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MINIMAL MODELS FOR NON-FREE CIRCLE ACTIONS

AGUSTÍ ROIG AND MARTINTXO SARALEGI-ARANGUREN

ABSTRACT. Let $\Phi\colon \mathbb{S}^1 \times M \to M$ be a smooth action of the unit circle $\mathbb{S}^1$ on a manifold M. In this work, we compute the minimal model of M in terms of the orbit space B and the fixed point set $F \subset B$, as a dg-module over the Sullivan's minimal model of B.

The question we treat in this work is the following: Given a smooth action $\Phi\colon \mathbb{S}^1 \times M \to M$, is it possible to construct a model of M using just basic data? The answer is well known when the fixed point set is empty (in particular, when the action is free). A dgc algebra model of M is given by a Hirsch extension of the dgc algebra Sullivan minimal model $\mathcal{A}(B)$ of the orbit space B

$$\mathcal{A}(B) \otimes \Lambda(x), \tag{1}$$

where the degree of x is 1 and dx defines the Euler form of the action (for example, see [8]). This formula does not apply when the fixed point set F is not empty. Roughly speaking this happens because the Euler form does not live on B.

Our answer to the above question is a minimal model of the deRham dgc algebra $\Omega(M)$ of M, which is a dg module over the Sullivan minimal model $\mathcal{A}(B)$ of B. Such structure is associated to M by means of the canonical projection $\pi\colon M \to B$. We prove that the minimal model of M, as an $\mathcal{A}(B)$-dg module, is the graded cone

$$\mathcal{M}(M) = \mathcal{A}(B) \oplus_{e'} \mathcal{M}(B, F), \tag{2}$$

where $\mathcal{M}(B, F)$ is a sort of relative minimal model of the pair (B, F) and $e'\colon \mathcal{M}(B, F) \to \mathcal{A}(B)$ is a degree 2 map. This map is determined by the Euler class of the action and it will be described below. We also prove that, for $F = \emptyset$, the formulas (1) and (2) coincide.

There are some algebraic invariants of M and F that are closely related: Poincaré characteristic, localization, rational homotopy, ... We add another item to this list: the minimal model of M and F. In fact, considering the $\mathcal{A}(B)$-dg module structure associated to F by means of the natural inclusion $\iota\colon F \hookrightarrow B$, we prove that the minimal model of F as an $\mathcal{A}(B)$-dg module is the graded cone

$$\mathcal{M}(F) = \mathcal{A}(B) \oplus_{i'} \mathcal{M}(B, F),$$

where $i'\colon \mathcal{M}(B, F) \to \mathcal{A}(B)$ is a degree 0 map. This map is determined by ι and will be described below. Observe that the minimal models $\mathcal{M}(M)$ and $\mathcal{M}(F)$, as $\mathcal{A}(B)$-graded modules, have the same basis except for a shift by 2.

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We are also interested in the Borel space $M_{\mathbb{S}^1} = M \times_{\mathbb{S}^1} \mathbb{S}^\infty$, where an $\mathcal{A}(B)$-dg module structure is defined by means of the canonical projection $p\colon M_{\mathbb{S}^1} \to B$. We prove that the minimal model of $M_{\mathbb{S}^1}$ is the graded cone

$$\mathcal{M}(M_{\mathbb{S}^1}) = [\mathcal{A}(B) \otimes \Lambda(e)] \oplus_{q'} [\mathcal{M}(B, F) \otimes \Lambda(e)],$$

where $\deg e = 2$, $de = 0$ and $q'(b \otimes e^n) = e'(b) \otimes e^n + i'(b) \otimes e^{n+1}$. This formula implies that the equivariant cohomology $H^{i}_{\mathbb{S}^1}(M)$ (i.e., the cohomology of $M_{\mathbb{S}^1}$) can be computed using just basic data by means of the long exact sequence

$$\begin{aligned} \cdots &\to [H(B) \otimes \Lambda(e)]^{i} \to H^{i}_{\mathbb{S}^1}(M) \\ &\to [H(B, F) \otimes \Lambda(e)]^{i-1} \xrightarrow{(q')^*} [H(B) \otimes \Lambda(e)]^{i+1} \to \cdots . \end{aligned}$$

Moreover, when the Euler class vanishes, we prove that the equivariant cohomology of M is just $H(B) \oplus \left[H(F) \otimes \Lambda^+(e)\right]$. We also translate some classic results (Localization Theorem, equivariant formality, ...) in terms of basic data. In connection with the minimal models of M, $M_{\mathbb{S}^1}$ and F the reader can consult [1] and [2].

Let us illustrate these results with the suspension of the Hopf action on $\mathbb{S}^3$. The north and south poles of the total space $\mathbb{S}^4$ are the fixed points of the action and the orbit space is $\mathbb{S}^3$. We work in the category of $\Lambda(a)$-dg modules, where $\deg a = 3$. From the above formul we get

$$\begin{aligned} \mathcal{M}(\mathbb{S}^4) = \Lambda(a) \quad &\otimes \quad \mathbb{R}\{1, c_n \,/\, n \in \mathbb{N}\}, \text{ with} \\ &\qquad \deg c_n = 2[\tfrac{n+3}{2}], \\ &\qquad dc_0 = a, dc_1 = 0, dc_{n+2} = a \cdot c_n, \\ \mathcal{M}(\mathbb{S}^0) = \Lambda(a) \quad &\otimes \quad \mathbb{R}\{1, \gamma_n \,/\, n \in \mathbb{N}\}, \text{ with} \\ &\qquad \deg \gamma_n = 2[\tfrac{n+1}{2}], \\ &\qquad d\gamma_0 = 0, d\gamma_1 = a, d\gamma_{n+2} = a \cdot \gamma_n, \\ \mathcal{M}(\mathbb{S}^4 \otimes_{\mathbb{S}^1} \mathbb{S}^\infty) = \Lambda(e, a) \quad &\otimes \quad \mathbb{R}\{1, c_n \,/\, n \in \mathbb{N}\}, \text{ with} \\ &\qquad \deg c_n = 2[\tfrac{n+3}{2}], \\ &\qquad dc_0 = a, dc_1 = e \cdot a, dc_{n+2} = a \cdot c_n. \end{aligned}$$

The minimal model $\mathcal{M}(\mathbb{S}^4)$ (resp. $\mathcal{M}(\mathbb{S}^0)$, resp. $\mathcal{M}(\mathbb{S}^4 \otimes_{\mathbb{S}^1} \mathbb{S}^\infty)$) is a free $\Lambda(a)$-graded module over the cohomology $H^*(Y_\pi)$ of the homotopy fiber Y_π of π (resp. Y_ι, resp. Y_p). So, we find the following relations between the Poincaré polynomials of these spaces:

$$P_{Y_\pi} = 1 - t^2 + t^2 P_{Y_\iota} = (1 - t^2) P_{Y_p}.$$

We prove that these relations are generic if B is simply connected and of finite type.

The main geometric tool used in this work are Verona's controlled forms [24]. In fact, when the set of fixed points F is not empty the orbit space B is not a regular manifold but a singular one, more precisely a stratified pseudomanifold. For such a space Z, Verona proved that the complex of controlled forms $\Omega_v^*(Z)$ compute the cohomology of Z. We prove more, namely that the minimal models $\mathcal{A}(Z)$ and $\mathcal{M}(Z)$ can be computed using controlled forms. It is important to notice that the Euler form is not a controlled form, nevertheless it appears in this context as a morphism of $\mathcal{A}(B)$-dg modules $e\colon \Omega_v^*(B, F) \to \Omega_v^{*+2}(B)$ (cf. [12]). In the writing of $\mathcal{M}(M)$ (resp. $\mathcal{M}(F)$) the operator e' (resp. i') is a model of e (resp. of the inclusion $i\colon \Omega_v^*(B, F) \hookrightarrow \Omega_v^*(B)$).

The starting point of the work is the observation that the cohomology of M can be computed by the graded cone $\Omega_v^*(B) \oplus_e \Omega_v^*(B, F)$. This formula also applies to semi-free actions of $\mathbb{S}^3$ [23]. So, all the results of this work extend to this kind of actions. A similar formula appears when one deals with an isometric action $\Phi\colon \mathbb{R} \times M \to M$, considering on B controlled basic forms instead of controlled forms [23]. Again, we conclude that the results of this work apply to isometric flows. In particular, we get the inequality

$$H^{r-1}((M, F)/\mathcal{F}) + \sum_{i=0}^{\infty} \dim H^{r+2i}(F) \leq \sum_{i=0}^{\infty} \dim H^{r+2i}(M),$$

when the flow is not trivial.

On the algebraic side, we develop to some extent the Theory of dg minimal modules. This kind of minimal objects was previously studied by the first author (cf. [18], [19], [20]) and independently by Kriz and May (cf. [13]).

The organization of the work is as follows. The first section is devoted to the algebraic tools we need to work with A-dg modules. In the second section we present the singular spaces we find when we deal with circle actions. Controlled forms are introduced in the third section. The main result of this paper is proved in the fourth and last section. Four technical lemmas are proved in the appendix.

A manifold is considered to be connected, without boundary and smooth (of class C^∞), unless otherwise is stated. The field of coefficients is $\mathbb{R}$.

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Convention. Throughout this paper, minimal models in the category of dgc algebras will be denoted by $\mathcal{A}(-)$. Minimal models in the category of dg modules will be denoted by $\mathcal{M}(-)$.

1. Dg-module minimal models

In this section we will develop the algebraic machinery necessary to prove Theorem 4.3: we define what is meant by a *minimal factorization of a morphism* of A-dg modules, prove its existence and uniqueness and a result concerning maps induced between them.

1.1. *A-dg modules.* Let A be a dgc algebra. An *A-dg module* M is a graded vector space together with a product $A \otimes M \longrightarrow M$ and a differential d: $M \longrightarrow M$ of degree $+1$ which satisfies Leibniz rule. Both graduations of A and M are over the non-negative integers. A *quasi-isomorphism* (*quis*) is an A-dg module morphism which induces an isomorphism in cohomology.

Let us begin with an immediate generalization of the cone of a morphism of complexes in the category of A-dg modules. Let φ: $M \to N$ be a morphism of A-dg modules of degree $p \in \mathbb{Z}$. This is the same as a degree 0 morphism of A-dg modules,

$$\varphi\colon\ M[-p] \to N,$$

where $M[-p]$ means the A-dg module M shifted by $-p$; it is graded by $M[-p]^n = M^{n-p}$ and the product $\mu_{M[-p]}$: $A \otimes M[-p] \to M[-p]$ and the differential $d_{M[-p]}$: $M[-p] \to M[-p]$ change signs according to

$$\begin{aligned} \mu_{M[-p]}(a \otimes m) &= (-1)^{|a|\cdot p}\mu_M(a \otimes m) \\ d_{M[-p]}(m) &= (-1)^p d_M(m). \end{aligned}$$

We shall denote by $N \oplus_\varphi M$ the A-dg module graded by

$$N \oplus_\varphi M = N \oplus M[1-p]$$

and with product and differential given by

$$a \cdot \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} a \cdot y \\ (-1)^{|a|\cdot(p-1)} a \cdot x \end{pmatrix}$$

$$\begin{pmatrix} d & \varphi \\ 0 & (-1)^{p-1}d \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} dy + \varphi x \\ (-1)^{p-1} dx \end{pmatrix},$$

where $a \cdot x$ denotes $\mu_M(a \otimes x)$. Finally, when we say that the sequence of A-dg module morphisms

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow 0$$

is *exact*, we simply mean that $0 \longrightarrow M^n \xrightarrow{\varphi^n} N^n \xrightarrow{\psi^n} P^n \longrightarrow 0$ are usual exact sequences of A^0-modules for all n.

1.1.1 *Remark.* A short exact sequence of A-dg modules as above is the same as the quis of A-dg modules

$$N \oplus_\varphi M \xrightarrow{(\psi,0)} P.$$

LEMMA 1.1.2. *Given a morphism φ: $M \to N$ of A-dg modules of degree p, we have a short exact sequence of A-dg modules*

$$0 \longrightarrow N \xrightarrow{\binom{1}{0}} N \oplus_{\varphi} M \xrightarrow{(0,1)} M[1-p] \longrightarrow 0.$$

Proof. Obvious. $\square$

Associated to the above exact sequence, there is the *long exact cohomology exact sequence* of φ:

$$\cdots \to H^n(N) \xrightarrow{\binom{1}{0}_*} H^n\big(N \oplus_{\varphi} M\big) \xrightarrow{(0,1)_*} H^{n+1-p}(M) \xrightarrow{\varphi_*} H^{n+1}(N) \to \cdots.$$

To end this elementary differential homological algebra, let us point out that a *(degree p) homotopy* between two A-dg module morphisms φ, ψ: $M \to N$ of degree p, is an A-dg module morphism h: $M \to N$ of degree $p-1$, such that

$$(-1)^p dh + hd = \psi - \varphi.$$

One can verify that this notion of homotopy coincides with the one defined in [20] using a path object.

1.2. *Minimal A-dg modules.* (cf. [10], [15], [18]). Let M be a A-dg module and n a non negative integer. A *degree n Hirsch extension* of M is an inclusion of A-dg modules $M \hookrightarrow M \oplus (A \otimes V^n)$ in which:

1. V is a homogeneous vector space of degree n.
2. $A \otimes V^n$ is the free A-graded module over V.
3. The differential of $M \oplus (A \otimes V^n)$ is induced by the differentials of M and A and the choice of a linear map d: $V^n \to M^{n+1}$.

A morphism of A-dg modules $M \oplus (A \otimes V^n) \longrightarrow N$ is given by a morphism of A-dg modules φ: $M \longrightarrow N$ and a linear map f: $V^n \longrightarrow N^n$ subjected to the condition $\varphi \circ d = d \circ f$.

A *minimal KS-extension* of M is an inclusion of A-dg modules ι: $M \longrightarrow N$ together with an exhaustive filtration $\{N(n,q)\}_{(n,q)\in I}$ of N, indexed by $I = \{(n,q) \in \mathbb{N} \times \mathbb{N}\}$ with lexicographical order, such that:

1. $N(0,0) = M$.
2. For $q > 0$, $N(n,q)$ is a degree n Hirsch extension of $N(n, q-1)$.
3. $N(n+1, 0) = \varinjlim_q N(n,q)$.

1.2.1 *Remark.* N is therefore of the form $M \oplus (A \otimes V)$, where V is a bigraded vector space. This kind of object plays the rôle of the KS-extensions of [10], denoted by $B \otimes \Lambda V$, with the tensor product replaced by the direct sum and the free dg-algebra over V replaced by the free A -dg module over V.

A *minimal KS-factorization* of an A-dg module morphism $\varphi\colon M \longrightarrow N$ is a commutative diagram of A-dg module morphisms

(3)
$$\begin{tikzcd} & M \arrow[dl, "\iota"'] \arrow[dr, "\varphi"] & \\ M \oplus (A \otimes V) \arrow[rr, "\rho"] & & N \end{tikzcd}$$

in which ρ is a quis and ι is a minimal KS-extension. If M is the zero A-dg module, we speak of *minimal KS-modules* and *minimal KS-models.*

1.3. *Models of A-dg modules.* Let now (3) be any commutative diagram of A-dg module morphisms. In this situation, we will say that ρ is an *M-morphism.* If ρ is also a quis, we will simply say that it is an M-quis. A homotopy between two M-morphisms which restricted to M is the identity will be called also a *M-homotopy.*

Alternatively, we could have said that ρ is a morphism of $M\backslash\mathbf{DGM}(A)$, *the category of A-dg modules under M.* So, a minimal KS-factorization is, simply, a minimal model in $M\backslash\mathbf{DGM}(A)$ (see [20] for the precise statement of this).

