(b) The chains have compact support, so we get (R1) and (R4). The case (R9) is immediate.

(d) Since $\mathcal{S}_U = \mathcal{I}$ implies $\mathcal{T}_U = \mathcal{I}$ then property (D) becomes a tautology.

(c) Consider a singular point $x \in X$. Following Remark 3.7 we distinguish three cases.

(C-a) $\boldsymbol{x \in S}$**, source stratum of** $\boldsymbol{\mathcal{S}}$. Considering Proposition 3.4 (a) and using the local calculations 1.5.b and 1.9.b, we need to prove

$$\begin{array}{llll} (R1) & H_*^{\overline{p}}(L,\mathcal{L}) \cong H_*^{I_\star\overline{p}}(L,\mathcal{L}') & \Longrightarrow & H_*^{\overline{p}}(\mathring{c}L,\mathring{c}\mathcal{L}) \cong H_*^{I_\star\overline{p}}(\mathring{c}L,\mathring{c}\mathcal{L}') \\ (R4) & \mathfrak{H}_*^{\overline{p}}(L,\mathcal{L}) \cong \mathfrak{H}_*^{I_\star\overline{p}}(L,\mathcal{L}') & \Longrightarrow & \mathfrak{H}_*^{\overline{p}}(\mathring{c}L,\mathring{c}\mathcal{L}) \cong \mathfrak{H}_*^{I_\star\overline{p}}(\mathring{c}L,\mathring{c}\mathcal{L}') \\ (R9) & \mathscr{H}_*^{\overline{p}}(L,\mathcal{L}) \cong \mathscr{H}_*^{I_\star\overline{p}}(L,\mathcal{L}') & \Longrightarrow & \mathscr{H}_*^{\overline{p}}(\mathring{c}L,\mathring{c}\mathcal{L}) \cong \mathscr{H}_*^{I_\star\overline{p}}(\mathring{c}L,\mathring{c}\mathcal{L}'). \end{array}$$

Since the perversity $\overline{p}$ verifies $\overline{p}(S) = I_\star\overline{p}(S^I)$ (cf. Lemma 4.3) then we have $\overline{p}(\mathtt{v}) = I_\star\overline{p}(\mathtt{v})$ (cf. (19)). The result comes now directly from the local calculations 1.5.b and 1.9.b.

(C-b) $\boldsymbol{x \in S}$**, exceptional stratum of** $\boldsymbol{\mathcal{S}}$. Considering Proposition 3.4 (b) and using the local calculations 1.5.b and 1.9.b, we need to prove

$$(R1)\ H_*^{\overline{p}}\left(\mathring{c}S^{b-1},\mathring{c}\mathcal{I}\right) \cong G, \qquad (R4)\ \mathfrak{H}_*^{\overline{p}}\left(\mathring{c}S^{b-1},\mathring{c}\mathcal{I}\right) \cong G, \qquad (R9)\ \mathscr{H}^*_{\overline{p}}\left(\mathring{c}S^{b-1},\mathring{c}\mathcal{I}\right) \cong R.$$

where $b = \operatorname{codim} S \geqslant 1$. Since $0 \leqslant \overline{p}(S) \leqslant \overline{t}(S) = b-2$ (cf. (25)) then we have $0 \leqslant \overline{p}(\mathtt{u}) \leqslant b-2$ (cf. (20)). The result comes now directly from the local calculations 1.5.b and 1.9.b.

