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A GYSIN SEQUENCE FOR SEMIFREE ACTIONS OF S^3

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ABSTRACT. In this work we shall consider smooth semifree (i.e., free outside the fixed point set) actions of $\mathbf{S}^3$ on a manifold M. We exhibit a Gysin sequence relating the cohomology of M with the intersection cohomology of the orbit space $M/\mathbf{S}^3$. This generalizes the usual Gysin sequence associated with a free action of $\mathbf{S}^3$.

Given a free action of the group of unit quaternions $\mathbf{S}^3$ on a differentiable manifold M, there exists a long exact sequence relating the deRham cohomology of the manifold M with the deRham cohomology of the orbit space $M/\mathbf{S}^3$; this is the Gysin sequence (see, e.g., [1, p. 179]):

$$(1)\qquad \cdots \to H^i(M) \xrightarrow{\oint^*} H^{i-3}(M/\mathbf{S}^3) \xrightarrow{\wedge[e]} H^{i+1}(M/\mathbf{S}^3) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots$$

where $\oint$ is the integration along the fibers of the natural projection $\pi\colon M \to M/\mathbf{S}^3$ and $[e] \in H^4(M/\mathbf{S}^3)$ is the Euler class of Φ. This paper is devoted to generalizing this relationship to the case where Φ is allowed to have fixed points (semifree action).

In this context the orbit space $M/\mathbf{S}^3$ is no longer a manifold but a stratified pseudomanifold, a notion introduced by Goresky and MacPherson in [7]. The Gysin sequence we get in this case is

$$(2)\qquad \cdots \to H^i(M) \xrightarrow{\oint^*} IH^{i-3}_{\bar{r}}(M/\mathbf{S}^3) \xrightarrow{\wedge[e]} IH^{i+1}_{\overline{r+4}}(M/\mathbf{S}^3) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots$$

where $\bar{r}$ and $\overline{r+4}$ are two perversities and $[e] \in IH^4_{\bar{4}}(M/\mathbf{S}^3)$ is the Euler class of Φ. The exact statement is given in Theorem 4.7. A similar sequence has been already found for circle actions [8]. Finally, we show a relationship between the existence of a section of π and the vanishing of the Euler class $[e]$. This result generalizes the situation of the free case.

The work is organized as follows. In §1 we introduce simple stratified spaces, which are singular spaces including the orbit space $M/\mathbf{S}^3$ as a special case. Section 2 is devoted to recalling the notion of intersection cohomology with the perversity introduced by MacPherson in [9]. The main tool we use to construct

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the Gysin sequence is the complex of invariant forms, which is studied in §3. Finally, we construct the Gysin sequence (2) in §4.

In this paper, a manifold is supposed to be without boundary and smooth (of class C^∞). From now on, we fix a manifold M with dimension m and $\Phi\colon \mathbf{S}^3 \times M \to M$ a smooth semifree action, that is, Φ is free out of the set $M^{\mathbf{S}^3}$ of fixed points (which will be different from M).

1. Simple stratified sets

We prove that the action Φ induces on M and $M/\mathbf{S}^3$ a particular structure of stratified set.

1.1. Let E be a stratified set [10]; we shall say that E is *simple* if there exists a stratum R with $E = \overline{R}$ (such R is said to be *regular*) and that any other stratum S is closed (S is said to be *singular*). The second condition implies that the singular strata are disjoint. The dimension of E is, by definition, $\dim R$. We shall write $\mathscr{S}$ to represent the family of singular strata.

1.2. We know (cf. [10]) that for each stratum $S \in \mathscr{S}$ there exist a neighborhood T_S of S, a compact manifold L_S, and a fiber bundle $\tau_S\colon T_S \to S$ satisfying:

(a) the fiber of τ_S is the cone $cL_S = L_S \times [0, 1[/L_S \times \{0\}$;
(b) the restriction map $\tau_{S|S}$ is the identity;
(c) the restriction $\tau_S\colon (T_S - S) \to S$ is a smooth fiber bundle with fiber $L_S \times]0, 1[$, whose structural group is $\mathrm{Diff}(L_S)$, the group of diffeomorphisms of L_S; and
(d) $T_S \cap T_{S'} = \varnothing$ if $S \neq S'$.

The family $\{T_S/S \in \mathscr{S}\}$ is said to be a *family of tubes*. Notice that, according to (c), there exists a smooth map $\lambda_S\colon (T_S - S) \to]0, 1[$ such that the restriction $\tau_S\colon \lambda_S^{-1}(]0, \varepsilon[) \to S$, for $\varepsilon \in [0, 1]$, is a fiber bundle with fiber $L_S \times]0, \varepsilon[$. We shall write $D_S = \lambda_S^{-1}(]0, 1/2[)$; in fact, D_S is the half of T_S.

1.3. The manifold M inherits from the action[1] Φ a natural structure of stratified set where the singular strata are the connected components of $M^{\mathbf{S}^3}$ and the regular stratum R is $M - M^{\mathbf{S}^3}$. This stratified set is simple because the open set $M - M^{\mathbf{S}^3}$ is dense.