THEOREM 1.3.1. *Let A be a dgc algebra and let $\varphi\colon M \longrightarrow N$ be an A-dg module morphism such that $\varphi^0_*\colon H^0(M) \longrightarrow H^0(N)$ is a monomorphism. Then there exists a minimal KS-factorization of φ.*

Proof. See Appendix. □

COROLLARY 1.3.2. *Let A be a dgc algebra and let N be an A-dg module. Then there exists a minimal KS-model of N.*

THEOREM 1.3.3 (cf. [11], [18]). *Let A be a dgc algebra and*

$$\begin{tikzcd} & M \arrow[dl, "\psi"'] \arrow[dr, "\iota"] & \\ X \arrow[rr, "\varphi"] & & M \oplus (A \otimes V) \end{tikzcd}$$

a commutative diagram of A-dg module morphisms in which ι is a minimal KS-extension and φ is a quis. Then there exists an A-dg module morphism $\sigma\colon M \oplus (A \otimes V) \longrightarrow X$ such that $\sigma\iota = \psi$ and $\varphi\sigma =$ id.

Proof. See Appendix. □

In other words, every M-quis whose target is a minimal KS-extension of M has a section which is also an M-morphism. This implies the uniqueness up to isomorphism of minimal models well known in other categories.

COROLLARY 1.3.4. *Two minimal KS-factorizations of the same M-morphism are M-isomorphic and the isomorphism is unique up to M-homotopies.*

Proof. It follows easily from Theorem 1.3.3, taking into account that the category $M\backslash\mathbf{DGM}(A)$ is a closed model category in which all objects are fibrant (cf. [17], [20, Corollary 2 to Proposition 1.15]). □

Particularly, if we take $M = 0$, the zero A-dg module, we obtain:

COROLLARY 1.3.5. *Two minimal KS-models of the same A-dg module are isomorphic and the isomorphism is unique up to homotopies.*

Given a morphism $\varphi\colon M \to N$ of A-dg modules we will need to construct a model of $N \oplus_\varphi M$. This will be done by means of the following results:

COROLLARY 1.3.6. *Let $\varphi\colon M \longrightarrow N$ be a morphism of A-dg modules and let $\rho_M\colon M' \longrightarrow M$ and $\rho_N\colon N' \longrightarrow N$ be two minimal models. Then there exists a morphism $\varphi'\colon M' \longrightarrow N'$, unique up to homotopies, that renders commutative up to homotopy the diagram*

$$\begin{array}{ccc} M' & \xrightarrow{\rho_M} & M \\ \Big\downarrow{\scriptstyle \varphi'} & & \Big\downarrow{\scriptstyle \varphi} \\ N' & \xrightarrow{\rho_N} & N \end{array} .$$

We will loosely say that φ' is a model *of φ.*

Proof. It follows from Theorem 1.3.3 and the model category structure [19, Proposition 1.16]. □

PROPOSITION 1.3.7. *Consider the above diagram. Let $h\colon M' \to N$ be a homotopy between $\varphi \circ \rho_M$ and $\rho_N \circ \varphi'$. Then*

$$\Phi = \begin{pmatrix} \rho_N & h \\ 0 & \rho_M \end{pmatrix} \colon N' \oplus_{\varphi'} M' \longrightarrow \dagger N \oplus_\varphi M$$

is a quis.

Proof. Let us verify that Φ commutes with differentials.

$$\begin{pmatrix} d & \varphi \\ 0 & (-1)^{p-1} d \end{pmatrix} \begin{pmatrix} \rho_N & h \\ 0 & \rho_M \end{pmatrix} - \begin{pmatrix} \rho_N & h \\ 0 & \rho_M \end{pmatrix} \begin{pmatrix} d & \varphi \\ 0 & (-1)^{p-1} d \end{pmatrix}$$
$$= \begin{pmatrix} d\rho_N - \rho_N' & dh + (-1)^p hd - (\rho_N \varphi' - \varphi \rho_M) \\ 0 & (-1)^{p-1} d\rho_M - (-1)^{p-1} \rho_M d' \end{pmatrix} .$$

And this is zero because ρ_M and ρ_N commute with differentials and h is a degree p homotopy between $\varphi \circ \rho_M$ and $\rho_N \circ \varphi'$. Next, we put Φ in the following obviously commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & N \oplus_\varphi M \arrow[r] & M[1-p] \arrow[r] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[u, "\rho_N"] & N' \oplus_{\varphi'} M' \arrow[r] \arrow[u, "\Phi"] & M'[1-p] \arrow[r] \arrow[u, "\rho_M"] & 0
\end{tikzcd}$$

And therefore Φ is a quis by the long exact sequence of φ and φ' and the Five Lemma. $\square$

Finally, we will need the following result concerning minimal dg modules and graded cones.

LEMMA 1.3.8. *Let φ: $M \to A$ be a degree p morphism of A-dg modules, with M a minimal A-dg module. If $M^{<(1-p)} = 0$ then $A \oplus_\varphi M$ is a minimal A-dg module.*

Proof. Let $M = A \otimes V$. Then, as a graded module, $A \oplus M = A \otimes (\mathbb{R} \oplus V)$. So we can define an exhaustive filtration in $A \oplus M$ as follows. Let $W(n,q)$ be the $\mathbb{R}$-vector spaces

$$W(n,q) = \begin{cases} 0 & \text{if } n = q = 0 \\ \mathbb{R} & \text{if } n = 0 \text{ and } q = 1 \\ V(n+1-p, q-1) & \text{if } n = 0 \text{ and } q > 1 \\ V(n+1-p, q) & \text{if } n \neq 0 \end{cases}$$

and put

$$(A \oplus_\varphi M)(n,q) = A \otimes \left(\bigoplus_{(m,r) \leq (n,q)} W(m,r) \right).$$

Then, all the inclusions $(A \oplus_\varphi M)(n, q-1) \hookrightarrow (A \oplus_\varphi M)(n,q)$ are degree n Hirsch extensions. $\square$

1.3.9. *Links with topology.* Before studying the relative case, let us show one example where these *minimal dg-modules* appear in topology. Let p: $E \to B$ be a continuous map between two topological spaces. We have the induced morphism p^*: $A_{\mathbb{R}}(B) \to A_{\mathbb{R}}(E)$ between the real algebras of polynomial forms making $A_{\mathbb{R}}(E)$ an $A_{\mathbb{R}}(B)$-dg module. Now, assume that B is connected, of finite type, with $\pi_1(B)$ acting trivially on $H^*(Y_p) = H^*(Y_p, \mathbb{R})$, where Y_p denotes the *homotopy fiber* of p. Then, by the second Theorem of Eilenberg-Moore (see [9] and [3]), we have

$$H^*(Y_p) \cong \mathrm{Tor}_{A_{\mathbb{R}}(B)}(\mathbb{R}, A_{\mathbb{R}}(E)).$$

By [20], this differential torsion product can be computed with a minimal model of $A_{\mathbb{R}}(E)$ as an $A_{\mathbb{R}}(B)$-dg module. Let $\mathcal{M}(E) = A_{\mathbb{R}}(B) \otimes_{\mathbb{R}} V$ be this minimal model. Then

$$H^*(Y_p) \cong H^*(\mathbb{R} \otimes_{A_{\mathbb{R}}(B)} \mathcal{M}(E)) \cong H^*(\mathbb{R} \otimes_{A_{\mathbb{R}}(B)} (A_{\mathbb{R}}(B) \otimes V)) = V$$

because $\mathbb{R} \otimes_{A_{\mathbb{R}}(B)} (A_{\mathbb{R}}(B) \otimes V) = V$ has zero differential due to minimality. So, the known Hirsch-Brown model of E,

$$\mathcal{M}(E) = A_{\mathbb{R}}(B) \otimes H^*(Y_p),$$

is a *minimal model* as $A_{\mathbb{R}}(B)$-dg modules. In particular, if we take B to be a point, we find that $H^*(E)$ is the minimal model of $A_{\mathbb{R}}(E)$ as a $\mathbb{R}$-dg module.

Extensive use of the minimal Hirsch-Brown model for the Borel construction is made in [2].

1.4. *Models of couples, existence and uniqueness.* In fact, we will need something more than simply dg-minimal models over a fixed $\mathbb{R}$-dg algebra. The process we are going to perform is the following: starting with an A-dg module M, we are going to compute first the dgc algebra minimal model of A:

$$\rho\colon A' \xrightarrow{\simeq} A$$

(i.e., the classical Sullivan minimal model). Then, by means of the dgc algebra quis ρ, we will make M an A'-dg module by defining the product with elements of A' by

$$a \cdot m = \rho^*(a) \cdot m,$$

where the product in the right-hand side is the product of M as an A-module. Let us note $\rho^*(M)$ for the dg module M with this structure of A'-module. Finally, we will compute the minimal model of $\rho^*(M)$ as an A'-dg-module:

$$\varphi\colon M' \xrightarrow{\simeq} \rho^*(M).$$

In other words, we will compute the minimal model of the couple (A, M) in the category **DGM** of *modules over all algebras*. The objects of this category are couples like (A, M). Morphisms are also couples

$$(f, \varphi)\colon (A, M) \to (B, N)$$

where $f\colon A \to B$ is a morphism of dgc algebras and $\varphi\colon M \to N$ is a dg-module f-morphism; that is to say, a morphism of A-dg modules $M \to f^*(N)$. In other words,

$$\varphi(a \cdot m) = f(a) \cdot \varphi(m) \quad \text{for all } a \in A, m \in M.$$

The algorithm we have described brings us the "true" minimal model

$$(\rho, \varphi)\colon\ (A', M') \to (A, M)$$

in the sense that the couple (A', M') is unique up to isomorphism (of **DGM**) and the couple of quis (ρ, φ) is also unique up to homotopies (of **DGM**): this follows from [20, Theorem 3.4], which tells us that the couple (A, M) is minimal in **DGM** if and only if A is a minimal $\mathbb{R}$-dgc algebra and M is an A-dg minimal module.

2. Stratifications and unfoldings

In this work we fix an effective smooth action $\Phi\colon \mathbb{S}^1 \times M \to M$ (non-trivial!). The orbit space of the action is B and $\pi\colon M \to B$ is the canonical projection. The action Φ induces on M a natural stratification by classifying the points of M according to their isotropy subgroups. This stratification is invariant by the action of $\mathbb{S}^1$, so the orbit space B also inherits a stratified structure. In this section we specify these facts.

2.1. *Stratifications.* A *stratification* of a paracompact topological space Z is a locally finite collection $\mathcal{S}_Z$ of disjoint connected manifolds called *strata*, such that:

(i) $Z = \bigsqcup_{S \in \mathcal{S}_Z} S$.
(ii) $S \cap \overline{S'} \neq \emptyset \iff S \subset \overline{S'}$ (and we write $S \leq S'$).
(iii) $(\mathcal{S}_Z, \leq)$ is a partially ordered set (*poset*).
(iv) There exists an open stratum R which is the maximum.

We shall say that Z is a *stratified space*. Note that R, called *regular stratum*, is necessarily dense. A *singular stratum* is an element of $\mathcal{S}_Z$ different from R. We shall write $\mathcal{S}_Z^{\mathrm{sing}}$ for the family of singular strata and $\Sigma_Z \subset Z$ for its union. The *length* of Z, written len Z, is the biggest integer n for which there exists a chain $S_0 < S_1 < \cdots < S_n$ of strata. In particular, len $Z = 0$ if and only if Z is a manifold endowed with the stratification $\mathcal{S}_Z = \{$connected components of $Z\}$. Notice that the length is always finite.

A continuous map (resp. homeomorphism) $f\colon Y \to Z$ between two stratified spaces is a morphism (resp. *isomorphism*) if it sends the strata of Y to the strata of Z smoothly (resp. diffeomorphically). We shall write $\mathrm{Iso}(Z)$ for the group of isomorphisms between Z and itself. A morphism $f\colon Y \to Z$ induces a poset morphism $f_{\mathcal{S}}\colon \mathcal{S}_Y \to \mathcal{S}_Z$ by putting $f_{\mathcal{S}}(S) \supset f(S)$. We shall say that f is a *strict morphism* if the map $f_{\mathcal{S}}$ is strictly increasing.

2.1.1. *Examples.* Through this work we shall find the following kinds of stratification.

(a) On a connected manifold N we always may consider the 0-length stratification $\mathcal{S}_N = \{N\}$. A stratum $S \subset Z$ inherits from $\mathcal{S}_Z$ such a stratification.

(b) Any open subset W of Z inherits naturally from $\mathcal{S}_Z$ a stratified structure satisfying len $W \leq$ len Z. The stratification is $\mathcal{S}_W = \{$connected components of $S \cap W \ / \ S \in \mathcal{S}_Z\}$. Notice that the inclusion $W \hookrightarrow Z$ is a strict morphism.

(c) Suppose Z compact. On the product $N \times cZ$, where cZ is the *cone* $Z \times [0,1[/ Z \times \{0\}$, we have the stratification $\mathcal{S}_{N\times cZ} = \{N \times S \times]0,1[\ / \ S \in \mathcal{S}_Z\} \cup \{N \times \{\text{vertex } \vartheta \text{ of } cZ\}\}$. Notice that len $N \times cZ =$ len $Z + 1$. A point of cZ will be denoted by $[\![x,t]\!]$ with $(x,t) \in Z \times [0,1]$. The vertex ϑ of cZ is $[\![x,0]\!]$.

Unless otherwise stated, we assume that the spaces W, cZ and $N \times cZ$ are endowed with the stratification described above. Later on, we shall show how Φ determines a natural stratification on M and B.

2.2. *Stratified pseudomanifolds.* When the strata are assembled conically we find stratified pseudomanifolds. We introduce this notion. An open subset W of a stratified space Z is said to be *modeled* on the stratified space L if there exists an isomorphism $\varphi \colon \mathbb{R}^n \times cL \to W$. The pair (W,φ) is said to be a *chart* of Z. A family of charts $\{(W,\varphi)\}$, where the family $\{W\}$ is a covering of Z, is called an *atlas*.

We shall say that the stratified space Z is a *stratified pseudomanifold* if there exists a family $\{L_S\}_{S\in\mathcal{S}_Z^{\text{sing}}}$ of stratified pseudomanifolds such that for any point $x \in \Sigma_Z$ we can find a chart (W,φ) modelled on L_S with $\varphi(0,\vartheta) = x$, where S is the stratum of Z containing x. The space L_S is the *link* of the stratum S.

This definition makes sense because it is made by induction on the length of Z (len $L_S <$ len Z). A stratified space with len $Z = 0$ is always a stratified pseudomanifold. Each of the examples given in 2.1.1 is a stratified pseudomanifold when Z is a stratified pseudomanifold. This definition is slightly more general than that of stratified pseudomanifold of [9] since we allow the singular strata to have codimension 1.

2.3. *Unfoldings.* The computation of the cohomology of a stratified pseudomanifold Z using differential forms is possible using the controlled forms of Verona [24]; but we need some extra data on Z so that these controlled forms will make sense. The original definition uses a system of neighborhoods of singular strata subjected to some compatibility conditions. A more comprehensive and less technical alternative is presented in [21] where a desingularisation of Z is used. With this blow-up, the controlled forms of M and B are more easily related.In this work we follow this point of view.