(C-c) $\boldsymbol{x \in S}$**, virtual stratum, with** $\boldsymbol{S^I}$ **singular stratum of** $\boldsymbol{\mathcal{S}}$. Considering Proposition 3.4 (c) and using the local calculations 1.5.b and 1.9.b, we need to prove

$$\begin{array}{llll} (R1) & H_*^{\overline{p}}\left(\mathring{c}(S^{b-1} * E),\mathring{c}\mathcal{E}_{\star b-1}\right) & \cong & H_*^{I_\star\overline{p}}(\mathring{c}E,\mathring{c}\mathcal{E}) \\ (R4) & \mathfrak{H}_*^{\overline{p}}\left(\mathring{c}(S^{b-1} * E),\mathring{c}\mathcal{E}_{\star b-1}\right) & \cong & \mathfrak{H}_*^{I_\star\overline{p}}(\mathring{c}E,\mathring{c}\mathcal{E}) \\ (R9) & \mathscr{H}^*_{\overline{p}}\left(\mathring{c}(S^{b-1} * E),\mathring{c}\mathcal{E}_{\star b-1}\right) & \cong & \mathscr{H}^*_{I_\star\overline{p}}(\mathring{c}E,\mathring{c}\mathcal{E}), \end{array} \tag{27}$$

where $b = \dim S^I - \dim S \geqslant 1$.

Since $S \preceq (S^I \backslash S)_{cc}$ (cf. (17)) and $\overline{p}((S^I \backslash S)_{cc}) \leqslant \overline{p}(S) \leqslant \overline{p}((S^I \backslash S)_{cc}) + b$ (cf. (22)) then we have $\overline{p}((S^{b-1})_{cc}) \leqslant \overline{p}(\mathtt{u}) \leqslant \overline{p}((S^{b-1})_{cc}) + b$ (cf. (21)). Similarly, we get $D\overline{p}((S^{b-1})_{cc}) \leqslant D\overline{p}(\mathtt{u}) \leqslant D\overline{p}((S^{b-1})_{cc}) + b$ (cf. (24)).

On the other hand, we have $I_\star\overline{p}(\mathtt{v}) = I_\star\overline{p}(S^I) = \overline{p}((S^I \backslash S)_{cc}) = \overline{p}\left(\left(S^{b-1}\right)_{cc}\right)$ (cf. Lemma 4.3 and (21))). Since $\dim(\mathbb{R}^a \times S^{b-1} \times]0,1[) = \dim(\mathbb{R}^a \times \mathbb{R}^b \times \{\mathtt{v}\})$ then $DI_\star\overline{p}(\mathtt{v}) = D\overline{p}\left(\left(S^{b-1}\right)_{cc}\right)$. We conclude that

$$DI_\star\overline{p}(\mathtt{v}) \leqslant D\overline{p}(\mathtt{u}) \leqslant DI_\star\overline{p}(\mathtt{v}) + b.$$

Applying the local calculations 1.5.b,c and 1.9.b,c the claim (27) becomes

$$(R1)\ H_*^{\overline{p}}(E,\mathcal{E}) \cong H_*^{I_\star\overline{p}}(E,\mathcal{E}), \qquad (R4)\ \mathfrak{H}_*^{\overline{p}}(E,\mathcal{E}) \cong \mathfrak{H}_*^{I_\star\overline{p}}(E,\mathcal{E}) \qquad (R9)\ \mathscr{H}^*_{\overline{p}}(E,\mathcal{E}) \cong \mathscr{H}^*_{I_\star\overline{p}}(E,\mathcal{E}),$$

The stratum S belongs to $\mathcal{V} = \mathcal{M}$ (cf. (15)). Since any other $R \in \mathcal{S}$ meeting the conical chart W verifies $S \prec R$ then R is a source stratum and then $\overline{p}(R) = I_\star\overline{p}(R)$ (cf. Lemma 4.3). From (21) we get $\overline{p} = I_\star\overline{p}$ on E. The claim is proved. ♣

Remark 4.7. The existence of 1-exceptional strata may impeach the above isomorphisms. This is the case for (R4), ... (R10). For example $\mathfrak{H}_*^{\overline{0}}\left(\mathring{c}S^0,\mathring{c}\mathcal{I}\right) = 0 \neq G = \mathfrak{H}_*^{\overline{0}}(]-1,1[,\mathcal{I})$. But we have