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# Fuzzy Filters

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In this paper a characterization of some fuzzy topological concepts, such as open sets, closed sets, adherent points, continuous functions,... is given by means of fuzzy filter convergence as defined in [2]. F-ultrafilters are also characterized and relations between F-filters and F-nets are studied, getting results analogous to those for general topology. © 1988 Academic Press, Inc.

## 1. PRELIMINARIES

In this section we will recall some of the definitions related to fuzzy points that will be used throughout the paper.

Let $X$ be a set and $I$ the unit interval. A fuzzy set in $X$ is an element of the set of all functions from $X$ into $I$.

DEFINITION 1.1. A fuzzy point $p$ in $X$ is a fuzzy set with membership function:

$$p(x)=\begin{cases} t_0 & \text{if } x=x_0 \\ 0 & \text{otherwise,} \end{cases}$$

where $0<t_0<1$. $p$ is said to have support $x_0$ and value $t_0$, and is noted by $p(x_0, t_0)$ or even $(x_0, t_0)$



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DEFINITION 1.2. Let $p$ be a fuzzy point in $X$ and $A$ an F-set (fuzzy set) in $X$; $p$ is said to be in the F-set $A$ (noted $p \in A$) if $p(x_0) < A(x_0)$, $x_0$ being the support of $p$.

DEFINITION 1.3. Let $p$ be an $F$-point (fuzzy point) in $X$ and $(X, \delta)$ an fuzzy topological space (f.t.s.). $N \in I^X$ is said to be an F-nhood (F-neighborhood) of $p$ if there is some open F-set $\mu$ in $X$ such that $p \in \mu$ and $\mu \leqslant N$.

The collection of all nhoods of an F-point $p$ in a topology $\delta$ is noted by $\mathcal{N}_p^\delta$ ($\mathcal{N}_p$ if no confusion is possible).

PROPOSITION 1.1 [1]. *Let $(X, \delta)$ be an f.t.s. An F-set $\mu$ of $X$ is open iff, for all $p \in \mu$, then $\mu \in \mathcal{N}_p$.*

DEFINITION 1.4. Let $(X, \delta)$ be an f.t.s. and $p$ an F-point. The collection $\mathcal{B}_p$ of subsets of $\mathcal{N}_p$ is a local basis at $p$ if, for all $N$ in $\mathcal{N}_p$, there is some $B \in \mathcal{B}_p$ with $p \in B$ and $B \leqslant N$.

DEFINITION 1.5 [2]. Let $A$ be an F-set and $(X, \delta)$ an f.t.s. An F-point $p$ is said to be in the adherence of $A$ (e.g., is adherent to $A$) if for each $N \in \mathcal{N}_{p'}$, $N \not\leqslant A^c$, where $p' = (x, 1-t)$.

PROPOSITION 1.2 [2]. $\bar{A} = \bigcup \{p/p$ *is in the adherence of* $A\}$

PROPOSITION 1.3 [2]. *An F-point $p$ is said to be in the closure of $A$, $p \in \bar{A}$, iff there is some F-point $q$ in the adherence of $A$ with $p \in q$.*

DEFINITION 1.6. Let $(X, \delta)$ and $(Y, \gamma)$ be f.t.s.'s and $f$ a map from $X$ into $Y$. $f$ is said to be F-continuous if for each F-point $p$ in $X$ and each nhood $N$ of $f(p)$, there is some nhood $M$ of $p$ with $f(M) \leqslant N$.

THEOREM 1.1 [1]. *Let $(X, \delta)$, $(Y, \gamma)$ be fuzzy topological spaces and $f$ a map from $X$ into $Y$. Then $f$ is F-continuous iff for each $\mu \in \gamma$, $f^{-1}(\mu) \in \delta$.*

## 2. FUZZY FILTERS

DEFINITION 2.1. A prefilter $\mathcal{F}$ on $X$ is a nonempty collection of subsets of $I^X$ with the properties:

(i) If $F_1, F_2 \in \mathcal{F}$ then $F_1 \wedge F_2 \in \mathcal{F}$

(ii) If $F \in \mathcal{F}$ and $F' \geqslant F$, then $F' \in \mathcal{F}$

(iii) $0 \notin \mathcal{F}$.