Since each singular stratum S of M is an invariant submanifold of M, we construct a tubular neighborhood $(T_S, \tau_S, S, \mathbf{S}^{l_S})$ satisfying:

(i) T_S is an open neighborhood of S;

(ii) $\tau_S\colon T_S \to S$ is a smooth fiber bundle with fiber the open disk D^{l_S+1} and $O(l_S + 1)$ as a structural group;

(iii) the restriction of τ_S to S is the identity;

(iv) τ_S is equivariant, that is, $\tau_S(g \cdot y) = g \cdot \tau_S(y)$;

(v) there exists an orthogonal action $\Psi^S\colon \mathbf{S}^3 \times \mathbf{S}^{l_S} \to \mathbf{S}^{l_S}$ and an atlas $\mathscr{A}_S = \{(U, \varphi)\}$ such that $\varphi\colon \tau_S^{-1}(U) \to U \times D^{l_S+1}$ is equivariant, that is, $\varphi(\Phi(g, x)) = (\tau_S(x), [\Phi^S(g, \theta), r])$ for each $g \in \mathbf{S}^3$ and $x = \varphi^{-1}(\tau_S(x), [\theta, r]) \in \tau_S^{-1}(U)$. Here we have identified D^{l_S+1} with the cone $c\mathbf{S}^{l_S} = \mathbf{S}^{l_S} \times [0, 1[/\mathbf{S}^{l_S} \times \{0\}$ and written $[\theta, r]$ an element of $c\mathbf{S}^{l_S}$.

[1] For the notions related with actions we refer to [3].

Notice that the action Φ^S is free and therefore the codimension of S is a multiple of 4. Consider, for each singular stratum S, a tubular neighborhood T_S verifying $T_S \cap T_{S'} = \varnothing$ if $S \neq S'$. Thus, the family $\{T_S\}$ is a family of tubes.

Let $\pi: M \to M/\mathbf{S}^3$ denote the canonical projection. The orbit space $M/\mathbf{S}^3$ inherits naturally from M a structure of stratified set, the strata are $\pi(R) = \pi(M - M^{\mathbf{S}^3})$, the regular stratum, with dimension $m-3$, and $\{\pi(S)/S \in \mathscr{S}\}$, the singular strata. The local description given by (v) shows that $M/\mathbf{S}^3$ is a simple stratified set.

For each $S \in \mathscr{S}$ the image $\pi(T_S)$ is a neighborhood of $\pi(S)$. The map $\rho_S: \pi(T_S) \to \pi(S)$ given by $\rho_S(\pi(x)) = \pi(\tau_S(x))$ is well defined. It is easy to show that $(\pi(\tau_S), \rho_S, \pi(S), \mathbf{S}^{l_S}/\mathbf{S}^3)$ verify 1.2(a)–(d). Then the family $\{\pi(T_S)/S \in \mathscr{S}\}$ is a family of tubes.

1.4. Consider the following commutative diagram:

$$\begin{array}{ccc} D_S - S & \xrightarrow{\tau_S} & S \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \pi(D_S - S) & \xrightarrow{\rho_S} & \pi(S) \end{array}$$

Since the restriction of π to the fibers of τ_s is a submersion $((\mathbf{S}^{l_S} \times]0, 1/2[) \mapsto (\mathbf{S}^{l_S}/\mathbf{S}^3 \times]0, 1/2[))$, we get the relation $\pi_*\{\operatorname{Ker}(\tau_S)_*\} = \operatorname{Ker}(\rho_S)_*$. This will be used in 3.3.

2. Intersection cohomology

We recall the notion of intersection cohomology [4] using the notion of perversity introduced by MacPherson in [9].

2.1. **Cartan's filtration.** Let $\kappa: N \to C$ be a smooth submersion between two manifolds N and C. For each differential form $\omega \not\equiv 0$ on N, we define the *perverse degree* of ω, written $\|\omega\|_C$, as the smallest integer k verifying:

> If $\xi_0, \dots, \xi_k$ are vectorfields on N tangents to the fibers of κ, then $i_{\xi_0} \cdots i_{\xi_k} \equiv 0$.

Here, i_{ξ_j} denotes the interior product by ξ_j. We shall write $\|0\|_C = -\infty$. For each $k \geq 0$ we put $F_k\Omega_N^* = \{\omega \in \Omega^*(N)/\|\omega\|_C \leq k \text{ and } \|d\omega\|_C \leq k\}$. This is the *Cartan's filtration* of κ [4]. Notice that, for $\alpha, \beta \in \Omega^*(N)$, we get the relations

(3) $\quad \|\alpha + \beta\|_C \leq \max(\|\alpha\|_C, \|\beta\|_C) \quad \text{and} \quad \|\alpha \wedge \beta\|_C \leq \|\alpha\|_C + \|\beta\|_C$.

2.2. Let E be a simple stratified set. A *perversity* is a map $\overline{q}: \mathscr{S} \to \mathbb{Z}$ (see [9]). A differential form ω on R is a *$\overline{q}$-intersection differential form* if for each $S \in \mathscr{S}$ the restriction $\omega|_{D_S}$ belongs to $F_{\overline{q}(S)}\Omega^*_{D_S}$. We shall denote by $\Omega^*_{\overline{q}}(E)$ the complex of $\overline{q}$-intersection differential forms of E. *Remark*: For the case $\mathscr{S} = \varnothing$, the complex $\Omega^*_{\overline{q}}(E)$ is exactly the deRham complex $\Omega^*(E)$ of E. The cohomology of the complex $\Omega^*_{\overline{q}}(E)$ is the *intersection cohomology* of E, written $IH^*_{\overline{q}}(E)$. This denomination is justified by 2.5.