Consider Z a stratified pseudomanifold. A continuous map $\mathcal{L} \colon \widetilde{Z} \to Z$, where $\widetilde{Z}$ is a (not necessarily connected) manifold, is an *unfolding* if the two following conditions hold:

1. The restriction $\mathcal{L}_M \colon \mathcal{L}_M^{-1}(R) \longrightarrow R$ is a local diffeomorphism.

2. There exist a family of unfoldings $\{\mathcal{L}_{L_S}\colon \widetilde{L_S} \to L_S\}_{S\in\mathcal{S}_Z^{sing}}$ and an atlas $\mathcal{A}$ of Z such that for each chart $(U,\varphi) \in \mathcal{A}$ there exists a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n \times \widetilde{L_S} \times]-1,1[& \xrightarrow{\widetilde{\varphi}} & \mathcal{L}_Z^{-1}(U) \\ {\scriptstyle Q}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{L}_Z} \\ \mathbb{R}^n \times cL_S & \xrightarrow{\varphi} & U \end{array}$$

where

(a) $\widetilde{\varphi}$ is a diffeomorphism and

(b) $Q(x_1,\ldots,x_n,\widetilde{\zeta},t) = (x_1,\ldots,x_n,[\![\mathcal{L}_{L_S}(\widetilde{\zeta}),|t|]\!])$.

This definition makes sense because it is made by induction on the length of Z. When len $Z = 0$ then $\mathcal{L}_Z$ is just a local diffeomorphism. The restriction $\mathcal{L}_Z\colon \mathcal{L}_S^{-1}(S) \to S$ is a fibration with $\widetilde{L_S}$ as a fiber, for any singular stratum S.

For each of the examples of 2.1.1 we have the following unfoldings:

(a) $\widetilde{N} = N$ and $\mathcal{L}_N =$ identity.
(b) $\widetilde{W} = \mathcal{L}_Z^{-1}(W)$ and $\mathcal{L}_W =$ restriction of $\mathcal{L}_Z$.
(c) $\widetilde{N \times cZ} = N \times \widetilde{Z} \times]-1,1[$ and $\mathcal{L}_{N\times cZ}(y,\widetilde{x},t) = (y,[\![\mathcal{L}_Z(\widetilde{x}),|t|]\!])$.

A morphism $f\colon Y \to Z$ between two stratified spaces, endowed with unfoldings $\mathcal{L}_Y\colon \widetilde{Y} \to Y$ and $\mathcal{L}_Z\colon \widetilde{Z} \to Z$, is a *liftable morphism* if there exists a smooth map $\widetilde{f}\colon \widetilde{Y} \to \widetilde{Z}$ with $\mathcal{L}_Z \circ \widetilde{f} = \mathcal{L}_Z \circ f$. Each φ is a liftable morphism. The inclusion $W \hookrightarrow Z$ is a liftable morphism.

From now on Z denotes a stratified pseudomanifold endowed with an unfolding $\mathcal{L}_Z\colon \widetilde{Z} \to Z$.

2.4. *Stratifications induced by the action.* We present the structure of stratified pseudomanifold of M and of the orbit space B. For technical reasons we need to consider *only in this paragraph* a smooth action $\Phi\colon G \times M \to M$ of a closed subgroup of the unit circle $\mathbb{S}^1$ on a manifold M. The properties listed below follow mainly from the Slice Theorem (see [12]).

• *Stratification.* Consider the equivalence relation $\sim$ defined on M by $x \sim y$ if G_x is equal to G_y, where $G_z = \{g \in G \;/\; \Phi(g,z) = z\}$ denotes the *isotropy subgroup* of a point $z \in M$. Each of the equivalence classes of $\sim$ is an invariant sub-manifold of M. The family $\mathcal{S}_M$ of the connected components of the equivalence classes given by this relation defines a stratification on M. The family $\mathcal{S}_B = \{\pi(S) \;/\; S \in \mathcal{S}_M\}$ defines a stratification on the orbit space B. When G is connected the map π_S is bijective and therefore π is a strict morphism.

We shall write G_S for the isotropy subgroup of a point (and therefore any point) of a stratum S. Notice that G_S is a closed subgroup of G. According to this subgroup there are three types of strata: *regular stratum* (G_S is 1), *exceptional stratum* (G_S is finite different from 1) and *fixed stratum* (G_S is $\mathbb{S}^1$). Notice that the restriction of the canonical projection $\pi\colon M \to B$ to S is a principal fibration over $\pi(S)$ with fiber G/G_S. We shall write F for the union of fixed strata. We shall identify $F \subset M$ with $\pi(F) \subset B$ by π.

• *Links*. For any singular stratum $S \subset M$ fix a point x on it and let $\mathbb{S}^{n_S}$ be the unit sphere of a slice transversal to the stratum S at x. The action Φ induces the orthogonal action $\Phi_S\colon G_S \times \mathbb{S}^{n_S} \to \mathbb{S}^{n_S}$; this action has no fixed points (*almost free action*). Notice that n_S is necessarily even for a fixed stratum. The link of S is the sphere $\mathbb{S}^{n_S}$ endowed with the stratification induced by Φ_S. The link of $\pi(S)$ is the quotient space $\mathbb{S}^{n_S}/G_S$. Notice that this link is homologically a sphere or a real projective space when S is an exceptional stratum [4] and a complex projective space when S is a fixed stratum [16].

• *Unfoldings*. It is proven in [12] that M possesses an equivariant unfolding $\mathcal{L}_M\colon \widetilde{M} \to M$ (relative to a free smooth action $\widetilde{\Phi}\colon G \times \widetilde{M} \to \widetilde{M}$) in such a way that the induced map $\mathcal{L}_B\colon \widetilde{M}/G \to B$ is an unfolding of B. Moreover, if $\widetilde{\pi}\colon \widetilde{M} \to \widetilde{M}/G$ is the canonical projection, we have $\mathcal{L}_B \circ \widetilde{\pi} = \pi \circ \mathcal{L}_M$. So, the morphism π is liftable.

3. Controlled forms

Controlled forms were introduced by Verona to compute the cohomology of a stratified pseudomanifold Z using differential forms [24]. We present this notion in this paragraph, following the approach of [21].

3.1. *Definitions*. A differential form ω on the regular stratum R of Z is said to be *liftable* if there exists a differential form $\widetilde{\omega}$ on $\widetilde{Z}$, called the *lifting* of ω, such that $\widetilde{\omega} = \mathcal{L}_Z^*\omega$ on $\mathcal{L}_Z^{-1}(R)$. By density the lifting is unique. The differential form ω can be tangential or transversal to the strata; in the first case we get controlled forms and in the second case we get perverse forms.

A liftable form ω is a *controlled form* if it induces a differential form ω_S on each singular stratum S, that is, if $\widetilde{\omega}\Big|_{\mathcal{L}_Z^{-1}(S)} = \mathcal{L}_Z^*\omega_S$. So, we can see ω as the family of differential forms $\{\omega_S \in \Omega(S)\}_{S \in \mathcal{S}_Z}$.

We shall write $\Omega_v(Z)$ the *complex of controlled forms* (or the *deRham-Verona complex*). This subcomplex of the deRham complex $\Omega(R)$ is in fact a dgc algebra. To see that, notice that if the differential forms ω and η are controlled, then the differential forms $\omega + \eta$, $\omega \wedge \eta$ and $d\omega$ are also controlled since, for each stratum S, they satisfy:

$$\widetilde{\omega + \eta}\Big|_{\mathcal{L}_Z^{-1}(S)} = \widetilde{\omega}\Big|_{\mathcal{L}_Z^{-1}(S)} + \widetilde{\eta}\Big|_{\mathcal{L}_Z^{-1}(S)}$$

$$= \omega_S + \eta_S, \widetilde{\omega \wedge \eta}\Big|_{\mathcal{L}_Z^{-1}(S)} = \widetilde{\omega}\Big|_{\mathcal{L}_Z^{-1}(S)} \wedge \widetilde{\eta}\Big|_{\mathcal{L}_Z^{-1}(S)} = \omega_S \wedge \eta_S,$$

and

$$\widetilde{d\omega}\Big|_{\mathcal{L}_Z^{-1}(S)} = d\widetilde{\omega}\Big|_{\mathcal{L}_Z^{-1}(S)} = d\omega_S.$$

For a stratum S, we have the restriction operator $R_S\colon \Omega_{_v}(Z) \to \Omega(S)$, defined by $R_S(\omega) = \omega_S$, which is a dgc algebra operator and therefore endows $\Omega(S)$ with a structure of $\Omega_{_v}(Z)$-dg module.

PROPOSITION 3.1.1. *When S is closed the restriction operator $R_S\colon \Omega_{_v}(Z) \to \Omega(S)$ is onto.*

Proof. Fix (U, ϕ) a chart of $\mathcal{A}$. Consider $\alpha\colon \mathbb{R}^n \to \mathbb{R}$ and $\beta\colon]-1,1[\to \mathbb{R}$, two smooth maps taking the value 1 on a neighborhood of 0. The map $f\colon Z - \Sigma \to \mathbb{R}$ defined from $f\phi(x_1, \ldots, x_n, [\zeta, t]) = \alpha(x_1, \ldots, x_n) \cdot \beta(t)$ is a controlled form. In fact, its lifting is the smooth map $\widetilde{f}\colon \widetilde{Z} \to \mathbb{R}$ defined from $\widetilde{f\phi}(x_1, \ldots, x_n, [\widetilde{\zeta}, t]) = \alpha(x_1, \ldots, x_n) \cdot \beta(|t|)$ which is constant on the fibers of $\mathcal{L}_Z$. A standard argument shows that there exists a partition of unity subordinated to $\mathcal{A}$.

We reduce the problem to show that $R_S\colon \Omega_{_v}(U) \to \Omega(U \cap S)$ is onto. Since S is closed then $U \cap S$ is the lowest stratum of U and therefore the question becomes this: Is the restriction operator $R_{\mathbb{R}^n}\colon \Omega_{_v}(\mathbb{R}^n \times cL_S) \to \Omega(\mathbb{R}^n)$ onto? And the answer is clearly yes. □

3.2. *Relative controlled forms.* Consider Y a union of strata of Z. A *relative controlled form* on (Z, Y) is a controlled form on Z vanishing on Y, that is, $\omega_S \equiv 0$ for each stratum $S \subset Y$. We shall write $\Omega_{_v}(Z, Y)$ the complex of relative controlled forms, which is a dgc algebra. If we consider the restriction operator

$$R_Y = \prod_{S \subset Y} R_S\colon \Omega_{_v}(Z) \to \Omega\left(Y = \bigsqcup_{S \subset Y} S\right) = \prod_{S \subset Y} \Omega(S)$$

then we can write $\Omega_{_v}(Z, Y) = \operatorname{Ker} R_Y$.

The wedge product $\wedge\colon \Omega_{_v}(Z) \times \Omega_{_v}(Z, Y) \longrightarrow \Omega_{_v}(Z, Y)$ endows $\Omega_{_v}(Z, Y)$ with a structure of $\Omega_{_v}(Z)$-dg module. The natural inclusion $\Omega_{_v}(Z, Y) \hookrightarrow \Omega_{_v}(Z)$ is a morphism in the category of $\Omega_{_v}(Z)$-dg modules.

3.3. *Controlled model.* The deRham-Verona complex $\Omega_{_v}(Z)$ depends on the unfolding chosen, but for a stratified space its cohomology does not: $H\left(\Omega_{_v}(Z)\right) \cong H(Z)$, the singular cohomology with real coefficients (cf. [24],[21] where *controlled* becomes *zero perversity*). But we have a stronger result at the level of dgc algebra minimal models. Recall that the dgc algebra minimal model $\mathcal{A}(Z)$ of Z is just the dgc algebra minimal model of $A_{\mathbb{R}}(Z)$, the dgc algebra of polynomial differential forms on the simplicial set $\overline{\mathrm{Sing}(Z)}$ of singular simplices of Z. The dgc algebra $\Omega_{v}(Z)$ is easier to handle than $A_{\mathbb{R}}(Z)$, but we need to know that their dgc algebra minimal models are the same. This follows immediately from the following Theorem (cf. [10]).

THEOREM 3.3.1. *Let Z be a stratified space. Then there exist two quis of dgc algebras*

$$\Omega_{v}(Z) \xrightarrow{\rho_1} \cdot \xleftarrow{\rho_2} A_{\mathbb{R}}(Z).$$

Proof. See Appendix. □

3.4. *Relative controlled model.* As in the absolute case (cf. Theorem 3.3), the model of a morphism $f\colon Z' \to Z$ between two stratified spaces can be computed, under certain conditions, using controlled forms instead of polynomial forms. These conditions involve the unfolding:

[P1] f preserves controlled forms: $f^*\omega \in \Omega_{v}(Z')$ for any $\omega \in \Omega_{v}(Z)$.
[P2] f preserves liftable simplices (see Appendix for the exact definition).

We shall say that an f satisfying the two conditions is *good*.

For good morphisms at least, we have a relative version of Theorem 3.3.

THEOREM 3.4.1. *Let $f\colon Z' \to Z$ be a good morphism of stratified spaces. Then there exists a commutative diagram of dgc algebra morphisms*

$$\begin{tikzcd}
\Omega_{v}(Z') \arrow[r, "\rho_1'"] & \cdot & A_{\mathbb{R}}(Z') \arrow[l, "\rho_2'"'] \\
\Omega_{v}(Z) \arrow[u, "f^*"] \arrow[r, "\rho_1"] & \cdot \arrow[u] & A_{\mathbb{R}}(Z) \arrow[l, "\rho_2"'] \arrow[u, "f^*"']
\end{tikzcd}$$

in which the horizontal arrows are quis.

Proof. See Appendix. □

The two examples of good morphisms used in this work are described in the following proposition.

PROPOSITION 3.4.2. *The projection $\pi\colon M \to B$ and the inclusion $\iota\colon F \hookrightarrow B$ are good morphisms.*

Proof. See Appendix. □

4. Models for M and F

This section is devoted to constructing the minimal models of M and F in the category of differential graduate models over the dgc algebra minimal model $\mathcal{A}(B)$ of B. As in other contexts (Euler-Poincaré characteristic, Localization Theorem, rational homotopy theory, ...), these two models are intimately related, in fact, they are free $\mathcal{A}(B)$-graded modules over the same (up to a shift) graded vectorial space.

4.1. *Model of a stratified space.* Let Z be a stratified space. As we showed in Theorem 3.3.1, its minimal model as a dgc algebra can be computed from $\Omega_v(Z)$. So let

$$\rho_Z\colon \mathcal{A}(Z) \xrightarrow{\simeq} \Omega_v(Z)$$

be a quis of dgc algebras, with $\mathcal{A}(Z)$ a minimal one. Next, let $f\colon Z' \to Z$ be a good morphism. Then we can endow $\Omega_v(Z')$ with a structure of $\mathcal{A}(Z)$-dg module by means of the composition

$$\mathcal{A}(Z) \xrightarrow{\rho_Z} \Omega_v(Z) \xrightarrow{f^*} \Omega_v(Z').$$

Since we will always consider this structure of module in $\Omega_v(Z')$ we shall not write $\rho_Z^* f^*(\Omega_v(Z'))$ but simply $\Omega_v(Z')$. In the same way, we also consider $A_{\mathbb{R}}(Z')$ as an $\mathcal{A}(Z)$-dg module.