DEFINITION 2.2. A collection $\mathscr{B}$ of subsets of $I^X$ is a base for some prefilter iff $\mathscr{B} \neq \varnothing$ and

(i) if $B_1, B_2 \in \mathscr{B}$ then $B_3 \leqslant B_1 \wedge B_2$ for some $B_3 \in \mathscr{B}$;

(ii) $0 \notin \mathscr{B}$.

The collection

$$\mathscr{F} = \{F \in I^X / \exists B \in \mathscr{B} \text{ s.t. } F \geqslant B\}$$

is a prefilter. $\mathscr{F}$ is said to be generated by $\mathscr{B}$ and is denoted $\langle \mathscr{B} \rangle$.

A collection $\mathscr{B}$ of subsets of $\mathscr{F}$ is a base for $\mathscr{F}$ iff for each $F \in \mathscr{F}$ there is some $B \in \mathscr{B}$ such that $B \leqslant F$.

If $\mathscr{F}_1$ and $\mathscr{F}_2$ are prefilters on $X$, $\mathscr{F}_1$ is said to be finer than $\mathscr{F}_2$ (equivalently: $\mathscr{F}_2$ is coarser than $\mathscr{F}_1$) iff $\mathscr{F}_1 \supset \mathscr{F}_2$.

DEFINITION 2.3. A prefilter $\mathscr{F}$ is said to converge to the fuzzy point $p$ (written $\mathscr{F} \to p$) iff $\mathscr{N}_p \subset \mathscr{F}$, that is, $\mathscr{F}$ is finer than the nhood prefilter at $p$ [2].

We say $\mathscr{F}$ has $p$ as a cluster point (written $\mathscr{F} \infty p$) iff $\forall N \in \mathscr{N}_p$, then $N \wedge F \neq 0$, $\forall F \in \mathscr{F}$.

We can express these notions in terms of bases of prefilters as follows:

(i) A base for a prefilter converges to a fuzzy point $p$ ($\mathscr{B} \to p$) iff each $N \in \mathscr{N}_p$ contains some $B \in \mathscr{B}$

(ii) A base for a prefilter has $p$ as a cluster point ($\mathscr{B} \infty p$) iff each $N \in \mathscr{N}_p$ meets each $B \in \mathscr{B}$.

These definitions are still valid if we use nhood bases at $p$, $\mathscr{B}_p$, instead of nhood systems at $p$, $\mathscr{N}_p$.

Clearly, if $\mathscr{F} \to p$, then $\mathscr{F} \infty p$.

PROPOSITION 2.1. *Let* $(X, \delta)$ *be a fuzzy topological space and* $\mathscr{F}$ *a prefilter on* $X$. *Then*: $\mathscr{F} \infty p$ *iff there is a prefilter* $\mathscr{G}$ *such that* $\mathscr{G} \supset \mathscr{F}$ *and* $\mathscr{G} \to p$.

*Proof.* Sufficiency follows trivially.

For necessity, if $\mathscr{B} = \{N \wedge F$, such that $N \in \mathscr{N}_p$ and $F \in \mathscr{F}\}$, then $\mathscr{B}$ constitutes a base for a prefilter $\mathscr{G}$. $\mathscr{G}$ is finer than $\mathscr{F}$ and converges to $p$.

*Remark.* If a prefilter $\mathscr{G}$ clusters at some fuzzy point $p$, then every coarser one does too.

PROPOSITION 2.2. *Let* $(X, \delta)$ *be an f.t.s.,* $A$ *a fuzzy set of* $X$, *and* $p$ *a fuzzy point. The following are equivalent*:

(i) $A \in \delta$ *iff for each prefilter* $\mathscr{F}$ *on* $X$ *converging to* $p \in A$, *we have* $A \in \mathscr{F}$;

(ii) *$p$ is adherent to $A$ iff there is some prefilter $\mathscr{F}$ on $X$ such that $A^c \notin \mathscr{F}$ and $\mathscr{F} \to p'$;*

(iii) *$A \in \delta^c$ iff whenever $\mathscr{F}$ is a prefilter on $X$ such that $A^c \notin \mathscr{F}$ and $\mathscr{F} \to p'$, then $p \subset A$.*

*Proof.* (i) Necessity follows from the definition of an open set.