Locally, the stratified set E looks like $\mathbb{R}^k \times cL_S$, where L_S is a compact manifold. Here, we have the following computational result:

Proposition 2.3. *For any perversity $\overline{q}$ we obtain*

$$IH^i_{\overline{q}}(\mathbb{R}^k \times cL_S) \cong \begin{cases} H^i(L_S) & \text{if } i \leq \overline{q}(S), \\ 0 & \text{if } i > \overline{q}(S). \end{cases}$$

Proof. Since the maps $pr\colon \mathbb{R}^k \times cL_S \to \mathbb{R}^{k-1} \times cL_S$ and $J\colon \mathbb{R}^{k-1} \times cL_S \to \mathbb{R}^k \times cL_S$ defined by $pr(x_1, \dots, x_k, [y, r]) = (x_2, \dots, x_k, [y, r])$ and $J(x_2, \dots, x_k, [y, r]) = (0, x_2, \dots, x_k, [y, r])$ verify $\|pr^*\omega\|_S \leq \|\omega\|_S$ and $\|J^*\eta\|_S \leq \|\eta\|_S$ for each $\omega \in \Omega^*(\mathbb{R}^{k-1} \times L_S \times]0, 1[)$ and $\eta \in \Omega^*(\mathbb{R}^k \times L_S \times]0, 1[)$, the induced operators

$$pr^*\colon \Omega^*_{\overline{q}}(\mathbb{R}^{k-1} \times cL_S) \to \Omega^*_{\overline{q}}(\mathbb{R}^k \times cL_S),$$
$$J^*\colon \Omega^*_{\overline{q}}(\mathbb{R}^k \times cL_S) \to \Omega^*_{\overline{q}}(\mathbb{R}^{k-1} \times cL_S)$$

are well defined. Notice that the composition J^*pr^* is the identity.

Consider the homotopy operator

$$h\colon \Omega^*(\mathbb{R}^k \times L_S \times]0, 1[) \to \Omega^{*-1}(\mathbb{R}^k \times L_S \times]0, 1[)$$

given by $h(\omega = \alpha + dx_1 \wedge \beta) = \int_0^- \beta \wedge dx_1$, where $\alpha, \beta \in \Omega^*(\mathbb{R}^k \times L_S \times]0, 1[)$ do not involve dx_1. It verifies

$$dh\omega + h\,d\omega = \omega - pr^*J^*\omega. \tag{4}$$

Now, the relation $\|h\omega\|_S \leq \|\omega\|_S$ implies that h is a homotopy between pr^*J^* and the identity on $\Omega^*_{\overline{q}}(\mathbb{R}^k \times cL_S)$. We have proved $IH^*_{\overline{q}}(\mathbb{R}^k \times cL_S) \cong IH^*_{\overline{q}}(cL_S)$. Moreover, by the equalities $\Omega^i_{\overline{q}}(cL_S) = \Omega^i(L_S \times]0, 1[)$, if $i < \overline{q}(S)$, $\Omega^{\overline{q}(S)}_{\overline{q}}(cL_S) \cap d^{-1}(0) = \Omega^{\overline{q}(S)}(L_S \times]0, 1[) \cap d^{-1}(0)$, and by previous calculation we get $IH^i_{\overline{q}}(\mathbb{R}^k \times cL_S) \cong H^i(L_S)$ for $i \leq \overline{q}(S)$.

It remains to prove that, given a cycle $\omega \in \Omega^i_{\overline{q}}(cL_S)$ with $i > \overline{q}(S)$, there exists $\eta \in \Omega^{i-1}_{\overline{q}}(cL_S)$ with $d\eta = \omega$. Write $\omega = \alpha + dr \wedge \beta$, where α, β do not involve dr (r variable of $]0, 1[$); observe that $\alpha \equiv \beta \equiv 0$ on $L_S \times]0, \frac{1}{2}[$. Then, since $\omega = d \int_0^- \beta \wedge dr$, it suffices to take $\eta = \int_0^- \beta \wedge dr$. □

The intersection cohomology satisfies the Mayer-Vietoris property as it is stated in [1, p. 94].

Proposition 2.4. *Given an open covering $\mathscr{U} = \{U\}$ of E, there exists a subordinated partition of the unity $\{f_U\}$ verifying $\omega \in \Omega^*_{\overline{q}}(U) \Rightarrow f_U\omega \in \Omega^*_{\overline{q}}(E)$.*

Proof. A *controlled map* $f\colon E \to \mathbb{R}$ is defined to be a continuous map, differentiable on each stratum, such that the restriction to the fibers of each $\tau_S\colon D_S \to S$ is a constant map [11]. Notice that we have the equality $\max(\|f\|_S, \|df\|_S) = 0$. Then the result follows from the fact that $\mathscr{U}$ possesses a subordinated partition of unity made up of controlled functions [11, p. 8]. □

Two perversities $\overline{p}$ and $\overline{q}$ are *dual* if $\overline{p}(S) + \overline{q}(S) = \dim L_S - 1$ for each $S \in \mathscr{S}$. For example, the *zero perversity* $\overline{0}$, defined by $\overline{0}(S) = 0$, and the *top perversity* $\overline{t}$, defined by $\overline{t}(S) = \dim L_S - 1$, are dual. The relationship between the intersection homology $IH^{\overline{p}}_*(E)$ of [6] and the intersection cohomology is given by

Proposition 2.5. $IH^*_{\overline{q}}(E) \cong IH^{\overline{p}}_*(E)$.

Proof. Consider the first case $E = \mathbb{R}^k \times cL_S$ as in 2.3. Following [6, 9] we get

$$IH^{\overline{p}}_i(\mathbb{R}^k \times cL_S) \cong \begin{cases} H_i(L_S) & \text{if } i \leq \dim L_S - 1 - \overline{p}(S), \\ 0 & \text{if } i \geq \dim L_S - \overline{p}(S), \end{cases}$$

which is isomorphic to $IH^i_{\overline{q}}(E)$ (see 2.3).

This shows that the intersection cohomology and the intersection homology are locally isomorphic. The passage from the local case to the global case cannot be made as in [7] because the axiomatic presentation of the intersection homology has not yet been extended to the new perversities, but we can proceed as in [2] by showing that the usual integration of differential forms over simplices induces a morphism between $IH^*_{\overline{q}}(E)$ and $\operatorname{Hom}(IH^{\overline{p}}_*(E), \mathbb{R})$; such a morphism turns out to be an isomorphism because of Mayer-Vietoris and previous local calculation. Since the proof is similar to that of [2], we leave this work to the reader. □

The following result has also been proved in [9].