PROPOSITION 4.1.1. *The minimal models of $\Omega_v(Z')$ and $A_{\mathbb{R}}(Z')$ as $\mathcal{A}(Z)$-dg modules are isomorphic.*

Proof. It follows from Theorem 3.4.1. The commutativity of the diagram given by this result means that ρ_1' and ρ_2' are quis of $\mathcal{A}(Z)$-dg modules. □

Let us denote by $\mathcal{M}(Z')$ this minimal model as $\mathcal{A}(Z)$-dg module. It obviously depends on f but not on the several choices we have made in this construction: the dgc algebra $\mathcal{A}(Z)$, the quis ρ_Z and the $\mathcal{A}(Z)$-dg module minimal model of $\Omega_v(Z')$ (or $A_{\mathbb{R}}(Z')$). Despite all these choices, the $\mathcal{A}(Z)$-dg module minimal model $\mathcal{M}(Z')$ is unique up to isomorphism by Section 1.3.

4.2. *Minimal model of F.* The fixed point set F plugs into the category of $\mathcal{A}(B)$-dg modules through the natural inclusion $\iota\colon F \hookrightarrow B$, which is a good morphism. We have already seen in Section 3.2 that $\Omega_v(B, F)$ is an $\Omega_v(B)$-dg module and therefore an $\mathcal{A}(B)$-dg module. We shall write $\mathcal{M}(B, F)$ for the *relative minimal model* of (B, F), that is, the minimal model of $\Omega_v(B, F)$ as $\mathcal{A}(B)$-dg module. Notice that the inclusion $i\colon \Omega_v(B, F) \hookrightarrow \Omega_v(B)$ is an $\mathcal{A}(B)$-dg module morphism. We shall write $i'\colon \mathcal{M}(B, F) \to \mathcal{A}(B)$ to represent any of its models (see Corollary 1.3.6). The degree of i and i' is 0.

PROPOSITION 4.2.1.

$$\mathcal{M}(F) = \mathcal{A}(B) \oplus_{i'} \mathcal{M}(B, F). \tag{4}$$

Proof. Consider the exact sequence $0 \to \Omega_v(B, F) \xrightarrow{i} \Omega_v(B) \xrightarrow{R_F} \Omega(F) \to 0$ (cf. Proposition 3.1.1). Then, by Remark 1.1.1 we have an $\mathcal{A}(B)$-dg module quis between $\Omega_v(B) \oplus_i \Omega_v(B, F)$ and $\Omega(F)$. So, by Proposition 1.3.7 we have an $\mathcal{A}(B)$-dg module quis between $\mathcal{A}(B) \oplus_{i'} \mathcal{M}(B, F)$ and $\Omega(F)$. Since $\mathcal{A}(B) \oplus_{i'} \mathcal{M}(B, F)$ is a minimal $\mathcal{A}(B)$-dg module (cf. Lemma 1.3.8) then by uniqueness we get $\mathcal{M}(F) = \mathcal{A}(B) \oplus_{i'} \mathcal{M}(B, F)$ (cf. Corollary 1.3.5). □

COROLLARY 4.2.2. *Let us suppose that B is of finite type and simply connected. Then $\mathcal{M}(B, F)$ is the free $\mathcal{A}(B)$-graded module generated by $\widetilde{H}^{*-1}(Y_t)$.*

Proof. From Section 1.3.9 we know that $\mathcal{M}(F)$ is a free $\mathcal{A}(B)$-graded module over $H^*(Y_t)$ and therefore $\mathcal{M}(F) = \mathcal{A}(B) \oplus_h [\mathcal{A}(B) \otimes \widetilde{H}(Y_t)]$ for some h of degree 1. From the above proposition we get the result wanted. □

4.2.3. *Remarks.* • M of finite type implies B of finite type. From [5] we know that if M is of finite type then F and (B, F) are of finite type. Using the long exact sequence associated to (B, F) one concludes that B is also of finite type.

• M simply connected implies B simply connected. Considering twisted neighborhoods of orbits (cf. [5]) one easily checks that a loop on B lifts in a path on M. Since the orbits of M are connected, we are done.

4.3. *Minimal model of M.* The fundamental vector field X of the action is defined by $X(x) = T_1\Phi_x(1)$, where $x \in M$ and $\Phi_x \colon \mathbb{S}^1 \to M$ is given by $\Phi_x(g) = \Phi(g, x)$. Since this vector field does not vanish on R we can consider the dual form $\chi \in \Omega^1(R)$, relatively a riemannian metric μ on R. When this metric is *good* (cf. [12]) the derivative $d\chi$ is a basic form relative to the projection $\pi \colon R \to \pi(R)$. So, there exists a differential form $e \in \Omega^2(\pi(R))$ such that $d\chi = \pi^* e$. Both differential forms, χ and e, are liftable. We shall say that e is an *Euler form*. They are not controlled forms because their restrictions to the links of fixed strata do not necessarily vanish. But the maps $\chi \colon \Omega_v(M, F) \to \Omega_v(M)$ and $e \colon \Omega_v(B, F) \to \Omega_v(B)$, given by $\gamma \mapsto \chi \wedge \gamma$ and $\omega \mapsto e \wedge \omega$, are well defined $\mathcal{A}(B)$-dg module morphisms. We shall write $e' \colon \mathcal{M}(B, F) \to \mathcal{A}(B)$ a model of e. Notice that the degree of e and e' is 2.

The main result of this work is the following:

THEOREM 4.3.1.

$$\mathcal{M}(M) = \mathcal{A}(B) \oplus_{e'} \mathcal{M}(B, F). \tag{5}$$

Proof. Let $I\Omega_v(M) = \{\omega \in \Omega_v(M)/\dagger L_X\omega = 0\}$, the complex of invariant controlled differential forms. We have seen in [12] that the inclusion $I\Omega_v(M) \hookrightarrow \Omega_v(M)$ is a dgc algebra quis. We endow $I\Omega_v(M)$ with the natural structure of $\mathcal{A}(B)$-dg module by means of the composition

$$\mathcal{A}(B) \xrightarrow{\rho_B} \Omega_v(B) \xrightarrow{\pi^*} \Omega_v(M),$$

which is well defined since $\pi^*(\Omega_v(B)) \subset I\Omega_v(M)$. The inclusion $I\Omega_v(M) \hookrightarrow \Omega_v(M)$ is now a quis of $\mathcal{A}(B)$-dg modules . Each invariant differential form ω is written uniquely as $\omega = \pi^*\alpha + X \wedge \pi^*\beta$; when ω is controlled then $\alpha \in \Omega_v(B)$ and $\beta \in \Omega_v(B, F)$ because X is tangent to the links of fixed strata. So, the operator

$$(6) \qquad \Delta\colon \Omega_v(B) \oplus_e \Omega_v(B, F) \longrightarrow I\Omega_v(M)$$

given by $\Delta(\alpha, \beta) = \pi^*\alpha + X \wedge \pi^*\beta$ is an $\mathcal{A}(B)$-dg module isomorphism and therefore $\mathcal{A}(B) \oplus_{e'} \mathcal{M}(B, F)$ is a model of M as $\mathcal{A}(B)$-dg module (cf. Proposition 1.3.7). Minimality again follows from Lemma 1.3.8. □

4.3.2. *Remarks.* (a) Formula (4) and (5) show that $\mathcal{M}(F)$ and $\mathcal{M}(M)$ are free $\mathcal{A}(B)$-graded modules over the same (up to a shift by 2) basis.

(b) This theorem contains the classic result saying that, when the action is *almost free* (that is, $F = \emptyset$), the dgc algebra minimal model of M is $\mathcal{A}(B) \otimes \Lambda(x)$ with $\deg x = 1$ (cf. [8]). Let us see that.

In this case $\mathcal{M}(M) = \mathcal{A}(B) \oplus_{e'} \mathcal{A}(B)$ and e' is the multiplication by a certain $e \in \mathcal{A}(B)$ of degree 2. The quis of $\mathcal{A}(B)$-dg modules we have constructed $\eta\colon \mathcal{M}(M) \to \Omega_v(M)$ satisfies $\eta(a, b) = a \cdot 1 + b \cdot \eta(0, 1)$. Consider the product on $\mathcal{M}(M)$ given by $(a, b) \cdot (a', b') = (a \cdot a', a \cdot b' + (-1)^{\deg a'} b \cdot a')$. A straightforward calculation shows that η is a quis of dgc algebras. But the two dgc algebras $\mathcal{M}(M)$ and $\mathcal{A}(B) \otimes \Lambda(x)$ are quasi isomorphic by using $(a, b) \mapsto (a \otimes 1 + b \otimes x)$.

We establish now some consequences of these results.

4.4. *Poincaré polynomials.* Given a topological space X we shall write P_X for its Poincaré polynomial, that is, $P_X(t) = \sum_{n \geq 0} \dim H^n(X) \cdot t^n$.

COROLLARY 4.4.1. *Suppose B of finite type and simply connected. Then* $1 - P_{Y_\pi} = t^2(1 - P_{Y_\iota})$.

Proof. We have seen in Proposition 4.2.2 that $\mathcal{M}(B, F)$ is a free $\mathcal{A}(B)$-graded module over $\widetilde{H}^{*-1}(Y_\iota)$. Applying the same method to π we conclude that $\mathcal{M}(B, F)$ is a free $\mathcal{A}(B)$-graded module over $\widetilde{H}^{*+1}(Y_\pi)$. So, $\dim H^0(Y_\pi) = 1$, $\dim H^1(Y_\pi) = 0$, $\dim H^2(Y_\pi) = \dim H^0(Y_\iota) - 1$ and finally $\dim H^n(Y_\pi) = \dim H^{n-2}(Y_\iota)$ for $n \geq 2$. This gives the result. □

4.5. *Vanishing of the Euler class.* Actions with vanishing Euler class $[e] \in H^2(B - F)$ have a particular status (cf. [12] for a geometrical interpretation). In the sequel we show how $\mathcal{M}(M)$ contains information about the Sullivan minimal model of M in this case.

When the Euler class vanishes we can choose a convenient riemannian metric on M so that e itself vanishes (cf. [12]). Thus $e' = 0$. The minimal model $\mathcal{M}(M)$ is of the form $\mathcal{A}(B) \otimes E$ with $E^0 = \mathbb{R}$ and $dE \subset \mathcal{A}(B) \otimes E^+$. It supports a dgc algebra structure by putting on E the trivial product: $1 \cdot v = v$ if $v \in E$ and $E^+ \cdot E^+ = 0$. This dgc algebra structure shall be called *naïve*. It contains the following information about M.

COROLLARY 4.5.1. *If the Euler class vanishes then the naïve dgc algebra structure of $\mathcal{M}(M)$ has the same real homotopy type of M. Moreover, $\pi^*_\psi(B)$ injects into $\pi^*_\psi(M)$.*

Proof. The operator $\Delta \colon \Omega_v(B) \oplus_0 \Omega_v(B, F) \longrightarrow \Omega_v(M)$ is a quis of $\mathcal{A}(B)$-dg modules which becomes a quis of dgc algebras when considering on the source the following product:

$$(\alpha, \beta) \cdot (\alpha', \beta') = (\alpha \cdot \alpha', (-1)^{\deg \alpha} \alpha \cdot \beta' + (-1)^{\deg \alpha' \cdot \deg \beta} \alpha' \cdot \beta). \tag{7}$$

This dgc algebra contains the real homotopy type of M.

Let $\rho_{(B,F)} \colon \mathcal{M}(B, F) \to \Omega_v(B, F)$ be the relative minimal model of (B, F). The operator

$$\rho_B \oplus \rho_{(B,F)} \colon \mathcal{M}(M) = \mathcal{A}(B) \oplus_0 \mathcal{M}(B, F) \longrightarrow \Omega_v(B) \oplus_0 \Omega_v(B, F)$$

is a quis of $\mathcal{A}(B)$-dg modules which becomes a quis of dgc algebras when considering the product (7) in both terms. This dgc algebra contains the real homotopy type of M. Notice that the dgc algebra structure on $\mathcal{M}(M)$ given by (7) is just the naïve structure. This gives the first part of the corollary.

We shall write $\mathcal{A}(B) = \Lambda Y$, with differential ∂, and $\mathcal{M}(M) = \Lambda Y \otimes E$, with differential d. We have $d_{|\Lambda Y} = \partial$ and $dE \subset \Lambda Y \otimes E^+$. This last property allows us to construct a KS-extension

$$\begin{tikzcd}
(\Lambda Y, \partial) \arrow[r] \arrow[dr] & (\Lambda Y \otimes E, d) \\
& (\Lambda Y \otimes \Lambda X, \delta) \arrow[u, "\varphi"']
\end{tikzcd}$$

such that $\varphi_{|\Lambda Y} = \mathrm{id}_{\Lambda Y}$ and $\varphi(X) \subset \Lambda Y \otimes E^+$. Recall that $\pi^*_\psi(B) = H(Y, \partial_0)$ and $\pi^*_\psi(M) = H(Y \oplus X, \delta_0)$, where ∂_0 and δ_0 are the linear part of ∂ and δ respectively.

Since $(\Lambda Y, \partial)$ is minimal we just have $\partial_0 = 0$ and therefore $\pi^*_\psi(B) = Y$. On the other hand, the composition $\delta_0 = X \xrightarrow{\delta} \Lambda Y \otimes \Lambda X \xrightarrow{\text{projection}} \Lambda Y \xrightarrow{\text{projection}} Y$ vanishes: if $x \in X$ then $\varphi(\delta x) = d(\varphi x) \in d(\Lambda Y \otimes E^+) \subset \Lambda Y \otimes E^+$. So, $\pi^*_\psi(M) = Y \oplus H(X, \delta_0)$ and the proof is finished since π^* becomes the inclusion $Y \hookrightarrow Y \oplus H(X, \delta_0)$. $\square$

4.5.2. *Remarks.* (a) When B is also contractible then $\mathcal{M}(M) = E$ is just a dgc algebra with trivial product and, by minimality, with zero derivative. In other words, M is a wedge of spheres.

(b) The Gysin sequence associated to (5) implies that the cohomology of B injects into the cohomology of M; in fact, we have the short exact sequence

$$0 \to H^*(B) \to H^*(M) \to H^{*-1}(B, F) \to 0.$$

The last statement of the corollary implies that, when M is of finite type and simply connected, we have the short exact sequence

$$0 \to \pi^*(B) \otimes \mathbb{R} \to \pi^*(M) \otimes \mathbb{R} \to \pi^*(Y_\pi) \otimes \mathbb{R} \to 0.$$

(c) The naïve structure of $\mathcal{M}(M)$ appears when $e' = 0$, but the previous result needs the vanishing of the Euler class itself as is shown in the following example. Consider the action Φ: $\mathbb{S}^1 \times \mathbb{CP}^n \to \mathbb{CP}^n$ given by $z \cdot [z_0, z_1, \ldots, z_n] = [z_0, z \cdot z_1, \ldots, z \cdot z_n]$ (in homogeneous coordinates). Here the fixed point set is $F = \mathbb{CP}^0 \cup \mathbb{CP}^{n-1}$ and the orbit space B is the closed cone over $\mathbb{CP}^{n-1}$. So, B is acyclic and $e' = 0$. The Euler class does not vanish since it generates $H^2(B - F) = H^2(\mathbb{CP}^{n-1} \times]0, 1[) = \mathbb{R}$. The minimal model we have computed in Theorem 4.3.1 is $\mathcal{M}(\mathbb{CP}^n) = \mathbb{R} \oplus_0 \widetilde{H}^{*-1}(\mathbb{CP}^0 \cup \mathbb{CP}^{n-1})$. Considering the naïve structure on it we get $\mathcal{M}(\mathbb{CP}^n) = H^*(\mathbb{S}^2 \vee \mathbb{S}^4 \vee \cdots \vee \mathbb{S}^{2n})$ as dgc algebras. But clearly this dgc algebra does not contain the real homotopy type of $\mathbb{CP}^n$.

