$\textbf{The}\quad \overbrace{\textbf{Basic}}^{\text{Reinhardt 59'}}\quad \overbrace{\textbf{Intersection Cohomology}}^{\text{Goresky-MacPherson, 80'}}$

of a

$\underbrace{\textbf{Singular}}_{\text{Molino, 86'}}\quad \underbrace{\textbf{Riemannian Foliation}}_{\text{Reinhardt 59'}}$

Link word : Stratification

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$\boxed{\textbf{Singular riemannian foliation } (M, \mathcal{F}, g)}$

- **The leaves are connected immersed submanifolds (with variable dimension)**
- **The module $\Theta_{\mathcal{F}}$ of smooth vectorfields tangent to the leaves are transitive on each leaf (*singular foliation of Sussman-Stefan).***
- **Every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets (*transnormal system of Bolton*).**

Exemple

$\Phi\colon G \times M \to M$ **isometric Lie group actions.**

$$L_x = G(x) \qquad \dim L_x = \dim G - \dim G_x.$$

$\Psi\colon \mathbb{R} \times \mathbb{S}^6 \to \mathbb{S}^6$ **with** $\Phi(t, [z_1, z_2, z_3, r]) = [e^{ti} \cdot z_1, e^{ti} \cdot z_2, e^{ti} \cdot z_3, r].$

$\boxed{\text{Stratification}}$

Partition $\mathsf{S}_{\mathcal{F}}$ of M given by

$$s \sim y \Longleftrightarrow \dim L_x = \dim L_y.$$

- **Each stratum $S \in \mathsf{S}_{\mathcal{F}}$ is a proper submanifold.**
- **The pair $(\mathsf{S}_{\mathcal{F}}, \preceq)$ is a poset where $S_1 \preceq S_2 \Leftrightarrow S_1 \subset \overline{S_2}$.**
- **The length of this poset is the depth of the foliation. It is 0 when $(M, \mathcal{F})$ is a regular foliation.**
- **There is a maximum stratum R, the regular stratum : open dense submanifold.**
- **The minimal strata are the closed strata.**
- **Each $(S, \mathcal{F}_S)$ is a regular riemannian foliation.**

Exemple

$\Phi\colon G \times M \to M$ **isometric Lie group actions.**

$$x \sim y \quad \Longleftrightarrow \quad \dim G_x = \dim G_y.$$

$S_1, S_2 \preceq R.$

Local Structure

- ***Riemannian foliation.***

$$(\mathbb{R}^{m+p}, \mathcal{H}) \textbf{ with } \mathcal{H} = \{dx_1 = \cdots = dx_p = 0\}.$$

- ***Singular riemannian foliation.***

$$(\mathbb{R}^{m+p} \times c\mathbb{S}^n, \mathcal{H} \times c\mathcal{K})$$

where $(\mathbb{S}^n, \mathcal{K})$ **is a SRF without 0-dimensional leaves (the link of** S**). The strata are**

  - $(\mathbb{R}^{m+p} \times$ **vertex**$)$
  - $(\mathbb{R}^{m+p} \times S \times]0,1[)$**, with** $S \in \mathsf{S}_{\mathcal{H}}$**.**

Exemple

$$\Psi \colon \mathbb{R} \times \mathbb{S}^6 \to \mathbb{S}^6 \textbf{ with } \Phi(t, [z_1, z_2, z_3, r]) = [e^{ti} \cdot z_1, e^{ti} \cdot z_2, e^{ti} \cdot z_3, r].$$

$\boxed{\text{Blow up}}$

- ***Elementary blow up.*** **We replace each point of a minimal stratum by its link, that is, we replace**

$$\mathbb{R}^{m+p} \times c\mathbb{S}^n \qquad \textbf{by} \qquad \mathbb{R}^{m+p} \times \mathbb{S}^n \times [0,1[.$$

We obtain the elementary blow up

$$\mathcal{E} \colon (\widehat{M}, \widehat{\mathcal{F}}) \longrightarrow (M, \mathcal{F})$$

where $(\widehat{M}, \widehat{\mathcal{F}})$ **is a singular riemannian foliation with**

$$\text{depth } \mathsf{S}_{\widehat{\mathcal{F}}} < \text{depth } \mathsf{S}_{\mathcal{F}}.$$

- ***Bow up.*** **Iterating the processus we obtain the Molino's blow up**

$$\mathcal{L} \colon (\widetilde{M}, \widetilde{\mathcal{F}}) \to (M, \mathcal{F}),$$

where $(\widetilde{M}, \widetilde{\mathcal{F}})$ **is a riemannian foliation.**

<u>Exemple of blow up</u>

$$\mathcal{L} \colon \mathbb{S}^5 \times [-1,1] \longrightarrow \mathbb{S}^6 \quad ((z_1, z_2, z_3), r) \mapsto [(z_1, z_2, z_3), r].$$

$\boxed{\text{Basic intersection cohomology}}$

The orbit espace $M/\mathcal{F}$ "is" a stratified pseudomanifold of Goresky-MacPherson.