Corollary 2.6. *Suppose that each link L_S is connected (that is, E is normal). Then $IH^*_{\overline{0}}(E) \cong H^*(E)$.*

Proof. It suffices to consider the isomorphism $IH^{\overline{t}}_*(E) \cong H_*(E)$ proved in [9]. □

Corollary 2.7. *If E is a manifold then $IH^*_{\overline{q}}(E) \cong H^*(E)$, for each perversity $\overline{0} \leq \overline{q} \leq \overline{t}$.*

Proof. Since E is normal, Corollary 2.6 reduces the problem to prove that the inclusion $\Omega^*_{\overline{0}}(E) \hookrightarrow \Omega^*_{\overline{q}}(E)$ induces an isomorphism in cohomology. Applying 2.4 and 2.3 and taking into account the inequalities $0 \leq \overline{q}(S) \leq \dim L_S - 1$ we transform the problem to showing $H^i(L_S) = 0$ for $0 < i \leq \overline{q}(S)$. But this is exactly the same as showing that L_S is a cohomology sphere, which follows from the fact that M is a manifold. □

3. Invariant forms

A good simplification in the construction of the Gysin sequence is the use of invariant forms.

3.1. The *fundamental vectorfields* X_1, X_2, X_3 of Φ are the vectorfields of M defined by $X_i(x) = T_e\Phi_x(l_i)$, $i = 1, 2, 3$, where $\{l_1, l_2, l_3\}$ is a basis of the Lie algebra of $\mathbf{S}^3$. These vectorfields can be chosen to verify $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, and $[X_3, X_1] = X_2$. The zero-set for each of them is exactly $M^{\mathbf{S}^3}$.

It is well known that the subcomplex of invariant forms

$$\begin{aligned} I\Omega^*(M) &= \{\omega \in \Omega^*(M)/g^*\omega = \omega \text{ for each } g \in \mathbf{S}^3\} \\ &= \{\omega \in \Omega^*(M)/L_{X_i}\omega = 0, \ i = 1, 2, 3\} \end{aligned}$$

computes the cohomology of M (see, e.g., [5]). We prove now a similar result for

$$I\Omega^*_{\overline{q}}(M) = \{\omega \in \Omega^*_{\overline{q}}(M)/L_{X_i}\omega = 0, \ i = 1, 2, 3\}.$$

Proposition 3.2. *For each perversity* $\overline{0} \le \overline{q} \le \overline{t}$ *we have* $H^*(I\Omega_{\overline{q}}(M)) \cong H^*(M)$.

Proof. We first apply 2.4 (with $\mathscr{U}$ made up of invariant sets and $\{f_U\}$ to be invariant controlled maps) and reduce the problem to $M = \mathbb{R}^k \times c\mathbf{S}^{l_S}$. Here, the action of $\mathbf{S}^3$ is given by

$$(g, (x_1, \dots, x_k, [y, r])) \mapsto (x_1, \dots, x_k, [\Phi^S(g, y), r]). \tag{5}$$

Consider $\mathbb{R}^k \times c\mathbf{S}^{l_S}$ as the product $\mathbb{R} \times (\mathbb{R}^{k-1} \times c\mathbf{S}^{l_S})$. Notice that the fundamental vectorfields of $\mathbb{R}^k \times c\mathbf{S}^{l_S}$ (resp. $\mathbb{R}^{k-1} \times c\mathbf{S}^{l_S}$) are

$$X_i = (\underbrace{0, \dots, 0}_{k}, Y_i, 0) \quad (\text{resp. } Z_i = (\underbrace{0, \dots, 0}_{k-1}, Y_i, 0))$$

where Y_i are the fundamental vectorfields of $\mathbf{S}^{l_S}$, $i = 1, 2, 3$. Write pr, J, and h as the operators given by 2.3 for this decomposition. The equalities $pr_* X_i = Z_i$, $J_* Z_i = X_i$, and $i_{X_i} h = h i_{X_i}$ show that these operators are equivariant. Proceeding as in 2.3, we first reduce the problem to the case $M = \mathbb{R}^{k-1} \times c\mathbf{S}^{l_S}$ and finally to the case $M = c\mathbf{S}^{l_S}$. Again, the operators used in 2.3 to reduce the problem to $\mathbf{S}^{l_S}$ are equivariant. Here, the inclusion $I\Omega^*(\mathbf{S}^{l_S}) \hookrightarrow \Omega^*(\mathbf{S}^{l_S})$ induces an isomorphism in cohomology because Φ^S is free. □

3.3. For any differential form $\alpha \in \Omega^*(\pi(M - M^{\mathbf{S}^3}))$ the pull-back $\pi^*\alpha$ is an invariant form. According to 1.4 it satisfies

$$\|\pi^*\alpha\|_S = \|\alpha\|_{\pi(S)} \tag{6}$$

for each $S \in \mathscr{S}$.

Let μ be a Riemannian metric on $R = M - M^{\mathbf{S}^3}$ invariant by the action of Φ and satisfying $\chi_i(X_j) = \delta_{i,j}$ for $i, j \in \{1, 2, 3\}$. The *fundamental forms* of Φ are the differential forms on $M - M^{\mathbf{S}^3}$ defined by $\chi_i = \mu(X_i, -)$, $i = 1, 2, 3$. They satisfy

$$\|\chi_i\|_S = 1. \tag{7}$$

Let $e \in \Omega^4(\pi(R))$ be a closed form representing the Euler class of the action $\Phi\colon \mathbf{S}^3 \times R \to R$. Then we can choose $\eta \in \Omega^3(R)$ so that $i_{X_3} i_{X_2} i_{X_1} \eta = 0$ and $d\eta = d(\chi_1 \wedge \chi_2 \wedge \chi_3) - \pi^* e$ (cf. [5, p. 322]). Notice that the relation $\|e\|_{\pi(S)} \le 4$ holds for each $S \in \mathscr{S}$. The class $[e] \in IH^4_{\overline{4}}(M/\mathbf{S}^3)$ is called the *Euler class* of Φ. It coincides with the usual one when the action Φ is free.