For sufficiency we consider for each $p \in A$, $\mathscr{N}_p = \mathscr{F}$. Then $A \in \mathscr{N}_p$, $\forall p \in A$, so $A \in \delta$.

(ii) If $p$ is adherent to $A$, then $\forall N \in \mathscr{N}_{p'}$, $N \not\subset A^c$. Considering $\mathscr{F} = \mathscr{N}_{p'}$, $\mathscr{F} \to p'$ and $A^c \notin \mathscr{N}_{p'}$. Conversely, let $\mathscr{F}$ be a prefilter with $\mathscr{F} \to p'$ and $A^c \notin \mathscr{F}$. Then $N \not\subset A^c$, $\forall N \in \mathscr{N}_{p'}$; otherwise $A^c \in \mathscr{N}_{p'} \subset \mathscr{F}$. It follows that $p$ is adherent to $A$.

(iii) Necessity follows from (ii) and the fact that

$$\bar{A} = \bigcup \{p/p \text{ adherent to } A\}.$$

Conversely, each fuzzy point $p$ that is adherent to $A$, belongs to $A$. Then $\bar{A} = \bigcup \{p/p \text{ adherent to } A\} \subset A$ and $A \in \delta^c$.

LEMMA 2.1. *Let $f$ be a map from $X$ into $Y$ and $\mathscr{F}$ a prefilter on $X$. Then, $\{f(F)/F \in \mathscr{F}\}$ is a base for a prefilter on $Y$.*

PROPOSITION 2.3 [1]. *Let $(X, \delta)$ and $(Y, \delta')$ be f.t.s.'s and $f$ a map from $X$ into $Y$. Then $f$ is fuzzy continuous (F-continuous) iff whenever $\mathscr{F}$ converges to $p$, $p$ being a fuzzy point, $f(\mathscr{F})$ converges to $f(p)$.*

*Proof.* Suppose $\mathscr{F} \to p$. Then each $M \in \mathscr{N}_p$ is in $\mathscr{F}$. Let $N \in \mathscr{N}_{f(p)}$. By F-continuity [1, Definition 3.1 and Theorem 3.1], there must be some $M' \in \mathscr{N}_p$ such that $f(M') \leqslant N$.

Conversely, let $\mathscr{F} = \mathscr{N}_p$. As $\mathscr{N}_p \to p$, $f(\mathscr{N}_p) \to f(p)$. Hence, for each $N \in \mathscr{N}_{f(p)}$, there is some $M \in \mathscr{N}_p$ such that $f(M) \leqslant N$ and $f$ is F-continuous.

PROPOSITION 2.4. *Let $(X, \delta) = \prod_{j \in J} (X_j, \delta_j)$ be a fuzzy product space [4], $\mathscr{F}$ a prefilter on $X$, and $p$ a fuzzy point in $X$. Then:*

(i) *$\mathscr{F} \to p$ iff for each $j \in J$, $p_j(\mathscr{F}) \to p_j = \pi_j(p)$.*

(ii) *If $\mathscr{F} \infty p$, then for each $j \in J$, $p_j(F) \infty p_j$.*

*Proof.* (i) Sufficiency follows from Proposition 2.3 and the F-continuity of the projection maps.

Necessity. We know that the collection

$$\beta = \{\inf_{j \in K} \{v_j \circ \pi_j / v_j \in \delta_j, \forall j \in K, K \subset I, \text{ and } K \text{ a finite set}\}$$

is a base for $\delta$.

We will prove that for each $\mu \in \beta$ such that $p \in \mu$, there is some $F \in \mathscr{F}$ such that $F \subset \mu$.

Let $p = ((x_i)_{i \in J}, t)$ be a fuzzy point in $X = \prod_{i \in J} X_i$ such that $p \in \mu$. Then $p((x_i)_{i \in J}) < \mu(x_i)_{i \in J}) = \mathrm{Inf}_{j \in K} \{v_j(\pi_j(x_i)_{i \in J})\} = \mathrm{Inf}_{j \in K} \{v_j(x_j)\}$; i.e., $t < \mathrm{Inf}_{j \in K} \{v_j(x_j)\}$, $\pi_j(p)(x_j) = t < \mathrm{Inf}_{j \in K} \{v_j(x_j)\} \leqslant v_j(x_j)$ $\forall j \in K$, with $v_j \in \delta_j$ and $p_j \in v_j$ $\forall j \in K$. Consequently, $\forall j \in K$, there is some $F_j \in \mathscr{F}$ such that $p_j(F_j) \leqslant v_j$. Choose $F = \mathrm{Inf}_{j \in K} \{F_j\} \in \mathscr{F}$, then $F \subset \mu$ and $\mu \in \mathscr{F}$.