4.6. *Cohomological dimension.* Write $\operatorname{dimc}(X)$ for the cohomological dimension of the topological space X; i.e., $\operatorname{dimc}(X) = \sup\{n \in \mathbb{N} \,/\, H^n(X) \neq 0\}$.

COROLLARY 4.6.1. *Under the hypothesis of Corollary* 4.4.1, *if the numbers* $\operatorname{dimc}(B)$ *and* $\operatorname{dimc}(Y_\pi)$ *are finite then* $\operatorname{dimc}(M) = \operatorname{dimc}(F)$ *or* $\operatorname{dimc}(F) + 2$.

Proof. From Corollary 4.4.1 we get $\operatorname{dimc}(Y_\pi) = \operatorname{dimc}(Y_\iota) = 0$ or $\operatorname{dimc}(Y_\pi) = \operatorname{dimc}(Y_\iota) + 2$. Now considering the homotopy fibrations associated to π and ι we get $\operatorname{dimc}(M) = \operatorname{dimc}(B) + \operatorname{dimc}(Y_\pi)$ and $\operatorname{dimc}(F) = \operatorname{dimc}(B) + \operatorname{dimc}(Y_\iota)$ and then we get the result. □

We find examples of this situation when $M = \mathbb{C}^n$, where $\mathbb{S}^1$ acts by complex multiplication, and $M = \mathbb{S}^{n+2} = \mathbb{S}^1 * \mathbb{S}^n$, where $\mathbb{S}^1$ acts by multiplication on the first factor and trivially on the second factor. When M is compact and oriented the condition $\operatorname{dimc}(M) = \operatorname{dimc}(F) = 0$ does not occur and the condition $\operatorname{dimc}(M) = \operatorname{dimc}(F) + 2$ is equivalent to saying that F possesses a connected component of codimension 2. This is also equivalent to the fact that B has a boundary. So, under the conditions of Corollary 4.4.1, if $\operatorname{dimc}(B) < \infty$ and B has no boundary then $\operatorname{dimc}(Y_\pi) = \operatorname{dimc}(Y_\iota) = \infty$.

4.7. *Equivariant cohomology.* The *equivariant cohomology* of M is the cohomology of the quotient space $M_{\mathbb{S}^1} = M \times_{\mathbb{S}^1} \mathbb{S}^\infty$, written $H_{\mathbb{S}^1}(M)$. The natural projection $p\colon M_{\mathbb{S}^1} \to B$ induces a natural structure of $\mathcal{A}(B)$-dg module on $M_{\mathbb{S}^1}$. Here we compute the *equivariant minimal model* of M, that is, $\mathcal{M}_{\mathbb{S}^\infty}(M) = \mathcal{M}(M_{\mathbb{S}^1})$.

We shall write $\Lambda(e)$ for the polynomial algebra generated by an element e of degree 2. The trivial $\mathcal{A}(B)$-dg module structure will be considered on it. The main result in this framework is the following:

THEOREM 4.7.1. *If the fixed point set F is not empty, then*

$$\mathcal{M}_{\mathbb{S}^\infty}(M) = [\mathcal{A}(B) \otimes \Lambda(e)] \oplus_{q'} [\mathcal{M}(B, F) \otimes \Lambda(e)], \tag{8}$$

where $q'(b \otimes e^n) = e'(b) \otimes e^n + i'(b) \otimes e^{n+1}$.

Proof. The equivariant cohomology can be computed using the complex $\Omega_{\mathbb{S}^1}(M) = I\Omega(M)\otimes\Lambda(e)$, endowed with the derivative $d(\omega\otimes e^n) = (d\omega)\otimes e^n + (i_X\omega)\otimes e^{n+1}$. Here i_X denotes the contraction by X. Proceeding as in [21, p. 213] one shows that the two dgc algebras $I\Omega(M)$ and $I\Omega_v(M)$ are quasi-isomorphic. Therefore the equivariant cohomology of M is computed by using $I\Omega_v(M) \otimes \Lambda(e)$ and p induces the operator $P\colon \Omega_v(B) \to I\Omega_v(M) \otimes \Lambda(e)$ defined by $P(\alpha) = \pi^*\alpha \otimes 1$. Under these transformations the $\mathcal{A}(B)$-dg module structure on $I\Omega_v(M) \otimes \Lambda(e)$ is given by

$$a \cdot (\omega \otimes e^n) = P(\rho_B(a)) \cdot (\omega \otimes e^n), \quad \text{with } a \in \mathcal{A}(B), \omega \in I\Omega_v(M).$$

Now we compute the minimal model of $I\Omega_v(M) \otimes \Lambda(e)$ relative to this structure.

The $\mathcal{A}(B)$-dg module isomorphism $\Delta\colon \Omega_v(B) \oplus_e \Omega_v(B, F) \longrightarrow I\Omega_v(M)$ induces the $\mathcal{A}(B)$-dg module isomorphism

$$\nabla\colon \left[\Omega_v(B) \otimes \Lambda(e)\right] \oplus_q \left[\Omega_v(B, F) \otimes \Lambda(e)\right] \longrightarrow I\Omega_v(M) \otimes \Lambda(e),$$

where $q(\beta \otimes e^n) = (\beta \wedge e) \otimes e^n + i(\beta) \otimes e^{n+1}$. Proceeding as in Theorem 4.3.1 we get that a model of $I\Omega_v(M) \otimes \Lambda(e)$ is just $[\mathcal{A}(B) \otimes \Lambda(e)] \oplus_{q'} [\mathcal{M}(B, F) \otimes \Lambda(e)]$, where $q'(b \otimes e^n) = e'(b) \otimes e^n + i'(b) \otimes e^{n+1}$. This model is minimal because of Lemma 1.3.8. □

Notice that for the almost free case ($F = \emptyset$) we have obtained the following non-minimal $\mathcal{A}(B)$-dg module model: $[\mathcal{A}(B) \otimes \Lambda(e)] \oplus_{i' \otimes e} [\mathcal{A}(B) \otimes \Lambda(e)]$; the minimal one is just $\mathcal{A}(B)$.

4.7.2. *Remarks.* (a) *Poincaré polynomial.* When $F \neq \emptyset$, B is of finite type and simply connected the relation between the homotopy fibers of p and π is given by $P_{Y_\pi} = (1 - t^2) P_{Y_p}$ (same proof as that of Corollary 4.4.1).

(b) *Extension of scalars.* The complexes $\Omega_{\mathbb{S}^1}(M)$ and $\mathcal{M}_{\mathbb{S}^\infty}(M)$ naturally support a structure of $\Lambda(e)$-dg module. The $\mathcal{A}(B)$-dg module quis we have constructed is in fact a $\Lambda(e)$-dg module quis. For this structure, the extension of scalars of $\mathcal{M}_{\mathbb{S}^\infty}(M)$ is just $\mathcal{M}(M)$, that is, $\mathbb{R} \otimes_{\Lambda(e)} \mathcal{M}_{\mathbb{S}^\infty}(M) = \mathcal{M}(M)$. In other words, the model of the fiber of $M \to M_{\mathbb{S}^1} \to B$ is "the fiber of the model" (cf. [10]).

(c) *Vanishing Euler class.* Since $e' = 0$ one has $\mathcal{M}_{\mathbb{S}^\infty}(M)$ isomorphic to $\mathcal{A}(B) \oplus \left[\mathcal{M}(F) \otimes \Lambda^+(e)\right]$, relative to both module structures. We conclude that the cohomology of B injects into the equivariant cohomology of M, that is, we have in fact the exact sequence $0 \to H(B) \to H_{\mathbb{S}^1}(M) \to H(F) \otimes \Lambda^+(e) \to 0$.

(d) *Equivariant cohomology.* Formula (8) says that we can compute the equivariant minimal model $\mathcal{M}_{\mathbb{S}^\infty}(M)$ in terms of basic data: i', e': $\mathcal{M}(B, F) \rightrightarrows \mathcal{A}(B)$. The equivariant cohomology $H^*_{\mathbb{S}^1}(M)$ can also be computed in terms of basic data. In fact, the short exact sequence $0 \to \mathcal{A}(B) \otimes \Lambda(e) \to \mathcal{M}_{\mathbb{S}^\infty}(M) \to \mathcal{M}(B, F) \otimes \Lambda(e) \to 0$ associated to (8) (cf. Lemma 1.1.2) gives the long exact sequence

$$\begin{aligned} \cdots \to\ & [H(B) \otimes \Lambda(e)]^i \to H^i_{\mathbb{S}^1}(M) \\ \to\ & [H(B, F) \otimes \Lambda(e)]^{i-1} \xrightarrow{q^*} [H(B) \otimes \Lambda(e)]^{i+1} \to \cdots, \end{aligned}$$

which determines $H_{\mathbb{S}^1}(M)$ in terms of i^*, e^*: $H(B, F) \rightrightarrows H(B)$.

(e) *Equivariantly formal spaces.* Put τ: $M \to M_{\mathbb{S}^1}$ the inclusion given by $\tau(x) =$ class of $(x, 1)$. The manifold M is *equivariantly formal* if the restriction map τ^*: $H_{\mathbb{S}^1}(M) \to H(M)$ is surjective (cf. [5], [6]). We can translate this condition in terms of basic data by considering the commutative diagram

$$\begin{array}{ccccccccc} \cdots \to & H^i_{\mathbb{S}^1}(M) & \longrightarrow & [H(B, F) \otimes \Lambda(e)]^{i-1} & \xrightarrow{q^*} & [H(B) \otimes \Lambda(e)]^{i+1} & \to & \cdots \\ & \downarrow{\scriptstyle \tau^*} & & \downarrow{\scriptstyle \mathfrak{R}} & & \downarrow{\scriptstyle \mathfrak{R}} & & \\ \cdots \to & H^i(M) & \longrightarrow & H^{i-1}(B, F) & \xrightarrow{e^*} & H^i(B) & \longrightarrow & \cdots \end{array}$$

where $\mathfrak{R}(\sum [\alpha_n] \otimes e^n) = [\alpha_n]$. Since the restriction $\mathfrak{R}$: Coker $q^* \to$ Coker e^* is surjective, the manifold M is equivariantly formal if and only if the restriction $\mathfrak{R}$: Ker $q^* \to$ Ker e^* is surjective. In other words, any string $[\alpha_0] \xrightarrow{e^*} 0$ fits into a string $0 \xleftarrow{i^*} [\alpha_n] \xrightarrow{e^*} [\beta_n] \xleftarrow{i^*} \cdots \xrightarrow{e^*} [\beta_0] \xleftarrow{i^*} [\alpha_0] \xrightarrow{e^*} 0$.

(f) *Localization Theorem.* This theorem asserts that the restriction map R_F: $M \to F$ induces an isomorphism between their localizations $S^{-1}H_{\mathbb{S}^1}(M)$ and $S^{-1}H_{\mathbb{S}^1}(F)$. We can translate this theorem in terms of basic data saying that the map

$$\nabla\colon\ H\big(B, F; S^{-1}\Lambda(e)\big) \longrightarrow H\big(B, F; S^{-1}\Lambda(e)\big),$$

defined from $\nabla([\omega]) = e \cdot [\omega] + [\omega \wedge e]$, is an isomorphism. This comes from the fact that the Localization Theorem is equivalent to the vanishing of $S^{-1}H_{\mathbb{S}^1}(M, F)$

and from the exact sequence

$$\begin{aligned}\cdots \to\ & [H(B,F)\otimes\Lambda(e)]^{i} \to H^{i}_{\mathbb{S}^1}(M,F)\\ \to\ & [H(B,F)\otimes\Lambda(e)]^{i-1} \xrightarrow{q^*} [H(B,F)\otimes\Lambda(e)]^{i+1} \to\end{aligned}$$

obtained as in (d). In fact, for $S = \{1, e^p, e^{2p}, \ldots\}$, the inverse of ∇ is given by $\nabla^{-1}([\omega]) = \sum_{n\geq 0}\sum_{j=0}^{p-1}(-1)^{(n+1)p-1-j}(e^{-p})^{n+1}e^{j}\cdot[\omega\wedge e^{p(n+1)-j-1}]$, which makes sense since the differential form e^m vanishes for large enough m.

4.8. *Semifree $\mathbb{S}^3$-actions.* The results developed until here for circle actions extend directly to semifree $\mathbb{S}^3$-actions. This comes essentially from the fact that formula (6) applies (up to a shift) here (cf. [22]). We don't give all the results but just the main one.

Consider $\Phi\colon \mathbb{S}^3\times M\to M$, a semifree smooth action of $\mathbb{S}^3$ on a smooth manifold M. Let F be the submanifold of fixed points. Semifreeness means that $\mathbb{S}^3$ acts freely on $M - F$. The orbit space B is a stratified pseudomanifold whose singular strata are the connected components of F. The inclusion $\iota\colon F\hookrightarrow B$ induces the inclusion operator $i\colon \Omega_{_v}(B,F)\longrightarrow\Omega_{_v}(B)$ which is an $\mathcal{A}(B)$-dg module morphism of degree 0. Here the Euler form e lies on $\Omega^4(B-F)$ and induces the $\mathcal{A}(B)$-dg module morphism $e\colon \Omega_{_v}(B,F)\longrightarrow\Omega_{_v}(B)$ of degree 4. The $\mathcal{A}(B)$-dg module minimal models of these operators are i' and e' respectively. The main result in this framework is the following:

THEOREM 4.8.1.

$$\begin{aligned}\mathcal{M}(M) &= \mathcal{A}(B)\oplus_{e'}\mathcal{M}(B,F)\\ \mathcal{M}(F) &= \mathcal{A}(B)\oplus_{i'}\mathcal{M}(B,F).\end{aligned}$$

Proof. Follow the path taken in the Theorem 4.3.1. □

4.9. *Isometric flows.* An *isometric flow* is a real smooth action $\Phi\colon\mathbb{R}\times M\to M$ preserving a riemannian metric μ on the smooth manifold M. The fundamental vector field X of the action is a Killing vector field. We shall write $\mathcal{F}$ for the singular foliation determined by the orbits of the action. The fixed point set is a manifold denoted F. Notice that in this case the orbit space $B = M/\mathbb{R}$ can be very wild (even totally disconnected!). For this reason it is customary to work with "basic objects" (objects living on M transverse to the flow and invariant by the flow) instead of working directly with the objects living on B. For example, a *basic form* is a differential form on M which is transverse to the flow ($i_X\omega = 0$) and invariant by the flow ($L_X\omega = 0$), a *basic controlled form* is a controlled form on M verifying $i_X\omega = i_X d\omega = 0,\ldots$ We shall write

$\Omega(M/\mathcal{F})$	for the complex of basic forms,
$\Omega_{_v}(M/\mathcal{F})$	for the complex of basic controlled forms,
$\Omega_{_v}((M,F)/\mathcal{F})$	for the complex of basic relative controlled forms.

When the action is periodic we have in fact a circle action and these complexes become, up to isomorphism, $\Omega(B)$, $\Omega_v(B)$ and $\Omega_v(B, F)$ respectively.

The three complexes above are in fact dgc algebras. The dgc algebra minimal model of $\Omega_v(M/\mathcal{F})$ and $\Omega(M/\mathcal{F})$ are the same and they will be denoted by $\mathcal{A}(M/\mathcal{F})$ (cf. [23]). We work in the category of $\mathcal{A}(M/\mathcal{F})$-dg modules. The $\mathcal{A}(M/\mathcal{F})$-dg module minimal model of $\Omega_v((M, F)/\mathcal{F})$ will be denoted by $\mathcal{M}((M, F)/\mathcal{F})$.