4. Gysin sequence

The Gysin sequence is constructed by using the integration along the fibers of π; this operator is very simple when we are dealing with invariant differential forms.

4.1. Consider ω to be an invariant differential form. The differential form $i_{X_3} i_{X_2} i_{X_1} \omega$ is also invariant $(L_{X_i} i_{X_j} = i_{X_j} L_{X_i} + i_{[X_i, X_j]})$; moreover, $i_{X_i} i_{X_3} i_{X_2} i_{X_1} \omega = 0$ for $i = 1, 2, 3$ and therefore $i_{X_3} i_{X_2} i_{X_1} \omega$ is a basic form. That is, there exists $\eta \in \Omega^*(\pi(R))$ with $i_{X_3} i_{X_2} i_{X_1} \omega = (-1)^{|\omega|} \pi^* \eta$ where $|\omega| =$ degree of ω. Notice that $i_{X_3} i_{X_2} i_{X_1} d\omega = -d i_{X_3} i_{X_2} i_{X_1} \omega$.

The *integration along the fibers* of π is defined to be the operator $⨍\colon I\Omega^*(R) \to \Omega^{*-1}(\pi(R))$, where $⨍\omega = \eta$; it is a differential operator. Notice that $⨍\pi^*\alpha = 0$ and $⨍(-1)^{|\alpha|} \chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \pi^*\alpha = \alpha$ for any $\alpha \in \Omega^*(\pi(R))$.

4.2. If the action is free, the short exact sequence

$$0 \to \Omega^*(M/\mathbf{S}^3) \xrightarrow{\pi^*} I\Omega^*(M) \xrightarrow{\fint} \Omega^{*-3}(M/\mathbf{S}^3) \to 0$$

induces the long exact sequence

$$\cdots \to H^i(M) \xrightarrow{\fint^*} H^{i-3}(M/\mathbf{S}^3) \xrightarrow{\wedge[e]} H^{i+1}(M/\mathbf{S}^3) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots ,$$

where $[e] \in H^4(M/\mathbf{S}^3)$ is the Euler class of Φ.

If the action Φ is not free, the previous section is no longer an exact one (see 4.9). But, we are going to show that by considering the intersection differential forms of M instead of the differential forms, we also get a Gysin sequence relating in this case the intersection cohomology of $M/\mathbf{S}^3$ with the cohomology of M. This sequence arises from the study of the short exact sequence

$$0 \to \operatorname{Ker} \fint \xrightarrow{\iota} I\Omega^*_{\bar{q}}(M) \xrightarrow{\fint} \operatorname{Im} \fint \to 0,$$

and more precisely, from the comparison of $\operatorname{Ker}\fint$ and $\operatorname{Im}\fint$ with $\Omega^*_{\bar{?}}(M/\mathbf{S}^3)$.

There will come out a shift on the perversities involved, due to the perverse degree of e. For this reason we fix three perversities: $\bar{q}$ (of M), and $\bar{r}$ and $\overline{r+4}$ (of $M/\mathbf{S}^3$) satisfying $\bar{r}(\pi(S)) = \bar{q}(S) - 4$, $\overline{r+4}(\pi(S)) = \bar{q}(S)$, and $\bar{0} \le \bar{q} \le \bar{t}$.

4.3. **Kernel of $\fint$.** By construction we have $\operatorname{Ker}\fint = \{\omega \in I\Omega^*_{\bar{q}}(M) / i_{X_3} i_{X_2} i_{X_1}\omega = 0\}$. For each $\alpha \in \Omega^*(\pi(R))$ we have $\|\pi^*\alpha\|_S = \|\alpha\|_{\pi(S)}$ (cf. 3.3) and $\fint \pi^*\alpha = 0$. Thus, the operator $\pi^*\colon \Omega^*_{\overline{r+4}}(M/\mathbf{S}^3) \to \operatorname{Ker}\fint$ is well defined. In fact, we have:

Proposition 4.4. *The operator $\pi^*\colon \Omega^*_{\overline{r+4}}(M/\mathbf{S}^3) \to \operatorname{Ker}\fint$ induces an isomorphism in cohomology.*

Proof. We first apply 2.4 (with $\mathcal{U}$ made up of invariant sets and $\{f_U\}$ invariant controlled maps) and reduce the problem to $M = \mathbb{R}^k \times c\mathbf{S}^{l_S}$. Consider $pr'\colon \mathbb{R}^k \times c\mathbf{S}^{l_S}/\mathbf{S}^3 \to \mathbb{R}^{k-1} \times c\mathbf{S}^{l_S}/\mathbf{S}^3$ the natural projection as in 2.3. Set $\pi\colon \mathbb{R}^k \times c\mathbf{S}^{l_S} \to \mathbb{R}^k \times c\mathbf{S}^{l_S}/\mathbf{S}^3$ and $\pi'\colon \mathbb{R}^{k-1} \times c\mathbf{S}^{l_S} \to \mathbb{R}^{k-1} \times c\mathbf{S}^{l_S}/\mathbf{S}^3$ the natural projections. With the notation of 3.2, we have $pr'\pi = \pi' pr$. The relations $pr_* X_i = Z_i$, $J_* Z_i = X_i$, and $i_{X_i} h = h i_{X_i}$ imply

$$\fint pr^* = pr^* \fint', \qquad \fint' J^* = J^* \fint, \qquad \fint h = h \fint, \tag{8}$$