(ii) Follows from (i) and Proposition 2.1.

DEFINITION 2.4. A prefilter $\mathscr{F}$ on $X$ is an F-ultrafilter if there is no other prefilter finer than $\mathscr{F}$ (i.e., it is maximal for the inclusion relation among prefilters).

DEFINITION 2.5. Let $\mathscr{F}$ be a prefilter on $X$. A subset $Y$ of $X$ is included in $\mathscr{F}$ if every fuzzy subset with support $Y$ is an element of $\mathscr{F}$.

PROPOSITION 2.5. *If $X$ is a set and $\mathscr{F}$ a prefilter on $X$, the following are equivalent*:

(i) *$\mathscr{F}$ is an F-ultrafilter.*

(ii) *Let $A \in I^X$. If $A \notin \mathscr{F}$ then there is some $F \in \mathscr{F}$ such that $A \wedge F = 0$.*

(iii) *Let $Y$ be a subset of $X$. Then either $Y$ or $Y^c$ is included in $\mathscr{F}$.*

*Proof.* (i)$\Rightarrow$(ii) If $A \notin \mathscr{F}$ and $A \wedge F \neq 0$ for each $F \in \mathscr{F}$, then the collection $\mathscr{B} = \{A \wedge F / F \in \mathscr{F}\}$ is a base for a prefilter $\mathscr{G}$ which is strictly finer than $\mathscr{F}$, since $A \in \mathscr{G}$.

(ii)$\Rightarrow$(iii) Let $Y \subset X$. If there are a fuzzy set $A$ with support $Y$ and a fuzzy set $B$ with support $Y^c$ not belonging to the prefilter, there must be, by hypothesis, two fuzzy sets $F_A, F_B \in \mathscr{F}$ such that $F_A \wedge A = 0$ and $F_B \wedge B = 0$. Then $F_A(x) = 0$, $\forall x \in Y^c$ and $F_B(x) = 0$, $\forall x \in Y$ is impossible since $F_A \wedge F_B \in \mathscr{F}$.

(iii)$\Rightarrow$(i) If $\mathscr{F}$ is not an F-ultrafilter, let $\mathscr{G} \supset \mathscr{F}$ and $G \in \mathscr{G}$ such that $G \notin \mathscr{F}$. If $Y = \mathrm{Supp}\, G$, any fuzzy set $B$ with support $Y^c$ is in $\mathscr{F}$ and then in $\mathscr{G}$. This is impossible since $B \wedge G = 0$.

We say a prefilter $\mathscr{F}$ is free iff $\bigcap \{F/F \in \mathscr{F}\} = \varnothing$ and is fixed iff $\bigcap \{F/F \in \mathscr{F}\} \neq \varnothing$.

PROPOSITION 2.6. *Every fuzzy ultrafilter $\mathscr{F}$ is free.*

*Proof.* If $\bigcap \{F/F \in \mathscr{F}\} \neq \varnothing$, there is some fuzzy point $p \in \bigcap \{F/F \in \mathscr{F}\}$ $(p = (x, t))$. By Proposition 2.5 the fuzzy point $q = (x, s) \in \mathscr{F}$ $\forall s \in (0, 1)$, and then $p \in q$. This is impossible since $s < t$.

This suggests the following definition:

DEFINITION 2.6. An F-ultrafilter $\mathscr{F}$ is strong free (or s-free for short) iff $\bigcap \{\text{Supp } F/F \in \mathscr{F}\} = \varnothing$.

Clearly, if $\bigcap \{\text{Supp } F/F \in \mathscr{F}\} \neq \varnothing$ and $\mathscr{F}$ is an F-ultrafilter, this intersection must be reduced to a point.

The only s-fixed F-ultrafilters are: $\mathscr{F}_x = \{F \in I^X/F(x) > 0\}$. We will call them trivial F-ultrafilters.