The inclusion $\iota\colon F \hookrightarrow M$ gives the inclusion operator $i\colon \Omega_v((M, F)/\mathcal{F}) \longrightarrow \Omega_v(M/\mathcal{F})$ which is an $\mathcal{A}(M/\mathcal{F})$-dg module morphism of degree 0. Here the Euler form e lies on $\Omega^2((M - F)/\mathcal{F})$ and it induces the $\mathcal{A}(M/\mathcal{F})$-dg module morphism $e\colon \Omega_v((M, F)/\mathcal{F}) \longrightarrow \Omega_v(M/\mathcal{F})$ of degree 2. The $\mathcal{A}(M/\mathcal{F})$-dg module minimal models of these operators are i' and e' respectively.

We shall write $X = \partial/\partial t$ if there exists a diffeomorphism $M \cong B \times \mathbb{R}$ sending X on a multiple of $\partial/\partial t$. The main result in this framework is the following:

THEOREM 4.9.1. *If $X \neq \partial/\partial t$ then*

$$\begin{aligned} \mathcal{M}(M) &= \mathcal{A}(M/\mathcal{F}) \oplus_{e'} \mathcal{M}((M, F)/\mathcal{F}) \\ \mathcal{M}(F) &= \mathcal{A}(M/\mathcal{F}) \oplus_{i'} \mathcal{M}((M, F)/\mathcal{F}). \end{aligned}$$

Proof. If we prove that the inclusion $I\Omega_v(M) \hookrightarrow \Omega_v(M)$ and the restriction $\Omega(M) \to \Omega_v(M)$ are dgc algebra quis, then it suffices to follow the path taken in the proof of the Theorem 4.3.1.

An isometric flow defines a singular riemannian foliation $\mathcal{F}$ where the orbits are all closed or the closures of the orbits are all tori (cf. [14]). In the first case the natural projection $\pi\colon M \to M/\mathcal{F}$ becomes a locally trivial fibration and, by orientability, a trivial one. So, $X = \partial/\partial t$ and we are in the second case. Using the Mayer-Vietoris argument we can replace M by a torus $\mathbb{T}$ endowed with an $\mathbb{R}$-linear action. The problem is then to prove that the inclusion $I\Omega_v(\mathbb{T}) \hookrightarrow \Omega_v(\mathbb{T})$ and the restriction $\Omega(\mathbb{T}) \to \Omega_v(\mathbb{T})$ are dgc algebra quis. Since the flow is regular then $\Omega_v(\mathbb{T}) = \Omega(\mathbb{T})$. By density, the complex $I\Omega_v(\mathbb{T}) = I\Omega(\mathbb{T})$ becomes $\Omega_{\mathbb{T}}(\mathbb{T}) = \{\omega \in \Omega(\mathbb{T})$ invariant by $\mathbb{T}\}$. Since $\mathbb{T}$ is compact, we already know that the inclusion $\Omega_{\mathbb{T}}(\mathbb{T}) \hookrightarrow \Omega(\mathbb{T})$ is a dgc algebra quis (cf. [7]). □

In the case $X = \partial/\partial t$ this result would imply $H^*(M) = H^*\left(B \times \mathbb{S}^1\right)$, which is false. We finish the work by extending to isometric flows a well-known result related to periodic actions.

COROLLARY 4.9.2. *If $X \neq \partial/\partial t$ then, for any $r \geq 0$,*

$$H^{r-1}((M, F)/\mathcal{F}) + \sum_{i=0}^{\infty} \dim H^{r+2i}(F) \leq \sum_{i=0}^{\infty} \dim H^{r+2i}(M).$$

Proof. From the above formula we get the following long exact sequences:

$$\cdots \to H^i(M/\mathcal{F}) \to H^i(M) \to H^{i-1}((M,F)/\mathcal{F}) \to H^{i+1}(M/\mathcal{F}) \to \cdots,$$
$$\cdots \to H^i((M,F)/\mathcal{F}) \to H^i(M/\mathcal{F}) \to H^i(F) \to H^{i+1}((M,F)/\mathcal{F}) \to \cdots.$$

Since the action is free out of F the above theorem admits the relative version $\mathcal{M}(M,F) = \mathcal{M}((M,F)/\mathcal{F}) \oplus_{e'} \mathcal{M}((M,F)/\mathcal{F})$. This gives the long exact sequence

$$\begin{aligned}\cdots &\to H^i((M,F)/\mathcal{F}) \to H^i(M,F)\\ &\to H^{i-1}((M,F)/\mathcal{F}) \to H^{i+1}((M,F)/\mathcal{F}) \to \cdots.\end{aligned}$$

Finally, by considering the long exact sequence associated to (M,F) one gets all the ingredients to proceed as in [5, pag. 161] to obtain the following Smith-Gysin sequence:

$$\begin{aligned}\cdots &\to H^i((M,F)/\mathcal{F}) \to H^i(M)\\ &\to H^{i-1}((M,F)/\mathcal{F}) \oplus H^i(F) \to H^{i+1}((M,F)/\mathcal{F}) \to \cdots\end{aligned}$$

Now it suffices to follow the procedure of [5, page 127]. $\square$

When M is compact the group of isometries of (M,μ) is a compact Lie group. So, the action of $\mathbb{R}$ extends to an action of a torus $\mathbb{T}$. For this action we have the inequality

$$H^{r-1}((M,F)/\mathbb{T}) + \sum_{i=0}^{\infty} \dim H^{r+2i}(F) \le \sum_{i=0}^{\infty} \dim H^{r+2i}(M),$$

for each $r \ge 0$ (cf. [5]), but notice that in general $H^*((M,F)/\mathcal{F})$ and $H^*((M,F)/\mathbb{T})$ are not equal.

5. Appendix

In this Appendix we give the proof of Theorem 1.3.1 (existence of a minimal KS-extension), Theorem 1.3.3 (uniqueness of a minimal KS-extension), Theorem 3.3.1 (minimal model versus controlled forms), Theorem 3.4.1 (relative minimal model versus controlled forms) and Proposition 3.4.2 (examples of good morphisms).

5.1. *Proof of Theorem* 1.3.1. To begin with, put $N(0,0) = M$, $\iota_{(0,0)} = \mathrm{id}_M$ and $\rho_{(0,0)} = \varphi$. Let us assume that we have already constructed

$$\iota_{(n,0)}\colon M \longrightarrow N(n,0) \qquad \text{and} \qquad \rho_{(n,0)}\colon N(n,0) \longrightarrow X$$

such that

(i)$_{(n,0)}$ $\iota_{(n,0)}$ is a minimal KS-extension,

(ii)$_{(n,0)}$ $\rho_{(n,0)}\iota_{(n,p)} = \varphi$, and

(iii)$_{(n,0)}$ $(\rho_{(n,0)})^i_*\colon H^i(N(n,0)) \to H^i(X)$ is an isomorphism for $0 \le i \le n-1$ and $(\rho_{(n,0)})^n_*$ is a monomorphism.

Now, for $q > 0$, take $V(n,q) = H^{n+1}(N(n,q-1),X)$ and consider it as a homogeneous vector space of degree n. Take also a linear section s of the natural projection $Z^{n+1}(N(n,q-1),X) \longrightarrow V(n,q)$. So for every $v \in V(n,q)$, we have $sv = (t_v, x_v) \in N(n,q-1)^{n+1} \oplus X^n$ such that

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix} d & 0\\ \rho_{(n,q-1)} & -d\end{pmatrix}\begin{pmatrix}t_v\\x_v\end{pmatrix}.$$

Define

$$N(n,q) = N(n,q-1) \oplus (A \otimes V(n,q))$$

with differential $dv = t_v$. Also define $\iota_{(n,q)}\colon M \longrightarrow N(n,q)$ as the composition of $\iota_{(n,q-1)}$ with the natural inclusion of $N(n,q-1) \hookrightarrow N(n,q)$. Finally, take $\rho_{(n,q)}\colon N(n,q) \longrightarrow X$ to be the morphism of A-dg modules induced by $\rho_{(n,q-1)}$ and the linear map $f\colon V(n,q) \longrightarrow X$ defined by $fv = x_v$.
Let us verify that

(i)$_{(n,q)}$ $\iota_{(n,q)}$ is a minimal KS-extension because of (i)$_{(n,q-1)}$ and the definition of $\iota_{(n,q)}$,

(ii)$_{(n,q)}$ $\rho_{(n,q)}\iota_{(n,q)} = \rho_{(n,q-1)}\iota_{(n,q-1)} = \varphi$ because of the definitions and (ii)$_{(n,q-1)}$,

(iii)$_{(n,q)}$ $(\rho_{(n,q)})^i_*\colon H^i(N(n,q)) \longrightarrow H^i(X)$ is an isomorphism for $i = 0,\dots,n$.

In fact, for $i < n$, $\rho_{(n,q)}$ coincides with $\rho_{(n,q-1)}$ and so the morphism induced in cohomology is an isomorphism by (iii)$_{(n,q-1)}$. In degree n, the natural inclusion $N(n,q-1) \hookrightarrow N(n,q)$ induces a monomorphism in cohomology because new generators can only kill cocycles in degree $> n$. So $(\rho_{(n,q)})^n_*$ is a monomorphism and all we need to prove is that it is also an epimorphism: Let $[x] \in H^n(X)$ and $[x] \notin \operatorname{Im}(\rho_{(n,q-1)})^n_*$ (otherwise we are done). Then x defines a non-zero relative cohomology class $v = [(0,x)] \in V(n,q)$. By definition, $v = [(dv, \rho_{(n,q)})v]$. So

$$\begin{pmatrix} dv\\ \rho_{(n,q)}v - x\end{pmatrix} = \begin{pmatrix} d & 0\\ \rho_{(n,q-1)} & -d\end{pmatrix}\begin{pmatrix}t\\y\end{pmatrix} = \begin{pmatrix} dt\\ \rho_{(n,q-1)}t - dy\end{pmatrix}$$

with $t \in N(n, q-1)^{n+1}$, $y \in X^n$. As a consequence,

$$(\rho_{(n,q)})^n_*[v - t] = [x]$$

and so $(\rho_{(n,q)})^n_*$ is an epimorphism. Lastly, put $N(n+1, 0) = \lim_{\overrightarrow{q}} N(n, q)$, $\iota_{(n+1,0)} = \lim_{\overrightarrow{q}} \iota_{(n,q)}$, $\rho_{(n+1,0)} = \lim_{\overrightarrow{q}} \rho_{(n,q)}$ and let us verify the induction hypothesis. Conditions $(\mathrm{i})_{(n+1,0)}$ and $(\mathrm{ii})_{(n+1,0)}$ are trivial. For $(\mathrm{iii})_{(n+1,0)}$, $(\rho_{(n+1,0)})^i_*$: $H^i(N(n+1,0)) \longrightarrow H^i(X)$ is an isomorphism for $i = 0, \ldots, n$ because all of the $(\rho_{(n,q)})^i_*$ are isomorphisms by $(\mathrm{iii})_{(n,q)}$ and $(\rho_{(n+1,0)})^{n+1}_*$ is a monomorphism, because if $a \in Z^{n+1}N(n+1, 0)$ is such that $[\rho_{(n+1,0)}a] = 0$, then, as $a \in N(n, q)$ for some q, we would have $\rho_{(n,q)}a = dx$ for some $x \in X$. So $v = [(a, x)] \in V(n, q+1)$ will kill the class $[a]$ in $H^{n+1}(N(n, q))$ and therefore in $H^{n+1}(N(n+1, 0))$. □

5.2. *Proof of Theorem* 1.3.3. We will confine ourselves to the case $M = 0$, since this is the only case we need in this paper. Let $N = A \otimes V$, and assume we have already built

$$\sigma_{(m,0)}\colon N(m, 0) \to X$$

in a such way that for all $m \leq n$,

$$(\mathrm{i})_m \quad \rho\sigma_{(m,0)} = \mathrm{id}_{N(m,0)}, \text{ and}$$

$$(\mathrm{ii})_m \quad \sigma_{(m,0)}\big|_{N(m',0)} = \sigma_{(m',0)} \text{ for all } m' \leq m.$$

We will define

$$\sigma_{(n+1,0)}\colon N(n+1, 0) \to X$$

satisfying $(\mathrm{i})_{n+1}$, $(\mathrm{ii})_{n+1}$ and then the section of ρ will be

$$\sigma = \lim_{\overrightarrow{n}} \sigma_{(n,0)}.$$

To do this, we will extend $\sigma_{(n,0)}$ to morphisms

$$\sigma_{(n,q)}\colon N(n, q) \to X$$

in such a way that for every $p \leq q$,

$$(\mathrm{i})_{(n,p)} \quad \rho\sigma_{(n,p)} = \mathrm{id}_{N(n,p)}, \text{ and}$$

$$(\mathrm{ii})_{(n,p)} \quad \sigma_{(n,p)}\big|_{N(n,p')} = \sigma_{(n,p')} \text{ for all } p' \leq p.$$

Once we have these $\sigma_{(n,q)}$ for all $q \geq 0$, we will put

$$\sigma_{(n+1,0)} = \lim_{\overrightarrow{q}} \sigma_{(n,q)},$$

which will satisify $(\mathrm{i})_{n+1}$ and $(\mathrm{ii})_{n+1}$.

So let us begin the construction of the $\sigma_{(n,q)}$. For $q = 0$, $\sigma_{(n,0)}$ exists by induction hypothesis. Assume we already have $\sigma_{(n,p)}$ for all $p \leq q$. Then, consider the A-dg module

$$X(n,q) = \operatorname{Im}\left(\sigma_{(n,q)}: N(n,q) \twoheadrightarrow X\right).$$

Since $\rho\sigma_{(n,q)} = \mathrm{id}_{N(n,q)}$, $\sigma_{(n,q)}$ is a monomorphism; then $\sigma_{(n,q)}: N(n,q) \to X(n,q)$ is an isomorphism. If we apply the Five Lemma to the long cohomology sequences of the following commutative diagram of exact sequences of A-dg modules

$$\begin{tikzcd}
0 \arrow[r] & X(n,q) \arrow[r] \arrow[d, "\sigma^{-1}_{(n,q)}"] & X \arrow[r] \arrow[d, "\rho"] & X/X(n,q) \arrow[r] \arrow[d, "\bar\rho"] & 0 \\
0 \arrow[r] & N(n,q) \arrow[r] & N \arrow[r] & N/N(n,q) \arrow[r] & 0
\end{tikzcd}$$

it follows that $\bar\rho$ is a quis of A-dg modules.

