcal{F}^\alpha)$ preserves the bifs. Let ω be a differential form on $\mathbf{R}^m$ such that the restriction to $\omega|_{\mathbf{R}^m - F_\alpha}$ is a bif. In other words, there exists a differential form η on F_α such that $\omega = \tau_\alpha^*\eta$ on $\check{U}_\alpha$. Since on $\check{U}_\alpha$ we have $F\omega = \displaystyle\int_{[0,1]} i_{\partial/\partial t} f^*\tau_\alpha^*\eta \wedge dt = \tau_\alpha^* \int_{[0,1]} i_{\partial/\partial t} f_0^*\eta \wedge dt$, where $f_0\colon F_\alpha \times [0,1] \to F_\alpha$ is defined by $f_0(u,t) = t \cdot u$ then the restriction $F\omega|_{\mathbf{R}^m - F_\alpha}$ is a bif.

• *Proving* $IH^*_{\overline{0}}(\mathbf{R}^m/\mathcal{F}^\alpha) \cong \mathbf{R}$. Consider the retraction $h\colon (\mathbf{R}^m - F_\alpha) \times [0,1] \to (\mathbf{R}^m - F_\alpha)$ defined by

$$(8) \qquad \left\{ \begin{array}{lcl} h(u_1,\ldots,u_{m-n_\alpha},v_1,\ldots,v_{n_\alpha},t) & = & \\ t\cdot(u_1,\ldots,u_{m-n_\alpha},v_1,\ldots,v_{n_\alpha}) & + & (1-t)/4|(v_1,\ldots,v_{n_\alpha})|\cdot(0,\ldots,0,v_1,\ldots,v_{n_\alpha}). \end{array} \right.$$

Put $H\colon \Omega^*(\mathbf{R}^m - F_\alpha) \to \Omega^{*-1}(\mathbf{R}^m - F_\alpha)$ the induced homotopy operator defined by

$$H\omega = \int_{[0,1]} i_{\partial/\partial t} h^*\omega \wedge dt,$$

where $\displaystyle\int_{[0,1]}$ is the integration along the fibers of the canonical projection $pr\colon (\mathbf{R}^m - F_\alpha) \times [0,1] \to (\mathbf{R}^m - F_\alpha)$. Since X comes from an orthogonal action then $X = h_*(X,0)$ and therefore $i_X H\omega = H i_X\omega$. This gives $H(\Omega^*(\mathbf{R}^m/\mathcal{F}^\alpha)) \subset \Omega^{*-1}(\mathbf{R}^m/\mathcal{F}^\alpha)$. Moreover, if ω is a bif there exists a differential form η on F_α verifying $\omega = \tau_\alpha^*\eta$ on $\check{U}_\alpha$. A straightforward calculation shows that on $\check{U}_\alpha$ we have: $H\omega = \displaystyle\int_{[0,1]} i_{\partial/\partial t} h^*\tau_\alpha^*\eta \wedge dt = \tau_\alpha^* \int_{[0,1]} i_{\partial/\partial t} h_0^*\eta \wedge dt$, where $h_0\colon F_\alpha \times [0,1] \to F_\alpha$ is defined by $h_0(w,t) = t \cdot w$. This shows that $H\omega$ is a bif.

The homotopy operator $H\colon \Omega^*_{\overline{0}}(\mathbf{R}^m/\mathcal{F}^\alpha) \to \Omega^{*-1}_{\overline{0}}(\mathbf{R}^m/\mathcal{F}^\alpha)$ is well defined and verifies $dH\omega + Hd\omega = \omega - \omega(1/4)$ for any $\omega \in \Omega^*(\mathbf{R}^m/\mathcal{F}^\alpha)$. Here, $\omega(1/4)$ is the restriction of ω to $Imh = \tau_\alpha^{-1}(0) \cup \rho_\alpha^{-1}(1/4)$. Notice that $\deg\omega(1/4) \leq ||\omega||_\alpha \leq 0$ implies that $\omega(1/4) \equiv 0$ or $\deg\omega(1/4) = 0$. Then H is a well defined homotopy operator which shows that the cohomology $IH^*_{\overline{0}}(\mathbf{R}^m/\mathcal{F}^\alpha)$ can be calculated with the complex $\{\omega(1/4) \ / \ \omega \in \Omega^0_{\overline{0}}(\mathbf{R}^m/\mathcal{F}^\alpha)\}$, that is, $IH^*_{\overline{0}}(\mathbf{R}^m/\mathcal{F}^\alpha) \cong \mathbf{R}$. ♣

The following result has the flavor of the characteristic local calculation in intersection homology theory.