PROPOSITION 2.7. *If $X$ and $Y$ are sets of points, $f: X \to Y$ a map from $X$ into $Y$, and $\mathscr{F}$ an F-ultrafilter on $X$, then $f(\mathscr{F})$ is an F-ultrafilter on $Y$.*

*Proof.* Let $Y_0 \subset Y$ and $X_0 = f^{-1}(Y_0)$. Suppose $X_0$ is included in $\mathscr{F}$ (obviously if $f^{-1}(Y_0) = \varnothing$, then $X_0^c$ is included in $\mathscr{F}$). Then $Y_0$ is included in $f(\mathscr{F})$.

Let $A \in I^Y$ such that $\text{supp } A = Y_0$ and $B = f^{-1}(A) = A \circ f$, then $\text{supp } B = \text{supp } f^{-1}(A) = f^{-1} \text{ supp } A = X_0$, and hence $B \in \mathscr{F}$. Besides $f(B) \leqslant A$ and $A \in f(\mathscr{F})$.

The following theorem shows the relation between F-ultrafilters and ultrafilters on a set $X$.

THEOREM 2.1. *Let $X$ be a set of points, $\mathbb{U}$ the family of all ultrafilters on $X$, and $\mathbb{U}_F$ the family of all* F*-ultrafilters. We define two maps:*

$$a: \mathbb{U} \to \mathbb{U}_F \text{ by: } a(\mathfrak{G}) = \{F \in I^X/\text{supp } F \in \mathfrak{G}\}, \qquad \forall \mathfrak{G} \in \mathbb{U}$$

and

$$b: \mathbb{U}_F \to \mathbb{U} \text{ by: } b(\mathscr{F}) = \{\text{supp } F/F \in \mathscr{F}\}, \qquad \forall \mathscr{F} \in \mathbb{U}_F;$$

*a and b are well defined and they are inverse to each other.*

*Proof.* It follows from Proposition 2.5(iii), from the fact that $a(\mathfrak{G})$ (resp. $b(\mathscr{F})$) is an F-ultrafilter (resp. ultrafilter) on $X$. The second assertion is evident.

## 5. RELATION BETWEEN F-NETS AND F-FILTERS

In the following, we will use the definitions and properties of F-nets and F-ultranets given in [3].

DEFINITION 3.1. Let $\mathscr{F}$ be a prefilter on $X$, $\mathscr{P}_F$ the collection of all fuzzy points in $X$, and

$$A_{\mathscr{F}} = \{(p, F)/p \in F,\ p \in \mathscr{P}_F(X),\ F \in \mathscr{F}\} \subset \mathscr{P}_F(X) \times \mathscr{F},$$

directed by the relation: $(p_1, F_1) \leqslant (p_2, F_2)$ iff $F_2 \leqslant F_1$. The map $\psi_{\mathscr{F}}: \Lambda_{\mathscr{F}} \to \mathscr{P}_F(X)$ defined by $\psi_{\mathscr{F}}(p, F) = p$ is an F-net in $X$. It is called the F-net based on $\mathscr{F}$.

DEFINITION 3.2. Let $\psi: D \to \mathscr{P}_F(X)$ be an F-net in $X$. The F-filter generated by the collection $\mathscr{B}$ of all the F-subsets in which the F-net $\psi$ is residually contained (this collection is a base for a prefilter) is called the F-filter (or prefilter) generated by $\psi$ (Noted $\mathscr{F}_\psi$ or $\mathscr{F}_{\{p_d\}}$, with $\psi(d) = p_d$).

THEOREM 3.1. *Let $(X, \delta)$ be an f.t.s., $\mathscr{F}_\psi$ an F-filter (F-net) on $X$, and $p$ a fuzzy point. Then*

(i) *$\mathscr{F}$ converges to $p$ iff $\psi_{\mathscr{F}}$ converges to $p$;*

(ii) *$\psi$ converges to $p$ iff $\mathscr{F}_\psi$ converges to $p$;*

(iii) *$\mathscr{F}$ has $p$ as a cluster point iff $\psi_{\mathscr{F}}$ has $p$ as a cluster point;*

(iv) *$\psi$ has $p$ as a cluster point iff $\mathscr{F}_\psi$ has $p$ as a cluster point.*

*Proof.* ((i)$\Rightarrow$) Let $N \in \mathscr{N}_p \subset \mathscr{F}$. Then $(p, N) \in \Lambda_{\mathscr{F}}$ and $\psi(q, M) = q \in M \subset N$ $\forall (q, M) \geqslant (p, N)$. Thus the F-net is residually in every nhood of $p$.