Next, let $V(n, q+1)$ be an $\mathbb{R}$-vector space such that

$$N(n, q+1) = N(n,q) \oplus (A \otimes V(n, q+1))$$

and let j be the composition $V \overset{i}{\hookrightarrow} N \to N/N(n,q)$. Since $dV \subset N(n,q)$, we have $jV \subset Z(N/N(n,q))$, and so we have an $\mathbb{R}$-linear morphism $H_j: V \to H(N/N(n,q))$ which we can lift to $Z(X/X(n,q))$:

$$\begin{tikzcd}
 & & Z(X/X(n,q)) \arrow[d, "p"] \\
V \arrow[rru, "\lambda"] \arrow[r, "H_j"] & H(N/N(n,q)) & H(X/X(n,q)). \arrow[l, "\cong"']
\end{tikzcd}$$

In fact, $N/N(n,q)$ is an A-dg module concentrated in degrees $\geq n$. So $B^n(N/N(n,q)) = 0$ and the previous diagram in degree n is just

$$\begin{tikzcd}
 & Z^n(X/X(n,q)) \arrow[d, "\bar\rho"] \\
V \arrow[ru, "\lambda"] \arrow[r, "j"] & Z^n(N/N(n,q)).
\end{tikzcd}$$

Next, consider the pull back

$$\begin{tikzcd}
(X/X(n,q)) \times_{N/N(n,q)} N \arrow[r] \arrow[d] & N \arrow[d] \\
X/X(n,q) \arrow[r, "\bar{\rho}"] & N/N(n,q)
\end{tikzcd}$$

and the morphism induced by $\pi \colon X \to X/X(n,q)$ and $\rho \colon X \to N$,

$$(\pi, \rho) \colon X \longrightarrow (X/X(n,q)) \times_{N/N(n,q)} N.$$

It is an epimorphism: if $(\tilde{x}, y) \in (X/X(n,q)) \times_{N/N(n,q)} N$ then $\rho x - y \in N(n,q)$. So,

$$\begin{aligned}
(\pi, \rho)(x - \sigma_{(n,q)}(\rho x - y)) &= (\tilde{x} - 0, \rho x - (\rho x - y)) \\
&= (\tilde{x}, y).
\end{aligned}$$

So, we can lift (λ, i) to X:

$$\begin{tikzcd}
 & X \arrow[d, "(\pi, \rho)"] \\
V \arrow[ur, "f"] \arrow[r, "(\lambda, i)"'] & (X/X(n,q)) \times_{N/N(n,q)} N.
\end{tikzcd}$$

Then f and $\sigma_{(n,q)}$ induce the morphism of A-dg modules $\sigma_{(n,q+1)} \colon N(n, q+1) \to X$ we were looking for:

$$\sigma_{(n,q+1)} \big|_{N(n,q)} = \sigma_{(n,q)} \text{ and } \sigma_{(n,q+1)} |_V = f.$$

To see that it is an A-dg morphism, it suffices, by definition of a Hirsch extension, to verify that

$$\sigma_{(n,q)} d = df.$$

So, let $v \in V$; then $\pi f v = \lambda v \in Z^n(X/X(n,q))$. So, $0 = d\pi f v = \pi d f v$. Hence $dfv \in X(n,q) = \operatorname{Im} \sigma_{(n,q)}$. Let $\omega \in N(n,q)$ be such that $dfv = \sigma_{(n,q)}\omega$. Apply ρ to both sides of this equality and, by the previous diagram, we get

$$\rho d f v = d \rho f v = dv$$

and, by induction hypothesis $(\mathrm{i})_{(n,q)}$,

$$\rho\sigma_{(n,q)}\omega = \omega.$$

So $dv = \omega$, as we wanted.

Finally, $\sigma_{(n,q+1)}$ satisfies $(\mathrm{i})_{(n,q+1)}$ and $(\mathrm{ii})_{(n,q+1)}$ by construction. □

5.3. *Proof of Theorem* 3.3.1. This result generalizes to stratified spaces the result asserting that the dgc algebra minimal model of a manifold can be computed using its deRham complex. The proof is adapted from [10] using the same notation. It will be sufficient to prove that the dgc algebra minimal model of $\Omega_v(Z)$ is that of $A_{\mathbb{R}}(Z)$. For this purpose we construct a commutative diagram

$$\begin{tikzcd}
\Omega_v(Z) \arrow[rr, "\rho_1"] \arrow[d, "\int_2"] & & E(\underline{LS}(Z)) \arrow[dll, "\int_3"'] & & A_{\mathbf{R}}(Z) \arrow[ll, "\rho_2"'] \arrow[d, "\int_1"] \\
C(\underline{LS}(Z)) & & & & C(\underline{\mathrm{Sing}}(Z)) \arrow[llll, "\rho_3"']
\end{tikzcd} \tag{9}$$

where $\int_1, \int_2, \int_3$, and ρ_3 induce cohomology isomorphisms and ρ_2 and ρ_1 are dgc algebra morphisms. This implies that ρ_2 and ρ_1 are also dgc algebra quasi-isomorphisms and the Proposition is proved. We construct the diagram in several steps.

I. Unfolding of Δ (see [21]). The *unfolding* of the standard simplex Δ, relative to the decomposition $\Delta = \Delta_0 * \cdots * \Delta_p$, is the map

$$\mu_\Delta\colon\ \widetilde{\Delta} = \bar{c}\Delta_0 \times \cdots \times \bar{c}\Delta_{p-1} \times \Delta_p \longrightarrow \Delta$$

defined by

$$\begin{aligned}
\mu_\Delta([x_0, t_0], \ldots, [x_{p-1}, t_{p-1}], x_p) \;=\; & t_0x_0 + (1-t_0)t_1x_1 + \cdots + (1-t_0)\cdots \\
& (1-t_{p-2})t_{p-1}x_{p-1} + (1-t_0)\cdots(1-t_{p-1})x_p.
\end{aligned}$$

Here $\bar{c}\Delta_i$ denotes the closed cone $\Delta_i \times [0,1] \big/ \Delta_i \times \{0\}$ and $[x_i, t_i]$ a point of it. This map is smooth and its restriction $\mu_\Delta\colon \operatorname{int}(\widetilde{\Delta}) \longrightarrow \operatorname{int}(\Delta)$ is a diffeomorphism (we write $\operatorname{int}(P) = P - \partial P$ for the *interior* of the polyhedron P). It sends a face U of $\widetilde{\Delta}$ on a face V of Δ and the restriction $\mu_\Delta\colon \operatorname{int}(U) \to \operatorname{int}(V)$ is a submersion.

This blow-up is compatible with face and degeneracy maps.

1. *Face.* Let $\delta_F\colon F \longrightarrow \Delta$ be a codimension one face of Δ; the induced decomposition is $F = \Delta_0 * \cdots * \Delta_{j-1} * F_j * \Delta_{j+1} * \cdots * \Delta_p$ (we have written $\emptyset * X = X$). The *lifting* of δ_F is the map $\widetilde{\delta}_F\colon \widetilde{F} \to \widetilde{\Delta}$ defined by

$$\widetilde{\delta}_F(z) = \begin{cases}
z & \text{if } F_j \neq \emptyset \\
(z_0, \ldots, z_{j-1}, \vartheta_j, z_{j+1}, \ldots, z_{p-1}, x_p) & \text{if } F_j = \emptyset,\ j \neq p,\ \vartheta_j \text{ vertex of } \bar{c}\Delta_j \\
 & \text{and } z = (z_0, \ldots, \widehat{z_j}, \ldots, z_{p-1}, x_p) \\
(z_0, \ldots, z_{p-2}, [x_{p-1}, 1]) & \text{if } F_j = \emptyset,\ j = p, \text{ and} \\
 & z = (z_0, \ldots, z_{p-2}, x_{p-1}).
\end{cases}$$

This map is smooth (affine in barycentric coordinates), sends $\widetilde{F}$ isomorphically on a face of $\widetilde{\Delta}$ and satisfies $\mu_\Delta \circ \widetilde{\delta}_F = \delta_F \circ \mu_F$.

2. *Degeneracy.* Let $\sigma\colon D = \Delta * \{P\} \to \Delta$ be a degeneracy map with $\sigma_D(P) = Q \in \Delta_j$. The induced decomposition is $D = \Delta_0 * \cdots * \Delta_{j-1} * (\Delta_j * \{P\}) * \Delta_{j+1} * \cdots * \Delta_p)$. The *lifting* of σ is the map $\widetilde{\sigma}_D\colon \widetilde{D} \to \widetilde{\Delta}$ defined by

$$\widetilde{\delta}_D(z) = \begin{cases} (z_0, \ldots, [tx_j + (1-t)Q, t_j], \ldots, z_{p-1}, x_p) & \text{if } j < p \text{ and } z \text{ is} (z_0, \ldots, \\ & [tx_j + (1-t)P, t_j], .., z_{p-1}, x_p), \\ (z_0, \ldots, z_{p-1}, tx_p + (1-t)Q) & \text{if } j = p \text{ and } z \text{ is} \\ & (z_0, \ldots, z_{p-1}, tx_p + (1-t)P). \end{cases}$$

This map is smooth (linear in barycentric coordinates) and verifies $\mu_\Delta \circ \widetilde{\sigma}_D = \sigma_D \circ \mu_D$.

On the boundary $\partial \widetilde{\Delta}$ we find not only the blow-up $\widetilde{\partial \Delta}$ of the boundary $\partial \Delta$ of Δ but also the faces $B_i = \bar{c}\Delta_0 \times \cdots \times \bar{c}\Delta_{i-1} \times (\Delta_i \times \{1\}) \times \bar{c}\Delta_{i+1} \times \cdots \times \bar{c}\Delta_{p-1} \times \Delta_p$ with $i \in \{0, \ldots, p-2\}$ or $i = p-1$ and $\dim \Delta_p > 0$, which we shall call *bad faces*. Notice that $\dim \mu_\Delta(B_i) = \dim(\Delta_0 * \cdots * \Delta_i) < n - 1 = \dim B_i$.

II. The simplicial set[1]

$\underline{LS}(Z)$. On a stratified pseudomanifold it is not possible to define smooth simplices directly as in [10]. For this reason we introduce the notion of liftable simplices. Let $Z = Z_{n=\dim R} \supset \Sigma_Z = Z_{n-1} \supset \cdots \supset Z_0 \supset Z_{-1} = \emptyset$ be the filtration of Z, that is, Z_i is the union of strata with dimension smaller than i. A *liftable simplex* is a singular simplex $\varphi\colon \Delta \to Z$ satisfying the following two conditions.

[LS1] *Each pull back $\varphi^{-1}(Z_i)$ is a face of Δ.*

Consider $\{i_0, \ldots, i_p\} = \{i \in \{0, \ldots, n\} \,/\, \varphi^{-1}(Z_i) \neq \varphi^{-1}(Z_{i-1})\}$ and let Δ_j be the face of Δ with $\varphi^{-1}(Z_{i_j}) = \varphi^{-1}(Z_{i_j - 1}) * \Delta_j$. This defines on Δ the *φ-decomposition* $\Delta = \Delta_0 * \cdots * \Delta_p$.

[LS2] *There exists a smooth map $\widetilde{\varphi}\colon \widetilde{\Delta} \to \widetilde{Z}$ with $\mathcal{L}_Z \circ \widetilde{\varphi} = \varphi \circ \mu_\Delta$, where the unfolding of Δ is taken relative to the φ-decomposition of Δ.*

The $\varphi \circ \delta_F$-decomposition (resp. $\varphi \circ \sigma_D$-decomposition) is just

$$F = \Delta_0 * \cdots * \Delta_{j-1} * F_j * \Delta_{j+1} * \cdots * \Delta_p$$
$$(\text{resp. } D = \Delta_0 * \cdots * \Delta_{j-1} * (\Delta_j * \{P\}) * \Delta_{j+1} * \cdots * \Delta_p).$$

So, for a liftable simplex φ the simplices $\partial_F(\varphi) = \varphi \circ \delta_F$ and $s_D(\varphi) = \varphi \circ \sigma_D$ are liftable simplices. The face map ∂_F and the degeneracy map s_D verify the usual compatibility conditions. Put $\underline{LS}(Z)$ the family of liftable simplices. So, $(\underline{LS}(Z), \partial, s)$ define a simplicial set.

[1]For the notions related with simplicial sets, local systems, ... we refer the reader to [10].

Notice that a liftable simplex φ sends the interior of a face A of Δ on a stratum S of Z and that the restriction φ: $\operatorname{int}(A) \to S$ is smooth (using face maps we can suppose $A = \Delta$ and there we know that μ_Δ: $\operatorname{int}(\tilde{\Delta}) \to \operatorname{int} \Delta$ is a diffeomorphism).

III. The local systems C and E. The local system C on $\underline{\mathrm{Sing}}(Z)$ (resp. $\underline{LS}(Z)$) is defined in [10, 14.2] in such a way that the space of global sections $C(\underline{\mathrm{Sing}}(Z))$ (resp. $C(\underline{LS}(Z))$) is the complex generated by the singular simplices (resp. liftable simplices) of Z.

Consider a simplex Δ endowed with the decomposition $\Delta_0 * \cdots * \Delta_p$. A *liftable form* on Δ is a family $\eta = \{\eta_A \in \Omega(\operatorname{int}(A))\}_{\{A \text{ face of } \Delta\}}$ of differential forms possessing a common lifting $\widetilde{\eta} \in \Omega(\widetilde{\Delta})$, that is,

$$\widetilde{\eta} = \mu_\Delta^* \eta_{\mu_\Delta(H)} \text{ on } \operatorname{int}(H), \quad \text{for each face } H \text{ of } \widetilde{\Delta}.$$

The lifting $\widetilde{\eta}$ is unique. Since μ_Δ is an onto submersion with connected fibers then the lifting forms are exactly the differential forms ω on $\widetilde{\Delta}$ satisfying $\omega(v, -) = d\omega(v, -) = 0$ for any vector of $\widetilde{\Delta}$ with $(\mu_\Delta)_*(v) = 0$, that is, the *basic forms* on $\widetilde{\Delta}$. In the sequel we shall use both of points of view.

A *liftable form* on $\{\varphi\colon \Delta \to Z\} \in \underline{LS}(Z)$ is a liftable form on Δ relative to its φ-decomposition. For any φ: $\Delta \to Z$ liftable simplex we shall write $E\varphi$ = {liftable forms on φ}, which is a dgc algebra complex.

Consider δ_F: $F \to \Delta$ a face map and σ_D: $D \to \Delta$ a degeneracy map. We define the face operator ∂_F: $E\varphi \longrightarrow E_{\partial_F(\varphi)}$ and the degeneracy operator s_D: $E\varphi \longrightarrow E_{s_D(\varphi)}$ by $\partial_F(\widetilde{\eta}) = \widetilde{\delta}_F^* \widetilde{\eta}$ and $s_D(\widetilde{\eta}) = \widetilde{\sigma}_D^* \widetilde{\eta}$ respectively. These operators verify the usual compatibility conditions and thus define a local system E on $\underline{LS}(Z)$. Notice that the space $E(\underline{LS}(Z))$ of global sections of E is a dgc algebra.

When Z is a manifold endowed with the stratification $\{Z\}$, liftable simplex becomes smooth simplex and so $\underline{LS}(Z) = \underline{\mathrm{Sing}}^\infty(Z)$. Moreover, E becomes the local system A_∞ of C^∞ differential forms.

IV. Operators ρ *and* $\int$.

• The operator ρ_3 is just the inclusion, which makes sense since any liftable simplex is a singular one Proceeding as in [21] one proves that this inclusion induces an isomorphism in homology and therefore that ρ_3 is a quasi-isomorphism in the category of graded vector spaces.

• The operator ρ_2 is defined as follows. For each $\omega \in A_{\mathbb{R}}(Z)$ and each liftable simplex φ of Z we let $\mu_\Delta^* \omega_\varphi$ be the lifting of $(\rho_2(\omega))_\varphi$. This operator is a dgc algebra operator.

• The operator ρ_1 is defined as follows. For each $\omega \in \Omega_{_v}(Z)$ and each liftable simplex φ of Z we let $\widetilde{\varphi}^* \widetilde{\omega}$ be the lifting of $(\rho_1(\omega))_\varphi$. This operator is a dgc algebra operator.

• The operator $\int_1$ is given by integration of differential forms on simplices:

$$\left(\int_1 \omega\right)(\varphi) = \int_\Delta \omega_\varphi,$$

where $\omega \in A_{\mathbb{R}}(Z)$ and φ is a singular simplex of Z. The deRham Theorem says that $\int_1$ is a quasi-isomorphism in the category of graded vector spaces [10].