Proposition 2.3.2 *Let* $\Psi\colon \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$ *be an orthogonal action having* 0 *as unique fixed point. Write* $\mathcal{F}$ *(resp.* $\mathcal{G}$*) the foliation induced on* $\mathbf{R}^m$ *(resp. on the unit sphere* $\mathbf{S}^{m-1}$*) by this action. For any perversity* $\overline{r}$ *we have:*

$$IH^i_{\overline{r}}(\mathbf{R}^m/\mathcal{F}) \cong \left\{ \begin{array}{cl} H^i(\mathbf{S}^{m-1}/\mathcal{G}) & \text{if } i \leq \overline{r}(0) \\ 0 & \text{if } i > \overline{r}(0). \end{array} \right.$$

Proof. We use $\{\tau_0 \times \rho_0\colon \mathbf{R}^m \to \{0\} \times [0,\infty[\}$, defined by $(\tau_0 \times \rho_0)(u) = (0,|u|)$, as control data. Consider the homotopy operator $F\colon \Omega^*(\mathbf{R}^m/\mathcal{F}) \to \Omega^{*-1}(\mathbf{R}^m/\mathcal{F})$ defined by (8). It verifies $dF\omega + Fd\omega = \omega - \omega(1/4)$. Since the vector field X is tangent to the fibers of ρ_0 we know that the differential form $\omega(1/4)$ belongs to $\Omega^*(\mathbf{S}^{m-1}/\mathcal{G})$. This shows that the cohomology $IH^*_{\overline{r}}(\mathbf{R}^m/\mathcal{F})$ can be computed with the complex $\{\eta \in \Omega^*(\mathbf{S}^{m-1}/\mathcal{G}) \ / \ P^*\eta \in \Omega^*_{\overline{r}}(\mathbf{R}^m/\mathcal{F})\}$, where $P\colon (\mathbf{R}^m - \{0\}) \to \mathbf{S}^{m-1}$ is defined by $P(u) = u/|u|$. We end the proof noticing that $||P^*\eta||_0 = \deg \eta$. ♣

In fact, since the flow X has no zeros on the sphere then the number m is odd, say $2n+1$. Moreover, the Gysin sequence constructed in [6] shows that $H^*(\mathbf{S}^{m-1}/\mathcal{G}) \cong H^*(\mathbf{CP}^n)$.

When particularizing the action Φ, the basic intersection cohomology becomes familiar. We fix $\overline{r}$ a perversity.

2.3.3 The action Φ is almost free. Since $\Delta = \emptyset$ then $\Omega^*_{\overline{r}}(M/\mathcal{F}) = \Omega^*(M/\mathcal{F})$ and therefore the basic intersection cohomology$IH^*_{\overline{r}}(M/\mathcal{F})$ is just the basic cohomology $H^*(M/\mathcal{F})$. In particular, if Φ is also periodic then the orbit space $M/\mathbf{S}^1$ is an orbifold and we have $IH^*_{\overline{r}}(M/\mathcal{F}) = H^*(M/\mathbf{S}^1)$, the deRham cohomology in the category of orbifolds [9]. Moreover, if the action is free and periodic the basic intersection cohomology $IH^*_{\overline{r}}(M/\mathcal{F})$ is the deRham cohomology $H^*(M/\mathbf{S}^1)$.

