The basic intersection cohomology of a singular riemannian foliation.

Finiteness and Poincaré duality

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We prove that the basic intersection cohomology $I\!\!H^{*}_{\overline{p}}(M/\mathcal{F})$ of the singular riemannian foliation $\mathcal{F}$ determined by an abelian isometric action

$$\Phi\colon G \times (M, \mu) \to (M, \mu)$$

is finite dimensional and verifies the Poincaré duality. This duality includes two well-known fareway situations:

- Poincaré duality for basic cohomology (the action Φ is almost free).
- Poincaré duality for intersection cohomology (the abelian Lie group G is compact).

Poincaré duality

For an orientable compact m-dimensional manifold M the cohomology $H^*(M)$ is finite dimensional and the pairing product

$$\int : H^*(M) \times H^{m-*}(M) \longrightarrow \mathbb{R},$$

defined by

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta,$$

is non degenerate.

Riemannian foliations

– The leaves of a foliation $\mathcal{F}$ have the same dimension ℓ.

– These leaves are locally at constant distance.

– The transverse structure $M/\mathcal{F}$ "is a manifold".

– An almost free isometric action $\Phi\colon G\times M \longrightarrow M$ produces a riemannian foliation.

– Working example.

$$\Phi\colon \mathbb{R} \times \mathbb{S}^5 \longrightarrow \mathbb{S}^5$$

with

$$\Phi(s,(z_1,z_2,z_3)) = (e^{ias}z_1, e^{ibs}z_2, e^{ics}z_3),$$

for $a,b,c \in \mathbb{R} - \{0\}$.

The basic cohomology $H^*(M/\mathcal{F})$

– $\mathcal{X}(\mathcal{F})$ vector fields of M tangent to $\mathcal{F}$.

– Basic differential forms $\omega \in \Omega^*(M)$ with

$$i_X\omega = i_X d\omega = 0 \text{ for each } X \in \mathcal{X}(\mathcal{F}).$$

– Complex of basic differential forms $\Omega^*(M/\mathcal{F})$.

– Basic cohomology $H^*(M/\mathcal{F})$. Finite dimensional.

– Poincaré duality. The pairing product

$$\int : H^*(M/\mathcal{F}) \times H^{n-*}(M/\mathcal{F}) \longrightarrow \mathbb{R},$$

defined by

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta \wedge \nu$$

is non degenerate, where $\nu \in \Omega^{n=\dim \mathcal{F}}(M)$ is a tangent volume form (orientability condition): a volume form on each leaf.

Example

$\Phi\colon \mathbb{R}\times\mathbb{S}^5 \longrightarrow \mathbb{S}^5$ with

$$\Phi(s,(z_1,z_2,z_3)) = (e^{ias}z_1, e^{ibs}z_2, e^{ics}z_3),$$

for $a,b,c\in\mathbb{R}-\{0\}$.

– Actors

– Fundamental vectorfield $X\in\mathcal{X}(\mathbb{S}^5)$.

– Connection form $\chi\in\Omega^1(M)$

– Curvature form $e = d\chi\in\Omega^2\left(\mathbb{S}^5/\mathcal{F}\right)$.

– Euler class $[e]\in H^2(M/\mathcal{F})$.

– $H^i\left(\mathbb{S}^5/\mathcal{F}\right) = \begin{cases} \mathbb{R}\cdot[1] & i=0\\ \mathbb{R}\cdot[e] & i=2\\ \mathbb{R}\cdot[e\wedge e] & i=4\end{cases}$

– Poincaré duality ($\nu=\chi$):

$$[1] \leftrightarrow [e\wedge e]$$
$$[e] \leftrightarrow [e]$$

Singular Riemannian foliations. SRF

– The leaves of a foliation $\mathcal{F}$ may have different dimensions.

– These leaves are locally at constant distance.

– Putting together the leaves with the same dimension one gets a stratification $\mathcal{S}$. For each stratum S, the restriction $(S, \mathcal{F}_{\mathcal{S}})$ is a riemannian foliated manifold.

– The transverse structure $M/\mathcal{F}$ "is a stratified pseudomanifold" in the sense of Goresky-MacPherson.

– An isometric action $\Phi \colon G \times M \longrightarrow M$ produces a singular riemannian foliation.