The basic intersection cohomology

$$I\!H^{*}_{\overline{p}}(M/\mathcal{F})$$

studies this structure. It is defined from the complex of $\overline{p}$-intersection differential forms. The parameter

$$\overline{p} \colon \mathsf{S}_{\mathcal{F}} \to \mathbb{Z}$$

is an allowability label. It indicates the singular degree we allow to a form $\omega \in \Omega^{*}_{\overline{p}}(M/\mathcal{F})$.

$\boxed{\textbf{First properties of } I\!H^{*}_{\overline{p}}(M/\mathcal{F}).}$

- $I\!H^{*}_{\overline{p}}(M/\mathcal{F})$ **is the usual intersection cohomology when the leaf espace is a true stratified pseudomanifolds (compact leaves).**
- $I\!H^{*}_{\overline{p}}(M/\mathcal{F})$ **is the usual basic cohomology** $H^{*}(M/\mathcal{F})$ **when the singular riemannian foliation is a regular one.**
- $I\!H^{*}_{\overline{0}}(M/\mathcal{F}) = H^{*}(M/\mathcal{F})$
- $I\!H^{*}_{\overline{p}}(M/\mathcal{F}) = H^{*}((M, M - R)/\mathcal{F})$ **if** $\overline{p} < \overline{0}$**.**
- $I\!H^{*}_{\overline{p}}(M/\mathcal{F}) = H^{*}(R/\mathcal{F})$ **if** $\overline{p} > \overline{t}$**.**
  The top perversity $\overline{t}$ **is** $\overline{t}(S) = \dim M - \dim S - 2$**.**

Main expected properties of $I\!\!H^{*}_{\overline{p}}(M/\mathcal{F})$.

- **The basic intersection cohomology of a compact singular riemannian foliations is finite dimensional.**
- ***Poincaré duality.*** **The wedge product**

$$\wedge\colon I\!\!H^{i}_{\overline{p}}(M/\mathcal{F}) \times I\!\!H^{n-i}_{\overline{q}}(M/\mathcal{F}) \longrightarrow I\!\!H^{n}_{\overline{t}}(M/\mathcal{F})$$

($\kappa = 0$**) is a non degenerate pairing.**

Exemple

$I\!\!H^{i}_{\overline{p}}(\mathbb{S}^6/\mathcal{F})$	$\overline{p} = -1$	$\overline{p} = 0$	$\overline{p} = 3$	$\overline{p} = 4$
$i = 0$		1	1	1
$i = 1$				
$i = 2$			e	e
$i = 3$	$e \wedge dt$	$e \wedge dt$		
$i = 4$				
$i = 5$	$e \wedge e \wedge dt$	$e \wedge e \wedge dt$	$e \wedge e \wedge dt$	

$$||dt||_S = -1 \quad ||e||_S = 2 \quad ||t \cdot e||_S = 2 \quad ||e \wedge dt||_S = -1$$

$$||e \wedge e||_S = 4 \quad ||t \cdot e \wedge e||_S = 4 \quad ||e \wedge e \wedge dt||_S = -1.$$

$\mathbb{S}^6/\mathcal{F} \equiv \Sigma\mathbb{CP}^2 \qquad \overline{t} = 3$

Description of $\omega \in \Omega^*_{\overline{p}}(M/\mathcal{F})$.

a) The form $\omega \in \Omega^*(R)$ is a basic form.

b) The form $\omega \in \Omega^*(R/\mathcal{F})$ lifts to a form $\widetilde{\omega} \in \Omega^*\left(\widetilde{M}/\widetilde{\mathcal{F}}\right)$.

$$
\begin{tikzcd}
\mathcal{L}^*\omega \arrow[dr] & & & \widetilde{\omega} \arrow[dr] & & & \widetilde{\omega}_{|\mathcal{L}^{-1}(S)} \arrow[dl] \\
& \mathcal{L}^{-1}(R) \arrow[rr, hook] \arrow[d, "\mathcal{L}"'] & & \widetilde{M} \arrow[d, "\mathcal{L}"'] & & \mathcal{L}^{-1}(S) \arrow[ll, hook'] \arrow[d, "\mathcal{L}"] & \\
& R \arrow[rr, hook] & & M & & S \arrow[ll, hook'] & \\
\omega \arrow[ur] & & & & & &
\end{tikzcd}
$$

The perverse degree $||\omega||_S$ is the vertical degree of the form $\widetilde{\omega}$ relatively to the fibration

$$\mathcal{L}\colon \mathcal{L}^{-1}(S) \longrightarrow S.$$

c) $||\omega||_S \leq \overline{p}(S)$ and $||d\omega||_S \leq \overline{p}(S)$