2.3.4 The action Φ is periodic. We have already noticed that the orbit space is a stratified manifold.

Consider first the case where the action is semifree. The orbit space is a simple stratified space (cf. 2.2.2 b)). The pull back $\pi^*\colon \Omega^*(M - F/\mathbf{S}^1) \to \Omega^*(M - F/\mathcal{F}^*)$ is an isomorphism. For any $\alpha \in \Delta$ the restriction of $\pi\colon \check{U}_\alpha \to \pi(\check{U}_\alpha)$ to the fibers of τ_α is a submersion $(\mathbf{S}^{n_\alpha-1}\times]0,1/4[\to \mathbf{S}^{n_\alpha-1}/\mathbf{S}^1\times]0,1/4[)$ and then $\pi_*\{Ker(\tau_\alpha)_*\} = Ker(\gamma_\alpha)_*$. So, for any differential form $\omega \in \Omega^*(M - F/\mathbf{S}^1)$, we have $||\pi^*\omega||_\alpha = ||\omega||_\alpha$. The map $\pi^*\colon \Omega^*_{\overline{r}}(M/\mathbf{S}^1) \to \Omega^*_{\overline{r}}(M/\mathcal{F})$ is then an isomorphism and therefore we get that the basic intersection cohomology $IH^*_{\overline{r}}(M/\mathcal{F})$ is isomorphic to the intersection cohomology $IH^*_{\overline{r}}(M/\mathbf{S}^1)$.

When the action of the circle is not free on $M-F$ the orbit space $M/\mathbf{S}^1$ is a stratified space but not simple: the singular strata (corresponding to finite isotropy subgroups) are not closed and their closures met each other. Put $\{F_\alpha\}_{\alpha\in\nabla}$ the family of singular strata, where ∇ strictly contains Δ. The definition of the intersection cohomology of $M/\mathbf{S}^1$ involves also a control data for the strata and is similar to the definition given here for the simple case; but it refers to all the strata and not only the strata of fixed points. For sake of simplicity we do not give the exact definition an we refer the reader to [7] and [8]. Following the same argument as above, one can show that the pull back $\pi^*\colon \Omega^*_{\overline{r}}(M/\mathbf{S}^1) \to \Omega^*_{\overline{r}}(M/\mathcal{F})$ is a well defined monomorphism, where $\overline{r}\colon \nabla \to \mathbf{Z}$ is the perversity extending $\overline{r}\colon \Delta \to \mathbf{Z}$ by zero. Nevertheless it is not an isomorphism: the right hand side involves the strata of fixed points $\{F_\alpha\}_{\alpha\in\Delta}$ whereas the left hand side refers to all the strata $\{F_\alpha\}_{\alpha\in\nabla}$. The induced map in cohomology $\pi^*\colon IH^*_{\overline{r}}(M/\mathbf{S}^1) \to IH^*_{\overline{r}}(M/\mathcal{F})$ is however an isomorphism (essentially because $M - F/\mathbf{S}^1$ is a homological manifold).

In conclusion, when the action is periodic the basic intersection cohomology $IH^*_{\overline{r}}(M/\mathcal{F})$ is isomorphic to the intersection cohomology $IH^*_{\overline{r}}(M/\mathbf{S}^1)$.

3 Gysin sequence

We construct and study the Gysin sequence associated to the action Φ. The main tool we use are the invariant forms and the integration along the fibers.

3.1 Invariant differential forms. In the presence of a compact Lie group is tempting to consider the invariant differential forms in order to simplify the calculations. Remind that the complex of invariant differential forms

$$I\Omega^{*}(M) = \{\omega \in \Omega^{*}(M)/g^{*}\omega = \omega \text{ if } g \in \mathbf{T}\} = \{\omega \in \Omega^{*}(M)/L_X\omega = \omega\}$$

computes the cohomology of M (see for example [1]). We prove a similar result for the invariant bifs

$$I\Omega^{*}_{\overline{r}}(M) = \{\omega \in \Omega^{*}_{\overline{r}}(M)/g^{*}\omega = \omega \text{ if } g \in \mathbf{T}\} = \{\omega \in \Omega^{*}_{\overline{r}}(M)/L_X\omega = \omega\}.$$

Proposition 3.1.1 *For any perversity $\overline{0} \leq \overline{r} \leq \overline{t}$ we have $H^{*}(I\Omega^{\cdot}_{\overline{r}}(M)) \cong H^{*}(M)$.*

Proof. Since the inclusion $IH^{*}_{\overline{r}}(M) \cong H^{*}(M)$ is an isomorphism (cf. 2.2.5) it suffices to prove that the inclusion $I\Omega^{*}_{\overline{r}}(M) \hookrightarrow \Omega^{*}_{\overline{r}}(M)$ is a quasi-isomorphism. The operators used in [1] to prove that the inclusion $I\Omega^{*}(M-F) \hookrightarrow \Omega^{*}(M-F)$ is a quasi-isomorphism are a composition of operators of type $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ described below. It suffices to show that these operators preserve the bifs.