where $\fint$ (resp. $\fint'$) is the integration along the fibers of π (resp. π'). We conclude that the diagram

$$\begin{array}{ccc}
\Omega^*_{\overline{r+4}}(\mathbb{R}^k \times c\mathbf{S}^{l_S}/\mathbf{S}^3) & \xrightarrow{\pi^*} & \operatorname{Ker}\left\{\fint\colon I\Omega^*_{\overline{r+4}}(\mathbb{R}^k \times c\mathbf{S}^{l_S}) \to \Omega^{*-3}(\mathbb{R}^k \times \mathbf{S}^{l_S}/\mathbf{S}^3 \times]0,1[)\right\} \\
\uparrow{\scriptstyle (pr')^*} & & \uparrow{\scriptstyle pr^*} \\
\Omega^*_{\overline{r+4}}(\mathbb{R}^{k-1} \times c\mathbf{S}^{l_S}/\mathbf{S}^3) & \xrightarrow{(\pi')^*} & \operatorname{Ker}\left\{\fint'\colon I\Omega^*_{\overline{r+4}}(\mathbb{R}^{k-1} \times c\mathbf{S}^{l_S}) \to \Omega^{*-3}(\mathbb{R}^{k-1} \times \mathbf{S}^{l_S}/\mathbf{S}^3 \times]0,1[)\right\}
\end{array}$$

is well defined and commutative. The vertical rows are quasi isomorphisms (same procedure as 2.3). This first reduces the problem to $M = \mathbb{R}^{k-1} \times c\mathbf{S}^{l_S}$ and finally to $M = c\mathbf{S}^{l_S}$.

In order to prove that

$$\pi^*: IH^{i}_{\overline{r+4}}(c\mathbf{S}^{l_S}/\mathbf{S}^3) \to H^i\left(\operatorname{Ker}\left\{ f\!\!\!-: I\Omega^*_{\overline{q}}(c\mathbf{S}^{l_S}) \to \Omega^*(\mathbf{S}^{l_S}/\mathbf{S}^3\times]0,1[)\right\}\right)$$

is an isomorphism in cohomology, we distinguish three cases.

- $i < \overline{r}(\pi(S)) + 4$. Here, we have $\Omega^i_{\overline{r+4}}(c\mathbf{S}^{l_S}/\mathbf{S}^3) = \Omega^i(\mathbf{S}^{l_S}/\mathbf{S}^3\times]0,1[)$ and

$$(\operatorname{Ker} f\!\!\!-)^i = \{\omega \in I\Omega^i(\mathbf{S}^{l_S}\times]0,1[)/ i_{(Y_3,0)} i_{(Y_2,0)} i_{(Y_1,0)}\omega = 0\}.$$

Contracting the second factor to a point and proceeding as before, we reduce the problem to prove that

$$\pi^*: H^i(\mathbf{S}^{l_S}/\mathbf{S}^3) \to H^i(\{\omega \in I\Omega^*(\mathbf{S}^{l_S})/ i_{Y_3} i_{Y_2} i_{Y_1}\omega = 0\})$$

is an isomorphism. But, since the action Φ^S is free, we already know that the map

$$\pi^*: \Omega^*(\mathbf{S}^{l_S}/\mathbf{S}^3) \to \operatorname{Ker}\left\{ f\!\!\!-: I\Omega^*(\mathbf{S}^{l_S}) \to \Omega^{*-3}(\mathbf{S}^{l_S}/\mathbf{S}^3)\right\}$$

induces an isomorphism in cohomology.

- $i = \overline{r}(\pi(S)) + 4$. We can proceed in the same way because

$$\Omega^i_{\overline{r+4}}(c\mathbf{S}^{l_S}/\mathbf{S}^3) \cap d^{-1}(0) = \Omega^i(\mathbf{S}^{l_S}/\mathbf{S}^3\times]0,1[) \cap d^{-1}(0)$$

and

$$(\operatorname{Ker} f\!\!\!-)^i \cap d^{-1}(0) = \{\omega \in I\Omega^i(\mathbf{S}^{l_S}\times]0,1[)/ i_{(Y_3,0)} i_{(Y_2,0)} i_{(Y_1,0)}\omega = 0\} \cap d^{-1}(0).$$

- $i > \overline{r}(\pi(S)) + 4$. Since $IH^{i}_{\overline{r+4}}(c\mathbf{S}^{l_S}/\mathbf{S}^3) = 0$, it suffices to prove that for any $\omega \in I\Omega^i(\mathbf{S}^{l_S}\times]0,1[)$ satisfying (1) $\omega = 0$ on $\mathbf{S}^{l_S}\times]0,\frac{1}{2}[$, (2) $i_{(Y_3,0)} i_{(Y_2,0)} i_{(Y_1,0)}\omega = 0$, and (3) $d\omega = 0$, there exists $\eta \in I\Omega^{i-1}(\mathbf{S}^{l_S}\times]0,1[)$ verifying (1) and (2) with $d\eta = \omega$. Write $\omega = \alpha + dr \wedge \beta$ where $\alpha, \beta \in I\Omega^*(\mathbf{S}^{l_S}\times]0,1[)$ do not involve dr. We define $\eta = \int_0^- \beta \wedge dr$, which clearly satisfies (1) and $d\eta = \omega$. Since Y_1, Y_2, Y_3 do not involve $\partial/\partial r$, we also have (2). □

4.5. **Image of $f\!\!\!-$.** For each differential form $\alpha \in \Omega^*(\pi(R))$ we get

$$\max(\|\chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \pi^*\alpha\|_S,\ \|d(\chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \pi^*\alpha)\|_S) \le 4 + \|\alpha\|_{\pi(S)}$$

(cf. (3) and (7)). Since $f\!\!\!-(-1)^{|\alpha|}\chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \pi^*\alpha = \alpha$, we conclude that $\Omega^*_{\overline{r}}(M/\mathbf{S}^3)$ is a subcomplex of $\operatorname{Im} f\!\!\!-$.