• The operator $\int_2$ is given by integration of differential forms on simplices:

$$\left(\int_2 \omega\right)(\varphi) = \int_{\mathrm{int}(\Delta)} \varphi^* \omega = \int_{\widetilde{\Delta}} \widetilde{\varphi}^* \widetilde{\omega},$$

where $\omega \in \Omega_{\bar{v}}(Z)$ and φ is a liftable simplex of Z. This operator is differential if we have $\int_{B_i} \widetilde{\varphi}^* \widetilde{\omega} = 0$. To prove this, we write S for the stratum of Z containing $\varphi(\mathrm{int}(\mu_\Delta(B_i)))$; since $\widetilde{\varphi}^* \widetilde{\omega} = \mu_\Delta^* \varphi^* \omega_S$ we get $\int_{B_i} \widetilde{\varphi}^* \widetilde{\omega} = 0$ because $\dim B_i > \dim \mu_\Delta(B_i)$. Proceeding as in [21] one proves that $\int_2$ is a quasi-isomorphism in the category of graded vector spaces.

• The operator $\int_3$ is is given by integration of differential forms on simplices:

$$\left(\int_3 \eta\right)(\varphi) = \int_{\mathrm{int}(\Delta)} (\eta_\varphi)_\Delta = \int_{\widetilde{\Delta}} \widetilde{\eta}_\varphi,$$

where $\eta \in E(\underline{LS}(Z))$ and φ is a liftable simplex of Z. This operator is differential since $\int_{B_i} \widetilde{\eta}_\varphi = 0$; this comes from the equality $\widetilde{\eta}_\varphi = \mu_\Delta^*(\eta_\varphi)_{\mu_\Delta(B_i)}$ on $\mathrm{int}(B_i)$. We already know that the local system C on $\underline{LS}(Z)$ is an extendable local system [10, Proposition 14.11]; if we prove that D is an extendable local system it will follow that $\int_3$ is a quasi-isomorphism in the category of graded vector spaces [10, Theorem 12.27]. This fact comes from the Poincaré Lemma and the Extension Property we prove below. For this purpose we fix a decomposition $\Delta = \Delta_0 * \cdots * \Delta_p$.

POINCARÉ LEMMA. *H ({liftable forms on Δ}) $= \mathbb{R}$.*

We prove $H\left(\{\text{basic forms on } \widetilde{\Delta}\}\right) = \mathbb{R}$. First suppose that $\dim \Delta_p > 0$. Let ϑ be a vertex of Δ_p. Consider the following maps:

h_1: $\Delta \times [0,1] \to \Delta$ defined by

$$h_1(r_0x_0 + \cdots + r_px_p, t) = r_0x_0 + \cdots + r_{p-1}x_{p-1} + r_p(t\vartheta + (1-t)x_p),$$

h_2: $\widetilde{\Delta} \times [0,1] \to \widetilde{\Delta}$ defined by

$$h_2(z_0, \ldots, z_{p-1}, x_p, t) = (z_0, \ldots, z_{p-1}, t\vartheta + (1-t)x_p).$$

They are smooth homotopy maps between Δ (resp. $\widetilde{\Delta}$) and $\Delta' = \Delta_0 * \cdots * \Delta_{p-1} * \{\vartheta\}$ (resp. $\widetilde{\Delta}'$). Since $\mu_\Delta(h_2(z_0, \ldots, z_{p-1}, x_p, t)) = h_1(\mu_\Delta(z_0, \ldots, z_{p-1}, x_p), t)$ the basic forms are preserved and h_2 induces a homotopy operator between the complex of {basic forms on $\widetilde{\Delta}$} and the complex of {basic forms on $\widetilde{\Delta}'$}.

Suppose $\dim \Delta_p = 0$, that is, $\Delta_p = \{\vartheta\}$. Write ∇ for the simplex Δ endowed with the decomposition $\Delta_0 * \cdots * \Delta_{p-2} * (\Delta_{p-1} * \{\vartheta\})$. Since the map τ: $\widetilde{\nabla} \to \widetilde{\Delta}$, defined by $\tau(z_0, \ldots, z_{p-2}, tx_{p-1} + (1-t)\vartheta) = (z_0, \ldots, z_{p-2}, [x_{p-1}, t], \vartheta)$, is a diffeomorphism verifying $\mu_\Delta \circ \tau = \mu_\nabla$, we get that the complexes {basic forms on $\widetilde{\Delta}$} and {basic forms on $\widetilde{\nabla}$} are isomorphic.

Alternatively applying these two procedures we arrive at the case $\Delta = \{\vartheta\}$ and here it is clear that the cohomology of {basic forms on $\widetilde{\Delta} \equiv \{\vartheta\}$} is $\mathbb{R}$.

EXTENSION PROPERTY. *Each liftable form on $\partial\Delta$ possesses an extension to a liftable form on Δ.*

Let $\eta = \{\eta_A \in \Omega(\mathrm{int}(A)) \ / \ A \text{ face of } \partial\Delta\}$ be a liftable form on $\partial\Delta$. Let $\widetilde{\eta} \in \Omega\left(\widetilde{\partial\Delta}\right)$ be the lifting, which is a basic form satisfying $\widetilde{\eta} = \mu_\Delta^* \eta_{\mu_\Delta(H)}$ on $\mathrm{int}(H)$ for each face H of $\widetilde{\Delta}$. Since the fibers of μ_Δ: $\widetilde{\partial\Delta} \to \partial\Delta$ are not necessarily connected then we cannot identify liftable forms with basic forms.

First notice that any form on $\partial\widetilde{\Delta}$ possesses an extension to a form on $\widetilde{\Delta}$. Moreover, if the form is basic then its extension is necessarily basic (the restriction μ_Δ: $\mathrm{int}(\widetilde{\Delta}) \to \mathrm{int}(\Delta)$ is a diffeomorphism). So, it suffices to extend $\widetilde{\eta}$ to a basic form defined on $\partial\widetilde{\Delta}$.

Consider δ_F: $F \longrightarrow \Delta$ where F is a face of $\partial\Delta$. Using face maps one constructs a smooth map $\widetilde{\delta}_F$: $\widetilde{F}\dagger \to \widetilde{\Delta}$ sending isomorphically $\widetilde{F}$ on a face of $\partial\widetilde{\Delta}$ and satisfying $\mu_\Delta \circ \widetilde{\delta}_F = \delta_F \circ \mu_F$. Notice that in this case the map $\widetilde{\delta}_F$ is not necessarily unique ($\mathrm{codim}_\Delta F$ can be greater than 2). This equality implies that $\eta|_F = \{\eta_A \ / \ A \text{ face of } F\}$, the restriction of η to F, is a liftable form with lifting $\widetilde{\delta}_F^* \widetilde{\eta}$. Recall that this form is μ_F-basic.

For each $i \in \{0, \ldots, p-1\}$ we put $\nabla_i = \mu_\Delta(B_i)$, whose induced decomposition is just $\nabla_i = \Delta_0 * \cdots * \Delta_i$. Define the projection pr_i: $B_i \to \widetilde{\nabla}_i$ by $pr_i(z_0, \ldots, z_{i-1}, (x_i, 1), z_{i+1}, \ldots, z_{p-1}, x_p) = (z_0, \ldots z_{i-1}, x_i)$. This map sends (the interior of) a face of B_i on (the interior of) a face of ∇_i. We define on B_i the form $\gamma_i = pr_i^* \widetilde{\delta}_{\nabla_i}^* \widetilde{\eta}$, which is a basic form because $\mu_\Delta = \mu_{\nabla_i} \circ pr_i$. Moreover, γ_i is the lifting of $\eta|_{\nabla_i}$: if H is a face of B_i then $\gamma_i = pr_i^* \delta_{\nabla_i}^* \mu_\Delta^* \eta_{\mu_\Delta(H)} = \mu_\Delta^* \eta_{\mu_\Delta(H)}$ on

$\operatorname{int}(H)$. Since the lifting is unique the forms $\{\gamma_0, \dots, \gamma_{p-1}\}$ define a basic form on $\partial\widetilde{\Delta} - \partial\widetilde{\Delta}$, the union of bad faces. Again, the uniqueness of the lifting implies that this form coincides with $\widetilde{\eta}$ on $\partial\widetilde{\Delta}$. Therefore, the extension of $\widetilde{\eta}$ is constructed.

• *Commutativity*. One easily checks that $\int_3 \circ \rho_1 = \int_2$ and $\int_3 \circ \rho_2 = \rho_3 \circ \int_1$, that is, diagram (9) is commutative. Since $\int_1, \int_2, \int_3, \rho_3$ induce isomorphisms in cohomology (as graded vector spaces) and ρ_2, ρ_1 are dgc algebra morphisms then ρ_2, ρ_1 are dgc algebra quasi-isomorphisms. So, the dgc algebra minimal model of Z is that of $\Omega_{_v}(Z)$. □

5.4. *Proof of Theorem* 3.4.1. We shall say that a morphism $f\colon Z' \to Z$ between two stratified spaces is *good* if it satisfies the two following conditions:

[P1] f preserves controlled forms: $f^*\omega \in \Omega_{_v}(Z')$ for any $\omega \in \Omega_{_v}(Z)$.
[P2] f preserves liftable simplices, $f\circ\varphi'$-decomposition = φ'-decomposition and $f\circ\varphi' \in \underline{LS}(Z)$ for any $\varphi' \in \underline{LS}(Z')$.

Now consider the diagram

$$\begin{array}{ccccc}
\Omega_{_v}(Z') & \xrightarrow{\rho_1'} & E(\underline{LS}(Z')) & \xleftarrow{\rho_2'} & A_{\mathbb{R}}(Z') \\
\Big\uparrow f^* & & f^*\Big\uparrow & & f^*\Big\uparrow \\
\Omega_{_v}(Z) & \xrightarrow{\rho_1} & E(\underline{LS}(Z)) & \xleftarrow{\rho_2} & A_{\mathbb{R}}(Z)
\end{array}$$

where the pull backs are defined as follows.

For each $\omega \in A_{\mathbb{R}}(Z)$ and each liftable simple φ of Z' we put $(f^*\omega)_{\varphi'} = \omega_{f\circ\varphi'}$. This operator is a dgc algebra morphism.

Since f is good then $f^*\colon \Omega_{_v}(Z) \to \Omega_{_v}(Z')$ is a well-defined dgc algebra operator. For each $\eta \in E(\underline{LS}(Z))$ and each liftable simple φ of Z' we put $(f^*\eta)_\varphi = \eta_{f\circ\varphi}$, which makes sense since f is good. One easily checks $f^*\circ\rho_1' = \rho_1\circ f^*$ and $f^*\circ\rho_2 = \rho_2'\circ f^*$, which ends the proof □

5.5. *Proof of Proposition* 3.4.2. We first prove the following lemma

LEMMA. *Let* $\varphi\colon \Delta \to M$ *be a simplex satisfying* [*LS*1]. *Then, the family of strata of* M *meeting* Im φ *is totally ordered.*

We prove that if F_1, F_2 are two faces of Δ and S_1, S_2 two strata of M with $\varphi(\operatorname{int}(F_i)) \cap S_i \neq \emptyset$, $i = 1, 2$, then $S_1 \leq S_2$ or $S_2 \leq S_1$.

Since $\varphi^{-1}(M_{\dim S_i})$ (resp. $\varphi^{-1}(M_{\dim S_i - 1})$) is a face of Δ meeting $\operatorname{int}(F_i)$ (resp. not containing $\operatorname{int}(F_i)$) then $\operatorname{int}(F_i) \subset \varphi^{-1}(M_{\dim S_i})$ (resp. $\operatorname{int}(F_i) \cap \varphi^{-1}(M_{\dim S_i - 1}) = \emptyset$). So, $\operatorname{int}(F_i) \subset \varphi^{-1}(M_{\dim S_i} - M_{\dim S_i - 1})$ and by connectivity we get $\operatorname{int}(F_i) \subset \varphi^{-1}(S_i)$. Notice that this implies $\varphi(F_i) \subset \overline{S_i}$.

Now consider F_3, the smallest face of Δ containing F_1 and F_2. Let S_3 be a stratum of M with $\varphi(\mathrm{int}\,(F_3)) \cap S_3 \neq \emptyset$ (it always exists!). From the previous paragraph we get $\varphi(\mathrm{int}\,(F_3)) \subset S_3$ and $\varphi(F_3) \subset \overline{S_3}$ and therefore $S_1 \cap \overline{S_3} \neq \emptyset$, $S_2 \cap \overline{S_3} \neq \emptyset$ and so $S_1 \leq S_3$ and $S_2 \leq S_3$.

Let us suppose $\dim S_1 \leq \dim S_2$. Since the face $\varphi^{-1}(M_{\dim S_2})$ contains $\mathrm{int}\,(F_1)$ and $\mathrm{int}\,(F_2)$ then it also contains F_3 by minimality. Thus $S_3 \subset M_{\dim S_2}$ which gives $\dim S_3 \leq \dim S_2$ and therefore $S_3 = S_2$. Finally, $S_1 \leq S_2$.

Now, in order to verify Proposition 3.4.2, we need to verify conditions [P1] and [P2].

• Projection π.

[P1] This holds since π is a liftable morphism.

[P2] Let $\varphi\colon \Delta \to M$ be a liftable simplex. Following the lemma the family of strata meeting $\mathrm{Im}\,\varphi$ can be written as $S_0 < S_1 < \cdots < S_{p-1} < S_p$. The φ-decomposition of Δ is $\Delta = \Delta_0 * \cdots * \Delta_p$ with $\Delta_0 * \cdots * \Delta_i = \varphi^{-1}(S_0 \cup \cdots \cup S_i)$. Since π is a strict morphism then the family of strata of B meeting $\mathrm{Im}\,(\pi\circ\varphi)$ is $\pi(S_0) < \pi(S_1) < \cdots < \pi(S_{p-1}) < \pi(S_p)$. So, $\Delta_0 * \cdots * \Delta_i = (\pi\circ\varphi)^{-1}(\pi(S_0) \cup \cdots \cup \pi(S_i))$ which implies that the $(\pi\circ\varphi)$-decomposition of Δ is the φ-decomposition. The lifting of $\pi\circ\varphi$ is just $\widetilde{\pi}\circ\widetilde{\varphi}$.

• Inclusion ι.

[P1] Since $\mathrm{len}\,(F) = 0$, we have $\Omega_{v}(F) = \Omega(F)$. Notice that the operator $\iota^*\colon \Omega_{v}(B) \to \Omega(F)$ is just R_F.

[P2] Let $\varphi\colon \Delta \to F$ be a liftable simplex, that is, a smooth map $\varphi\colon \Delta \to S$ where S is a fixed stratum. So, the φ- and the $(\iota\circ\varphi)$-decompositions are just $\Delta = \Delta$. On the other hand, since $\mathcal{L}_B\colon \mathcal{L}_B^{-1}(S) \to S$ is a fibration and Δ is contractible then we can construct a smooth map $\psi\colon \Delta \to \widetilde{Z}$ with $\mu_Z\circ\psi = \iota\circ\varphi$. So, this map is the lifting of $\iota\circ\varphi$. $\square$

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Agustí Roig, Departament de Mathemàtica Aplicada, E.T.S.E.I.B. Universitat Politènica de Catalunya, Diagonal 647, 08028 Barcelona, Spain
roig@mal.upc.es

Martintxo Saralegi-Aranguren, Laboratoire Géométrie-Algèbre, Faculté Jean Perrin, Université d'Artois, 62307 Lens Cedex, France
saralegi@euler.univ-artois.fr