Put N a manifold. Consider the action $\Gamma\colon \mathbf{T} \times (N \times M) \to (N \times M)$ defined by $\Gamma(g,(x,y)) = (x, \widetilde{\Phi}(g,y))$. The singular strata are $\{N \times F_\alpha\}_{\alpha\in\Delta}$. We work with the following control data: $\{\tau_{N\times F_\alpha} \times \rho_{N\times F_\alpha}\colon N \times M \longrightarrow N \times F_\alpha \times [0,\infty[\}_{\alpha\in\Delta}$ defined by $\tau_{N\times F_\alpha}(x,y) = (x,\tau_\alpha(y))$ and $\rho_{N\times F_\alpha}(x,y) = \rho_\alpha(y)$. For a manifold N we shall write $pr_N\colon N \times M \to N$ and $pr_M\colon N \times M \to M$ the canonical projections.

- $\mathcal{L}_1\colon \Omega^{*}(N \times (M-F)) \to \Omega^{*}(M-F)$ is defined by $\mathcal{L}_1\omega = \int_N \omega \wedge pr_N^{*}\lambda$, where $\lambda \in \Omega^{*}(N)$ is a differential form with compact support and $\int_N$ is the integration along the fibers of pr_M.
- $\mathcal{L}_2\colon \Omega^{*}(N \times (M-F)) \to \Omega^{*-1}(N \times (M-F))$ is defined by $\mathcal{L}_2\omega = \int_{[0,1]} i_{\partial/\partial t} u^{*}\omega \wedge dt$, where $u\colon N \times [0,1] \times M \to N \times M$ is a smooth map with $u(x,t,y) = (u_0(x,t),y)$ and $\int_{[0,1]}$ is the integration along the fibers of the canonical projection $pr_{N\times M}\colon N \times M \times [0,1] \to N \times M$.
- $\mathcal{L}_3\colon \Omega^{*}(M-F) \to \Omega^{*}(\mathbf{T} \times (M-F))$ is defined by $\mathcal{L}_3\omega = \Phi^{*}\omega$.

These operators preserve the bifs.

- Let $\omega \in \Omega^{*}_{\overline{r}}(N \times M)$. Since the fibers of $\tau_{N\times F_\alpha}$ are included on the fibers of pr_N then $pr_N^{*}\lambda \in \Omega^{*}_{\overline{0}}(N \times M)$ and therefore $\omega \wedge pr_N^{*}\lambda \in \Omega^{*}_{\overline{r}}(N \times M)$ (see (4)). The equality $(pr_M)_{*}\Big\{Ker(\tau_{N\times F_\alpha})_{*}\Big\} = Ker(\tau_\alpha)_{*}$ implies that the operator $\int_N$ maps $\Omega^{*}_{\overline{r}}(N \times M)$ into $\Omega^{*}_{\overline{r}}(M)$. Thus $\mathcal{L}_1\omega \in \Omega^{*}_{\overline{r}}(M)$.
- Let $\omega \in \Omega^{*}_{\overline{r}}(N \times M)$. The map u maps the fibers of $\tau_{N\times[0,1]\times F_\alpha}$ into the fibers of $\tau_{N\times F_\alpha}$. Then $u^{*}\omega \in \Omega^{*}_{\overline{r}}(N \times [0,1] \times M)$. The equality $(pr_{N\times M})_{*}\Big\{Ker(\tau_{N\times F_\alpha})_{*}\Big\} = Ker(\tau_{N\times[0,1]\times F_\alpha})_{*}$ implies that $\mathcal{L}_2\omega$ belongs to $\Omega^{*}_{\overline{r}}(N \times M)$.
- Let $\omega \in \Omega^{*}_{\overline{r}}(M)$. The maps τ_α are invariant and therefore Φ maps the fibers of $\tau_{\mathbf{T}\times F_\alpha}$ into the fibers of τ_α. Then $\mathcal{L}_3\omega = \Phi^{*}\omega \in \Omega^{*}_{\overline{r}}(\mathbf{T} \times M)$. ♣