Proposition 4.6. *The inclusion $\Omega^*_{\overline{r}}(M/\mathbf{S}^3) \hookrightarrow \operatorname{Im} f\!\!\!-$ induces an isomorphism in cohomology.*

Proof. Given an invariant function $f = \pi^* f_0: M \to \mathbb{R}$ and an invariant differential form $\omega \in I\Omega^*(M)$, we get $f\!\!\!- f\omega = f_0 f\!\!\!- \omega$. We can therefore apply 2.4 and reduce the problem to the case $M = \mathbb{R}^k \times c\mathbf{S}^{l_S}$, where the action is given by (5).

Proceeding as in 4.4 we arrive at the case $M = c\mathbf{S}^{l_S}$. Here, in order to prove that the induced map

$$IH^i_{\overline{r}}(c\mathbf{S}^{l_S}/\mathbf{S}^3) \to H^i\left(\operatorname{Im}\left\{ f\!\!\!-: I\Omega^*_{\overline{q}}(c\mathbf{S}^{l_S}/\mathbf{S}^3) \to \Omega^{*-3}(\mathbf{S}^{l_S}/\mathbf{S}^3\times]0,1[)\right\}\right)$$

is an isomorphism for $i \ge 0$, we distinguish four cases:

- $i < \overline{r}(\pi(S))$. In this case we have $I\Omega^i_{\overline{r}}(c\mathbf{S}^{l_S}/\mathbf{S}^3) = \Omega^i(\mathbf{S}^{l_S}/\mathbf{S}^3\times]0,1[)$ and $(\operatorname{Im} f\!\!\!-)^i = \{f\!\!\!-\omega/\omega \in I\Omega^{i+3}(\mathbf{S}^{l_S}\times]0,1[)\}$, which is exactly $\Omega^i(\mathbf{S}^{l_S}/\mathbf{S}^3\times]0,1[)$.

• $i = \overline{r}(\pi(S))$. We can proceed in the same way because

$$I\Omega^i_{\overline{r}}(c\mathbf{S}^{l_S}/\mathbf{S}^3) \cap d^{-1}(0) = \Omega^i(\mathbf{S}^{l_S}/\mathbf{S}^3 \times]0,1[) \cap d^{-1}(0)$$

and

$$(\operatorname{Im} ⨍)^i \cap d^{-1}(0) = \left\{ ⨍ \omega / \omega \in I\Omega^{i+3}(\mathbf{S}^{l_S} \times]0,1[) \right\} \cap d^{-1}(0).$$

• $i = \overline{r}(\pi(S)) + 1$. Since $IH^i_{\overline{r}}(c\mathbf{S}^{l_S}/\mathbf{S}^3) = 0$ and $i + 3 = \overline{q}(S)$, we need to prove that for any $\omega \in I\Omega^{i+3}(\mathbf{S}^{l_S} \times]0,1[)$ verifying (1) $d\omega = 0$ on $\mathbf{S}^{l_S} \times]0, \frac{1}{2}[$ and (2) $d ⨍ \omega = 0$, there exists $\eta \in I\Omega^{i+2}(\mathbf{S}^{l_S} \times]0,1[)$ with $d ⨍ \eta = ⨍ \omega$. We project $\mathbf{S}^{l_S} \times]0,1[$ onto $\mathbf{S}^{l_S} \times \{1/4\} \equiv \mathbf{S}^{l_S}$. Relations (4) and (8) give $⨍ \omega = ⨍ pr^* J^* \omega + d ⨍ h\omega - hd ⨍ \omega = ⨍ pr^* J^* \omega + d ⨍ h\omega$, where $pr^* J^* \omega$, $h\omega \in I\Omega^{i+3}(\mathbf{S}^{l_S} \times]0,1[)$. By construction, the differential form $J^*\omega$ is a cycle of $I\Omega^{i+3}(\mathbf{S}^{l_S})$. Since $0 < i + 3 = \overline{q}(S) \le l_S - 1$, we find $\gamma \in I\Omega^{i+2}(\mathbf{S}^{l_S})$ with $d\gamma = J^*\omega$. Now, we can choose $\eta = pr^*\gamma + h\omega$.

• $i > \overline{r}(\pi(S)) + 1$. Since $IH^i_{\overline{r}}(c\mathbf{S}^{l_S}/\mathbf{S}^3) = 0$ and $i + 3 > \overline{q}(S)$, we need to prove that for any $\omega \in I\Omega^{i+3}(\mathbf{S}^{l_S} \times]0,1[)$ verifying (1) $\omega = 0$ on $\mathbf{S}^{l_S} \times]0, \frac{1}{2}[$, and (2) $d ⨍ \omega = 0$, there exists $\eta \in I\Omega^{i+2}(\mathbf{S}^{l_S} \times]0,1[)$ satisfying (1) with $d ⨍ \eta = ⨍ \omega$. It suffices to choose $\eta = (-1)^i \chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \pi^* \int_0^- \beta \wedge dr$, where $⨍ \omega = \alpha + dr \wedge \beta$ as in 4.4. □

We arrive at the main result of this work.