($\Leftarrow$) Let $N \in \mathscr{N}_p$. There is some $(q, F) \in \Lambda_{\mathscr{F}}$ such that if $(p', F') \geqslant (q, F)$ then $\psi_{\mathscr{F}}(p', F') = p' \in N$. Hence $F' \subset N$ and $N \in \mathscr{F}$.

((ii)$\Rightarrow$) Let $N \in \mathscr{N}_p$. There is some $d_0 \in D$ such that $p_d \in N$, $\forall d \geqslant d_0$ $(p_d = (x_d, t_d))$. Consider the fuzzy set:

$$A = \left\{ q_d \equiv \left( x_d, \frac{t_d + N(x_d)}{2} \right), \; d \geqslant d_0 \right\}.$$

$A \in \mathscr{F}_\psi$, since $p_d(x_d) = t_d < q_d(x_d) \leqslant A(x_d)$, $\forall d \geqslant d_0$ and $A \subset N$ since $\operatorname{Supp} A = \{x_d / d \geqslant d_0\}$ and $A(x_d) = \operatorname{Sup}_{d \geqslant d_0} \{q_d(x_d)\} \leqslant N(x_d)$. Hence $N \in \mathscr{F}$.

($\Leftarrow$) Let $N \in \mathscr{N}_p \subset \mathscr{F}_\psi$. There is some F-subset $B$ such that $p_d \in B$ $\forall d \geqslant d_B$ and $B \subset N$. Then $p_d \in N$ $\forall d \geqslant d_B$.

((iii)$\Rightarrow$) Let $N$ be an nhood of $p$ and $(p_0, F_0)$ an element of $\Lambda_{\mathscr{F}}$. By hypothesis, there is some $q \in N \wedge F_0$. Hence, there is $(q, F_0) \geqslant (p_0, F_0)$ such that $\psi_{\mathscr{F}}(q, F_0) = q \in N$. So $\psi_{\mathscr{F}} \propto p$.

($\Leftarrow$) Let $N$ be an nhood of $p$; $F$ an element of $\mathscr{F}$, and $p \in F$. Then $(p, F) \in \Lambda_{\mathscr{F}}$ and there is, by hypothesis, some $(p_0, F_0) \geqslant (p_{,F})$ such that $\psi_{\mathscr{F}}(p_0, F_0) = p_0 \in N$. Then $p_0 \in F \wedge N$ and $\mathscr{F} \propto p$.

((iv)$\Rightarrow$) Let $N$ be an nhood of $p$ and $A \in I^X$ such that there is some $d_0 \in D$ with $\psi(d) \in A$ $\forall d \geqslant d_0$. By hypothesis, there is some $d' \geqslant d_0$ such that $\psi(d') \in N$. Since $\psi(d')(x_{d'}) < A(x_{d'}) \wedge N(x_{d'})$ and $N \wedge A \neq \varnothing$ for each fuzzy subset which contains a tail of the net, $\mathscr{F}_\psi \propto p$.

($\Leftarrow$) Let $N \in \mathscr{N}_p$ and $d \in D$. Consider the F-subset: $A = \{q_{d'} = (x_{d'}, (1 + t_{d'})/2) / d' \geqslant d\}$, where $(x_{d'}, t_{d'}) = \psi(d')$. Clearly $\psi(d')(x_{d'}) =$

$t_{d'} < (1+t_{d'}/2 \leqslant A(x_{d'})\ \forall d' \geqslant d$, and then $A \in \mathscr{B}$. Besides, $A \wedge N \neq \varnothing$, as, by hypothesis, $\operatorname{supp} A = \{x_{d'}/d' \geqslant d_0\}$.

DEFINITION 3.3. An F-ultranet $\psi$ in $X$ is a trivial F-ultranet if $\exists d_0 \in D$ such that $x_d = x$, $\forall d \geqslant d_0$ (with $\psi(d) = (x_d, t_d)$).