3.2 Construction of the Gysin sequence. The *integration along the fibers* is the differential operator $\oint : I\Omega^*_{\overline{r}}(M) \to \Omega^{*-1}_{\overline{r}-\overline{1}}(M/\mathcal{F})$ defined by linearity from $\oint \omega = (-1)^{\deg \omega} i_X\omega$. It is well defined because ω is invariant and because $\max\{||i_X\omega||_\alpha, ||d(i_X\omega)||_\alpha\} \leq \max\{||\omega||_\alpha - 1, ||d\omega||_\alpha - 1\}$ for any $\alpha \in \Delta$.The sign appears in order to get $d\oint\omega = \oint d\omega$. This operator gives rise to the short exact sequence

$$0 \longrightarrow \Omega^*_{\overline{r}}(M/\mathcal{F}) \xhookrightarrow{\iota} I\Omega^*_{\overline{r}}(M) \xrightarrow{\oint} Im\oint \longrightarrow 0, \tag{9}$$

since $Ker\oint = \{\omega \in I\Omega^*_{\overline{r}}(M)/i_X\omega = 0\} = \Omega^*_{\overline{r}}(M/\mathcal{F})$. The *Gysin sequence* is the associated long exact sequence

$$\cdots \longrightarrow H^i(M) \xrightarrow{\oint^*} H^{i-1}(Im\oint) \xrightarrow{\delta} IH^{i+1}_{\overline{r}}(M/\mathcal{F}) \xrightarrow{\iota^*} H^{i+1}(M) \longrightarrow \cdots, \tag{10}$$

where δ is the connecting homomorphism.

3.2.1 Euler class. Define the riemannian metric ν on $M - F$ by $\nu = (\mu(X,X))^{-1} \cdot \mu$. Write $\chi \in \Omega^1(M - F)$ the dual form to X with regard to ν, that is, $\chi(Y) = \nu(X,Y)$. The differential form χ is invariant and verifies $i_Xd\chi = -di_X\chi + L_X\chi = -d1 + 0 = 0$. The differential form $e = d\chi$ is therefore a closed bif. It defines a class $[e] \in IH^2_{\overline{2}}(M/\mathcal{F})$ called the *Euler class*.

Notice that the Euler class is not necessarily 0 in the intersection basic cohomology although e is exact ($i_X\chi \neq 0$!). This class is independent of the choice of the metric. For another riemannian metric ν' on $M - F$, with X unitary vector field, we get $i_X(\chi - \chi') = 0$ and therefore $\chi - \chi' \in \Omega^1_{\overline{2}}(M/\mathcal{F})$ which implies $[e] = [e']$ on $IH^2_{\overline{2}}(M/\mathcal{F})$.

3.2.2 Image of $\oint$. For each differential form $\omega \in \Omega^*_{\overline{r}-\overline{2}}(M/\mathcal{F})$ the product $\omega \wedge \chi$ is an invariant form satisfying $\max\{||\omega \wedge \chi||_\alpha, ||d(\omega \wedge \chi)||_\alpha\} \leq \max\{||\omega||_\alpha + 2, ||d\omega||_\alpha + 1\} \leq \overline{r}(\alpha)$ (cf. (4))) and $\oint \chi \wedge \omega = \omega$. So, the complex $\Omega^*_{\overline{r}-\overline{2}}(M/\mathcal{F})$ is a subcomplex of $Im\oint$.

Proposition 3.2.3 *For any perversity $\overline{0} \leq \overline{r} \leq \overline{t}$ the inclusion $\Omega^*_{\overline{r}-\overline{2}}(M/\mathcal{F}) \hookrightarrow Im\oint$ is a quasi-isomorphism.*

Proof. Given a smooth function $f \in \Omega^0_{\overline{0}}(M)$ and $\omega \in \Omega^*_{\overline{r}}(M)$ we have $f\omega \in \Omega^*_{\overline{r}}(M)$ and $\oint f\omega = f\oint\omega$. The Mayer-Vietoris argument applies in this context and we can proceed as in Proposition 2.3.1. The problem is reduced to prove, using the notations of 2.3.1:

$$\Omega^*_{\overline{r}-\overline{2}}(\mathbf{R}^m/\mathcal{F}^\alpha) \overset{\sim}{\hookrightarrow} Im\left\{\oint : I\Omega^{*+1}_{\overline{r}}(\mathbf{R}^m) \longleftarrow \Omega^*_{\overline{r}-\overline{1}}(\mathbf{R}^m/\mathcal{F}^\alpha)\right\}. \tag{11}$$