Theorem 4.7. *Let $\Phi: \mathbf{S}^3 \times M \to M$ be a semifree action. Then there exists a long exact sequence*
(9)

$$\cdots \to H^i(M) \xrightarrow{⨍^*} IH^{i-3}_{\overline{r}}(M/\mathbf{S}^3) \xrightarrow{\wedge[e]} IH^{i+1}_{\overline{r+4}}(M/\mathbf{S}^3) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots,$$

where

(a) *$⨍$ is the integration along the fibers of the natural projection $\pi: M \to M/\mathbf{S}^3$,*
(b) *$\overline{r}$ is a perversity of $M/\mathbf{S}^3$ verifying $-4 \le \overline{r}(\pi(S)) \le l_S - 5$,*
(c) *$\overline{r+4}$ is the perversity of $M/\mathbf{S}^3$ defined by $\overline{r+4}(\pi(S)) = \overline{r}(\pi(S)) + 4$, and*
(d) *$[e] \in IH^4_{\overline{4}}(M/\mathbf{S}^3)$ is the Euler class of Φ.*

Proof. Consider $\overline{q}$ the perversity of M defined by $\overline{q}(S) = \overline{r}(\pi(S)) + 4$. The short exact sequence $0 \to \operatorname{Ker} ⨍ \xrightarrow{\iota} I\Omega^*(M) \xrightarrow{⨍^*} \operatorname{Im} ⨍ \to 0$ induces the long exact sequence (cf. 2.7)

$$\cdots \to H^i(M) \xrightarrow{⨍^*} H^{i-3}(\operatorname{Im} ⨍) \xrightarrow{\delta} H^{i+1}(\operatorname{Ker} ⨍) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots.$$

The connecting homomorphism is defined by $\delta[\alpha] = [(-1)^{|\alpha|} d(\chi_1 \wedge \chi_2 \wedge \chi_3) \wedge \pi^*\alpha]$, which is $[(-1)^{|\alpha|}\pi^*(e \wedge \alpha)]$ on $H^*(\operatorname{Ker} ⨍)$ (cf. 3.3). It suffices now to apply 4.4 and 4.6. □

Corollary 4.8. *Let $\Phi: \mathbf{S}^3 \times M \to M$ be a semifree action. The long exact sequences*

$$\cdots \to H^i(M) \xrightarrow{f^*} H^{i-3}(M/\mathbf{S}^3, M^{\mathbf{S}^3}/\mathbf{S}^3) \xrightarrow{\wedge[e]} H^{i+1}(M/\mathbf{S}^3) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots$$

and

$$\cdots \to H^i(M) \xrightarrow{f^*} H^{i-3}(M/\mathbf{S}^3) \xrightarrow{\wedge[e]} IH^{i+1}_{\bar{4}}(M/\mathbf{S}^3) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots$$

are exact, where for the second sequence we have assumed $M/\mathbf{S}^3$ to be without boundary.

Proof. In both cases we apply the previous theorem taking into account Corollary 2.6. For the first one we consider the perversity $\bar{r}$ defined by $\bar{r}(\pi(S)) = -4$. By definition, $IH^*_{\bar{r}}(M/\mathbf{S}^3)$ is the cohomology of the complex made up of differential forms on $M - M^{\mathbf{S}^3}/\mathbf{S}^3$ vanishing on a neighborhood of $M^{\mathbf{S}^3}/\mathbf{S}^3$; therefore,

$$IH^*_{\bar{r}}(M/\mathbf{S}^3) \cong H^*(M/\mathbf{S}^3, M^{\mathbf{S}^3}/\mathbf{S}^3).$$

For the second case, we consider the perversity $\bar{r} = \bar{0}$. This perversity satisfies condition (c) of the previous theorem because if $M/\mathbf{S}^3$ has no boundary then $l_S > 5$ for each $S \in \mathcal{S}$. □

4.9. The sequence (1) does not become necessarily (9). Let us give an example. Consider the unit sphere $\mathbf{S}^{4l+3}$ of $\mathbb{HP}^{l+1}$, where $\mathbb{HP}$ are the quaternions. The product by quaternions induce the action $\Psi: \mathbf{S}^3 \times \mathbf{S}^{4l+3} \to \mathbf{S}^{4l+3}$. Identify $\mathbf{S}^{4l+4}$ with the suspension $\Sigma\mathbf{S}^{4l+3} = \mathbf{S}^{4l+3} \times [-1, 1]/\{\mathbf{S}^{4l+3} \times \{1\}, \mathbf{S}^{4l+3} \times \{-1\}\}$. Consider the action $\Phi: \mathbf{S}^3 \times \mathbf{S}^{4l+4} \to \mathbf{S}^{4l+4}$ defined by $\Phi(\theta, [x, t]) = [\Psi(\theta, x), t]$. The sequence (1) becomes

$$\cdots \to H^i(\mathbf{S}^{4l+4}) \to H^{i-3}(\Sigma\mathbb{HP}^l) \to H^{i+1}(\Sigma\mathbb{HP}^l) \to H^{i+1}(\mathbf{S}^{4l+4}) \to \cdots,$$

which cannot be exact because $\chi(\mathbf{S}^{4l+4}) \neq 0$.

We finish the work with a geometrical interpretation of the vanishing of the Euler class, generalizing [5, p. 321].

Proposition 4.9. *If the principal fibration $\pi: (M - M^{\mathbf{S}^3}) \to (M - M^{\mathbf{S}^3})/\mathbf{S}^3$ has a section, then $[e] = 0$.*

The existence of a section of $\pi: (M - M^{\mathbf{S}^3}) \to (M - M^{\mathbf{S}^3})/\mathbf{S}^3$ implies the vanishing of the Euler class $[e']$ of the action $\Phi': \mathbf{S}^3 \times (M - M^{\mathbf{S}^3}) \to (M - M^{\mathbf{S}^3})$. Thus, the singular strata must have at most codimension four and, therefore, $F_4\Omega^*_{D_S} = \Omega^*(D_S - S)$ for each $S \in \mathcal{S}$. This implies $IH^*_{\bar{4}}(M/\mathbf{S}^3) = H^*((M - M^{\mathbf{S}^3})/\mathbf{S}^3)$. We have finished the proof because $[e] = [e']$.

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