– Working example.

$$\Psi \colon \mathbb{R} \times \mathbb{S}^6 \longrightarrow \mathbb{S}^6$$

with

$$\Psi(s, [z_1, z_2, z_3, t]) = [e^{ias} z_1, e^{ibs} z_2, e^{ics} z_3, t],$$

for $a, b, c \in \mathbb{R} - \{0\}$.

The basic cohomology $H^*(M/\mathcal{F})$

– $\mathcal{X}(\mathcal{F})$ vector fields of M tangent to $\mathcal{F}$.

– Basic differential forms $\omega \in \Omega^*(M)$ with

$$i_X\omega = i_X d\omega = 0 \text{ for each } X \in \mathcal{X}(\mathcal{F}).$$

– Complex of basic differential forms $\Omega^*(M/\mathcal{F})$.

– Basic cohomology $H^*(M/\mathcal{F})$. Finite dimensional.

– Poincaré duality. The pairing product

$$\int : H^*(M/\mathcal{F}) \times H^{n-*}(M/\mathcal{F}) \longrightarrow \mathbb{R},$$

defined by

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta \wedge \nu$$

…it does not work!!

Example

$\Psi\colon \mathbb{R}\times\mathbb{S}^6 \longrightarrow \mathbb{S}^6$ with

$\Psi(s,[z_1,z_2,z_3,t]) = [e^{ias}z_1, e^{ibs}z_2, e^{ics}z_3, t]$,

for $a,b,c \in \mathbb{R}-\{0\}$.

– $H^i(\mathbb{S}^6/\mathcal{F}) = \begin{cases} \mathbb{R}\cdot[1] & i=0 \\ \mathbb{R}\cdot[e\wedge dt] & i=3 \\ \mathbb{R}\cdot[e\wedge e\wedge dt] & i=5 \end{cases}$

– The Poincaré duality does not work since

$$\dim H^2\left(\mathbb{S}^6/\mathcal{F}\right) \neq \dim H^3\left(\mathbb{S}^6/\mathcal{F}\right)$$

Why??

In order to understand the situation, consider the blow up

$$\mathcal{L} \colon \mathbb{S}^5 \times [-1,1] \longrightarrow \mathbb{S}^6$$
$$(z_1, z_2, z_3 t) \mapsto [z_1, z_2, z_3 t]$$

– We compute the basic cohomology $H^*(\mathbb{S}^6/\mathcal{F})$ by using the Verona forms of the blow up. A Verona form is a form $\omega \in \Omega^*(\mathbb{S}^5 \times [-1,1]/\mathcal{F})$ verifying

$$\deg\left(\omega_{|boundary}\right) = \deg\left(d\omega_{|boundary}\right) = 0.$$

– Examples

$$\begin{array}{cc} f(z_1, z_2, z_3) & f(z_1, z_2, z_3)\sin 2\pi t \\ e & e \wedge dt \\ te & (1-t^2)e \end{array}$$

– Failure of Poincaré duality:

$$\dim H^3\left(\mathbb{S}^6/\mathcal{F}\right) = 1 \neq 0 = \dim H^2\left(\mathbb{S}^6/\mathcal{F}\right)$$

since

$$e \wedge dt = d(te).$$

Recovering Poincaré duality

(Goresky-MacPherson)

– The perverse degree of a differential form ω of $\Omega^*(\mathbb{S}^5 \times [-1,1]/\mathcal{F})$ is $||\omega|| = \deg\left(\omega_{|boundary}\right)$

– A perversity is an integer $\overline{p} \in \mathbb{Z}$.

– ω is a $\overline{p}$-intersection differential form when

$$||\omega|| \leq \overline{p} \text{ and } ||d\omega|| \leq \overline{p}.$$

– Basic intersection cohomology $I\!\!H^*_{\overline{p}}(\mathbb{S}^6/\mathcal{F})$

– Verona's case: $\overline{p} = 0$.

– Poincaré duality.