Remind that $\mathcal{F}^\alpha$ is the foliation on $\mathbf{R}^m$ defined by the orthogonal action Φ^α. Identify $\mathbf{S}^{n_\alpha-1}$ with the sphere $\tau_\alpha^{-1}(0) \cap \rho_\alpha^{-1}(1/4)$. Since the action Φ^α is orthogonal it induces an orthogonal action $\Phi_\alpha : \mathbf{R} \times \mathbf{S}^{n_\alpha-1} \to \mathbf{S}^{n_\alpha-1}$ (notice that this action is the restriction of the action $\Phi_\alpha : \mathbf{T} \times \mathbf{S}^{n_\alpha-1} \to \mathbf{S}^{n_\alpha-1}$ of 1.1). Write $\mathcal{F}_\alpha$ the foliation on $\mathbf{S}^{n_\alpha-1}$ induced by Φ_α. Since Φ_α is an almost free action then $\mathcal{F}_\alpha$ is a regular foliation.

Consider the homotopy operator $H\colon \Omega^*(\mathbf{R}^m - F_\alpha) \to \Omega^{*-1}(\mathbf{R}^m - F_\alpha)$ induced by the retraction $h\colon (\mathbf{R}^m - F_\alpha) \times [0,1] \to (\mathbf{R}^m - F_\alpha)$ defined by (8). Suppose that for any perversity $\overline{p}$ the restriction $H\colon \Omega^*_{\overline{p}}(\mathbf{R}^m/\mathcal{F}^\alpha) \to \Omega^{*-1}_{\overline{p}}(\mathbf{R}^m/\mathcal{F}^\alpha)$ is well defined. For any $\omega \in \Omega^*_{\overline{p}}(\mathbf{R}^m/\mathcal{F}^\alpha)$ we have:

$$dH\omega + Hd\omega = \omega - \omega(1/4) \quad \text{and} \quad \deg\omega(1/4) \le ||\omega||_\alpha \le \overline{p}(\alpha).$$

Then, the basic intersection cohomology $IH^*_{\overline{r}-\overline{2}}(\mathbf{R}^m/\mathcal{F}^\alpha)$ can be calculated by using the complex:

$$\mathcal{E}_0^* = \Omega^{<\overline{r}(\alpha)-2}(\mathbf{S}^{n_\alpha-1}/\mathcal{F}_\alpha) \oplus \left\{\Omega^{\overline{r}(\alpha)-2}(\mathbf{S}^{n_\alpha-1}/\mathcal{F}_\alpha) \cap d^{-1}(0)\right\}.$$

On the other hand, for any $\omega \in I\Omega^*_{\overline{r}}(\mathbf{R}^m)$ we have $\left(\oint\omega\right)(1/4) = \oint\omega(1/4)$ and therefore the cohomology $Im\left\{\oint\colon I\Omega^{*+1}_{\overline{r}}(\mathbf{R}^m) \longleftarrow \Omega^*_{\overline{r}-\overline{1}}(\mathbf{R}^m/\mathcal{F}^\alpha)\right\}$ can be calculated with the complex:

$$\begin{aligned}\mathcal{E}_1^* &= Im\left\{\oint\colon I\Omega^{*+1<\overline{r}(\alpha)}(\mathbf{S}^{n_\alpha-1}) \to \Omega^{*<\overline{r}(\alpha)-1}(\mathbf{S}^{n_\alpha-1}/\mathcal{F}_\alpha)\right\} \\ &\oplus Im\left\{\oint\colon I\Omega^{\overline{r}(\alpha)}(\mathbf{S}^{n_\alpha-1}) \cap d^{-1}(0) \to \Omega^{\overline{r}(\alpha)-1}(\mathbf{S}^{n_\alpha-1}/\mathcal{F}_\alpha) \cap d^{-1}(0)\right\}.\end{aligned}$$

We have reduced (11) to $\mathcal{E}_0^* \overset{\sim}{\hookrightarrow} \mathcal{E}_1^*$.