PROPOSITION 3.1 [3]. *Let $\{p_d\}$ be an F-net. Then $\{p_d\}$ is a trivial F-ultranet if $\exists d_0 \in D$ and $\{x_d\}_{d \geqslant d_0}$ is a trivial ultranet and $t_d \to 0$.*

PROPOSITION 3.2. *Let $(X, \delta)$ be an f.t.s., $\mathscr{F}$ (resp. $\psi$) an F-filter (resp. F-net) on X. Then*

(i) *The F-net based on an F-ultrafilter is an F-ultranet.*

(ii) *The F-filter based on an F-ultranet is an F-ultrafilter.*

(iii) *If $\psi$ is a trivial F-ultranet, $\mathscr{F}_\psi$ is a trivial F-ultrafilter.*

(iv) *If $\mathscr{F}$ is a trivial F-ultrafilter, $\psi_{\mathscr{F}}$ is a trivial F-ultranet.*

*Proof.* (i) Let $\mathscr{F}$ be an $F$-ultrafilter and $Y \subset X$. Suppose that $Y$ is included in $\mathscr{F}$. Then, for each $A \in I^X$ with $\operatorname{supp} A = Y$, $A \in \mathscr{F}$. Let $p \in A$ ($A \neq \varnothing$). Evidently $(p, A) \in \Lambda_{\mathscr{F}}$ and for each $(q, F) \in \Lambda_{\mathscr{F}}$ with $(q, F) \geqslant (p, A)$, $\psi_{\mathscr{F}}(q, F) = q \in F \leqslant A$. Then, $\psi_{\mathscr{F}}$ is residually in $A$, $A$ being a fuzzy set with support $Y$.

(ii) It follows from Definition 3.2.

(iii) If $\psi \equiv \{p_d\}$ is a trivial F-ultranet, there is some $d_0 \in D$ with $x_d = x$ whenever $d \geqslant d_0$. Besides, for each $s > 0$, there is some $d_1 \in D$ with $t_d < s$ whenever $d \geqslant d_1$.

We will prove that $\mathscr{F}_\psi = \{F \in I^X/F(x) > 0\}$. If $F \in \mathscr{F}_\psi$, there is some $B \in I^X$ and some $d_2 \in D$ such that $p_d \in B\ \forall d \geqslant d_2$ and $B \subset F$. Choose $d^* \in D$ such that $d^* \geqslant d_0$ and $d^* \geqslant d_2$. Then, $p_d \in B\ \forall d \geqslant d^*$ and $x_d = x$. Then, $p_d(x_d = x) < B(x) \leqslant F(x)$ if $d \geqslant d^*$. Consequently, $F(x) > 0$.

If $F$ is a fuzzy set and $x \in \operatorname{Supp} F$, there is some $s > 0$ such that $s < F(x)$ and, by hypothesis, $\exists d_1 \in D$ such that $t_d < s < F(x)\ \forall d \geqslant d_1$. This is to say that $F \in \mathscr{F}_\psi$ (it is even an element of the base).

(iv) If $\mathscr{F}$ is a trivial F-ultrafilter, $\mathscr{F} = \{F \in I^X/F(x) > 0\}$. Then, any F-point with support $x$ is an element of the F-filter $\mathscr{F}$. Let $p \equiv (x, t_p)$, $q \equiv (x, t_q)$ be two F-points with support $x$ and such that $p \in q$. Then, $(p, q) \in \Lambda_{\mathscr{F}}$ and for each $(p', F) \in \Lambda_{\mathscr{F}}$ with $(p', F) \geqslant (p, q)$, $\operatorname{Supp} F = x$ and hence $\operatorname{Supp} \psi_{\mathscr{F}}(p', F) = x$. Therefore, the net $\{\operatorname{supp} \psi_{\mathscr{F}}(p', F)\}_{(p', F) \geqslant (p, q)}$ is a trivial ultranet.

Besides, for each $s > 0$, let $p_s \equiv (x, s/4)$ and $q_s \equiv (x, s/2)$ be two F-points with $p_s \in q_s$. Then, $(p_s, q_s) \in \Lambda_{\mathscr{F}}$, and for every $(p, F) \in \Lambda_{\mathscr{F}}$ with $(p, F) \geqslant (p_s, q_s)$, $F \equiv (x, t)$ with $t \leqslant s/2$ and $p \equiv (x, r)$ with $r < t$. Hence $(p, F)(x) = r < s$.

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