	$\overline{p} = 0,1$	$\overline{p} = 2,3$
$i = 0$	$\mathbb{R}[\ 1\ ]$	$\mathbb{R}[\ 1\ ]$
$i = 1$	0	0
$i = 2$	0	$\mathbb{R}[\ e\ ]$
$i = 3$	$\mathbb{R}[\ e \wedge dt\ ]$	0
$i = 4$	0	0
$i = 5$	$\mathbb{R}[\ e \wedge e \wedge dt\ ]$	$\mathbb{R}[\ e \wedge e \wedge dt\ ]$

$$e \wedge dt = d(te).$$

Basic intersection cohomology
BIC

– Blow up.

$$\mathcal{L} \colon (\widetilde{M}, \widetilde{\mathcal{F}}) \longrightarrow (M, \mathcal{F}).$$

– Perverse degree. For each stratum S

$$\| - \|_S \colon \Omega^* \left(\widetilde{M} / \widetilde{\mathcal{F}} \right) \longrightarrow \mathbb{Z}$$

– A perversity is a map $\overline{p} \colon \mathcal{S} \to \mathbb{Z}$. The zero perversity is $\overline{0}(S) = 0$. The top perversity is $\overline{t}(S) = \operatorname{codim}_M S - 3$.

– ω is a $\overline{p}$-intersection differential is a form when

$$\|\omega\|_S \leq \overline{p}(S) \text{ and } \|d\omega\|_S \leq \overline{p}(S),$$

for each stratum S.

– Basic $\overline{p}$-ntersection cohomology $I\!\!H^*_{\overline{p}}(M/\mathcal{F})$

– $I\!\!H^*_{\overline{0}}(M/\mathcal{F}) = H^*(M/\mathcal{F})$.

– $I\!\!H^*_{\overline{p}}(M/\mathcal{F})$ is the intersection cohomology of Goresky-MacPherson if $\overline{0} \leq \overline{p} \leq \overline{t}$ and $\mathcal{F}$ is compact.

Poincaré duality

– The BIC $I\!H^{*}_{\overline{p}}(M/\mathcal{F})$ is finite dimensional and the pairing

$$\int : I\!H^{*}_{\overline{p}}(M/\mathcal{F}) \times I\!H^{n-*}_{\overline{q}}(M/\mathcal{F}) \longrightarrow \mathbb{R},$$

defined by

$$(\alpha, \beta) \mapsto \int_{\widetilde{M}} \alpha \wedge \beta \wedge \nu$$

is non degenerate, where $\overline{p} + \overline{q} = \overline{t}$.

– This result has been proven for the case where $\mathcal{F}$ comes from an isometric action

$$\Phi \colon G \times M \to M,$$

where G is an abelian Lie group.

Idea of the proof

We use the Mayer-Vietoris technics in order to reduce the question to the following twisted product:

$$\mathbb{T} \times_H \mathbb{R}^n,$$

where

- $\mathbb{T}$ is a compact Lie group containing G as a dense subgroup. Notice that G acts on $\mathbb{T}/H$ producing a riemannian foliation $\mathcal{F}_1$.
- H is a closed subgroup of $\mathbb{T}$ acting by isometries on $\mathbb{R}^n$. We write $\mathcal{F}_2$ the induced SRF.

The cohomology $I\!H^*_{\overline{p}}\big(\mathbb{T} \times_H \mathbb{R}^n/\mathcal{F}\big)$ can be computed with the complex of $\mathbb{T}$-invariant forms

$$\Big(\Omega^*_{\overline{p}}\big(\mathbb{T} \times_H \mathbb{R}^n/\mathcal{F}\big)\Big)^{\mathbb{T}},$$

which is quasi-isomorphic to (not easy!!)

$$\bigwedge^*(\gamma_{c+1}, \ldots, \gamma_f) \otimes \Big(\Omega^*_{\overline{p}}(\mathbb{R}^n/\mathcal{F}_1)\Big)^H$$

where

- $\{\gamma_1, \ldots \gamma_a, \gamma_{a+1}, \ldots, \gamma_b\}$ is a basis of $\mathfrak{g}$,

– $\{\gamma_{a+1}, \dots \gamma_b, \gamma_{b+1}, \dots, \gamma_c\}$ is a basis of $\mathfrak{h}$, and
– $\{\gamma_1, \dots, \gamma_a, \dots, \gamma_b, \dots, \gamma_c, \gamma_{c+1}, \dots, \gamma_f\}$ is a basis of $\mathfrak{t}$.

Finally, $I\!\!H^{*}_{\overline{p}}(\mathbb{T} \times_H \mathbb{R}^n/\mathcal{F})$ is

$$H^{*}(\mathbb{T}/H/\mathcal{F}_1) \otimes I\!\!H^{*}_{\overline{p}}(\mathbb{R}^n/\mathcal{F}_2).$$

Both terms are finite dimensional and verify the Poincaré duality.