The integration operator $\oint\colon I\Omega^{*+1}(\mathbf{S}^{n_\alpha-1}) \to \Omega^*(\mathbf{S}^{n_\alpha-1}/\mathcal{F}_\alpha)$ is onto. In fact, for any $\omega \in \Omega^*(\mathbf{S}^{n_\alpha-1}/\mathcal{F}_\alpha)$ the product $\chi \wedge \omega$ is an invariant form satisfying $\oint\chi\wedge\omega = \omega$. Nevertheless, notice that even if ω is closed we don't get $d(\chi\wedge\omega) = 0$. So, $\mathcal{E}_0^* \overset{\sim}{\hookrightarrow} \mathcal{E}_1^*$ becomes

$$\mathcal{E}_0^* \overset{\sim}{\hookrightarrow} \Omega^{*<\overline{r}(\alpha)-1}(\mathbf{S}^{n_\alpha-1}/\mathcal{F}_\alpha) \oplus Im\left\{\oint\colon \left\{I\Omega^{\overline{r}(\alpha)}(\mathbf{S}^{n_\alpha-1}) \cap d^{-1}(0)\right\} \to \left\{\Omega^{\overline{r}(\alpha)-1}(\mathbf{S}^{n_\alpha-1}/\mathcal{F}_\alpha) \cap d^{-1}(0)\right\}\right\}.$$

We end the proof if we show:

a) For any closed form $\omega \in I\Omega^{\overline{r}(\alpha)}(\mathbf{S}^{n_\alpha-1})$ there exists $\eta \in I\Omega^{\overline{r}(\alpha)-1}(\mathbf{S}^{n_\alpha-1})$ with $\oint\omega = d\oint\eta$.

b) $H\colon \Omega^*_{\overline{p}}(\mathbf{R}^m/\mathcal{F}_\alpha) \longrightarrow \Omega^{*-1}_{\overline{p}}(\mathbf{R}^m/\mathcal{F}_\alpha)$ is well defined for any perversity $\overline{p}$.

We prove this.

a) Remind that we have $0 \le \overline{r}(\alpha) \le n_\alpha - 2$. When $\overline{r}(\alpha) = 0$ we have $\oint\omega = 0$ and it is enough to take $\eta = 0$. For the other cases we have $H^{\overline{r}(\alpha)}(\mathbf{S}^{n_\alpha-1}) = 0$ and therefore we can find $\eta \in I\Omega^{\overline{r}(\alpha)-1}(\mathbf{S}^{n_\alpha-1})$ with $d\eta = \omega$. We have finished because $\oint d\eta = d\oint\eta$.

b) We have already seen in the proof of the Proposition 2.3.1 that H maps $\Omega^*(\mathbf{R}^m/\mathcal{F}_\alpha)$ into $\Omega^{*-1}(\mathbf{R}^m/\mathcal{F}_\alpha)$. It remains to verify that H maps $\Omega^*_{\overline{p}}(\mathbf{R}^m)$ into $\Omega^{*-1}_{\overline{p}}(\mathbf{R}^m)$. We proceed as in the proof of Proposition 3.1.1, whose notations we use. The map h sends the fibers of $\tau_{F_\alpha\times[0,1]}$ into the fibers of τ_α and then $h^*\omega \in \Omega^*_{\overline{p}}(\mathbf{R}^n \times [0,1])$. Put $pr_{\mathbf{R}^n}\colon \mathbf{R}^n \times [0,1] \to \mathbf{R}^n$ the canonical projection. The equality $(pr_{\mathbf{R}^n})_*\left\{Ker(\tau_{F_\alpha\times[0,1]})_*\right\} = Ker(\tau_\alpha)_*$ implies that $H\omega$ belongs to $\Omega^*_{\overline{p}}(\mathbf{R}^m)$. ♣

We arrive at the main result of this work.

Theorem 3.2.4 *Let $\Phi\colon \mathbf{R} \times M \to M$ be an isometric action. Then there exists a long exact sequence*

$$\text{(12)} \qquad \cdots \longrightarrow H^i(M) \longrightarrow IH^{i-1}_{\overline{r}-\overline{2}}(M/\mathcal{F}) \xrightarrow{\wedge[e]} IH^{i+1}_{\overline{r}}(M/\mathcal{F}) \longrightarrow H^{i+1}(M) \longrightarrow \cdots,$$

where $\overline{r}$ is any perversity verifying $\overline{0} \le \overline{r} \le \overline{t}$.

Proof. We have already seen that the short exact sequence (9) gives rise to the long exact sequence (10). The connecting homomorphism is defined by $\delta[\alpha] = [e] \wedge [\alpha]$ (cf. 3.2.2). It suffices now to apply the previous result. ♣

This long exact sequence is called the *Gysin sequence*.

3.2.5 Remarks.

a) The Gysin sequence does not depend on the choice of the riemannian metric μ.

b) When the action Φ is almost free the intersection basic cohomology becomes the basic cohomology (cf. 2.3.3) and (12) becomes (2). When the action Φ is periodic the intersection basic cohomology becomes the intersection cohomology (cf. 2.3.4) and (12) becomes (1). When the action is almost free and periodic the intersection basic cohomology becomes the deRham cohomology (cf. 2.3.3) and (12) becomes the usual Gysin sequence (1).

c) Taking $\overline{r} = \overline{0}$ the Gysin sequence becomes:

$$\cdots \longrightarrow H^{i}(M) \longrightarrow H_{c}^{i-1}((M-F)/\mathcal{F}) \xrightarrow{\wedge[e]} H^{i+1}(M/\mathcal{F}) \longrightarrow H^{i+1}(M) \longrightarrow \cdots,$$

(cf. (2.3.1)). Taking $\overline{r} = \overline{2}$ the Gysin sequence becomes:

$$\cdots \longrightarrow H^{i}(M) \longrightarrow H^{i-1}(M/\mathcal{F}) \xrightarrow{\wedge[e]} IH_{\overline{2}}^{i+1}(M/\mathcal{F}) \longrightarrow H^{i+1}(M) \longrightarrow \cdots.$$

This shows that the deRham cohomology is not enough to extend the Gysin sequence to the singular case.

d) A more sofisticated stratification can be considered on M by classifying its points following the isotropy subgroups. This leads to another approach of the basic intersection cohomology, where this finer stratification is considered. But the final result is the same because the new links are cohomologycally trivial.

3.3 Vanishing of the Euler class. Consider Φ almost free. The Euler class $[e] \in H^{2}(M/\mathcal{F})$ vanishes if and only if there exists a locally trivial fibration whose fibers are transverse to the orbifold Φ (cf. [10]).

A singular foliation $\mathcal{G}$ on M is said to be *transverse* to Φ if:

1) The singular leaves of $\mathcal{G}$ are the points of F.

2) For each point $M - F$ the leaf of $\mathcal{G}$ and the orbit of Φ passing through the point are transverse.

When Φ is periodic, we have proved in [6] that the vanishing of the Euler class $[e] \in IH_{\overline{2}}^{2}(M/\mathbf{S}^1)$ is equivalent to the existence of a singular foliation $\mathcal{G}$, transverse to the orbits of Φ, whose restriction to $M - F$ is a locally trivial fibration. Using the same strategy on can show the next result.

Proposition 3.3.1 *The two following statements are equivalent.*

a) The Euler class $[e] \in IH_{\overline{2}}^{2}(M/\mathcal{F})$ *vanishes.*

b) There exists a singular foliation $\mathcal{G}$*, transverse to the orbits of* Φ*, whose restriction to* $M - F$ *is a locally trivial fibration.*

Proof. Sketch. We blow up Φ into an almost free isometric action $\hat{\Phi}\colon \mathbf{R} \times \hat{M} \to \hat{M}$ whose Euler class also vanishes. We find a locally trivial fibration $\hat{\mathcal{G}}$ on $\hat{M}$ transverse to the orbits of $\hat{\Phi}$. The foliation $\mathcal{G}$ is just the push down of $\hat{\mathcal{G}}$. The reciprocal comes from the fact that the Euler class vanishes on $IH_{\overline{2}}^{2}(M/\mathcal{F})$ if and only if it vanishes on $H^{2}((M-F)/\mathcal{F})$ ♣

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