Dos cohomologías de intersección
y
una conjetura de Goresky y Pardon

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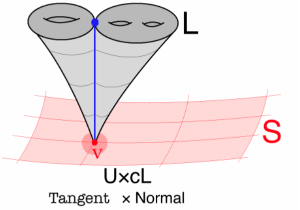
INTRODUCCION

Pseudovariedad estratificada: $X = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset \cdots \supset X_1 \supset X_0$

Estratos : $X_k \backslash X_{k-1}$ (variedades)

Ligaduras 

Ligadura $=$ Toro $\vee$ (Toro $\#$Toro)

. . . .

Perversidad : $\overline{p} = (0, 0, 0, p_3, p_4, \ldots, p_k, \ldots)$ con $p_{k+1} = p_k$ or $p_k + 1$.

Perversidad tope $\overline{t}$: $(0, 0, 0, 1, 2, 3, \ldots, k-2, \ldots)$

Perversidad dual $D\overline{p} = \overline{t} - \overline{p}$.

Introduction

Program para una variedad M	Programa para X
$C_*(M;\mathbb{Z})$	$C_*^{\overline{p}}(X;\mathbb{Z})$
$\rightsquigarrow$	$\rightsquigarrow$
$C^*(M;\mathbb{Z}) = \mathrm{Hom}(C_*(M;\mathbb{Z});\mathbb{Z})$	$C^*_{\overline{p}}(X;\mathbb{Z}) = \mathrm{Hom}(C_*^{\overline{p}}(X;\mathbb{Z});\mathbb{Z})$
$\rightsquigarrow$	$\rightsquigarrow$
producto cap	producto cap
$\rightsquigarrow$	$\rightsquigarrow$
$\cap[\gamma_X] \colon H^*(M;\mathbb{Z}) \xrightarrow{\cong} H_{n-*}(M;\mathbb{Z})$	$\cap[\gamma_X] \colon H^*_{\overline{p}}(X;\mathbb{Z}) \xrightarrow{\cong} H^{D\overline{p}}_{n-*}(X;\mathbb{Z})$

Si X localmente libre de $\overline{p}$-torsion :

$$\mathrm{Tors}\, H^{\overline{p}}_{\overline{p}(\dim L+1)}(Link;\mathbb{Z}) = 0$$

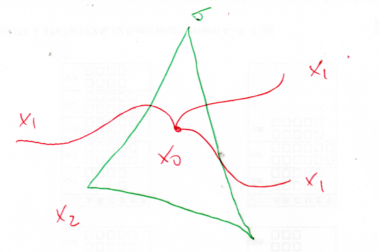
$$((X = \Sigma\mathbb{RP}^3))$$

$\boxed{\text{Objetivo de la charla}}$

- Construir la cohomología de intersección explotada $\mathscr{H}^{*}_{\overline{p}}(X)$. Propiedades.
- Comparar ambas cohomologías : $H^{*}_{\overline{p}}(X)$ y $\mathscr{H}^{*}_{\overline{p}}(X)$
- Dualidad de Poincaré
- Conjetura de Goresky-Pardon.

HOMOLOGIA DE INTERSECCION

Homología de intersección

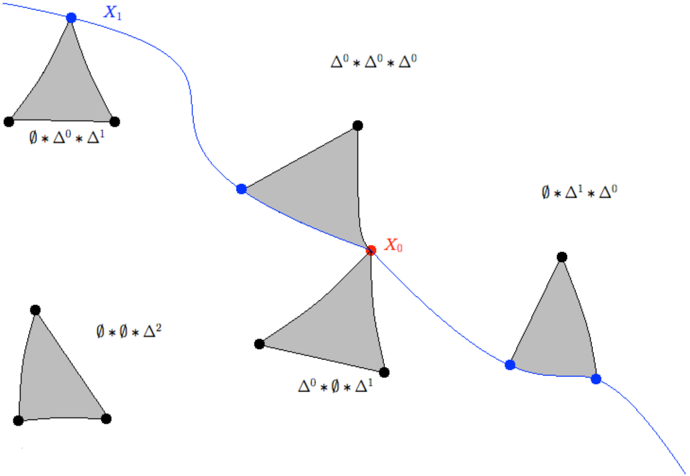


- $\sigma : \Delta \to X$.
- Grado perverso : $||\sigma||_k = \dim \sigma^{-1}(X_{n-k})$.
- Condición de permisibilidad: $||\sigma||_k \leqslant \dim \Delta - k + \overline{p}(k)$.
- Cadenas de intersección ξ : ξ y $\partial\xi$ son cadenas permisibles.
- Homología de intersección $H_*^{\overline{p}}(X)$.
- $H_*^{\overline{t}}(X) = H_*(X)$ si X es normal.

Símplices filtrados : $\sigma \colon \Delta \to X$ con $\sigma^{-1}(X_k)$ a face of Δ

$$\Delta = \underbrace{\Delta_0 * \cdots * \Delta_k}_{\sigma^{-1}(X_k)} * \cdots * \Delta_n$$

Homología de intersección



Homología de intersección

Símplices filtrados : $\sigma\colon \Delta \to X$ con $\sigma^{-1}(X_k)$ cara de Δ

$$\Delta = \underbrace{\Delta_0 * \cdots * \Delta_k}_{\sigma^{-1}(X_k)} * \cdots * \Delta_n$$

$$||\sigma||_k = \dim(\Delta_0 * \cdots * \Delta_{n-k}).$$

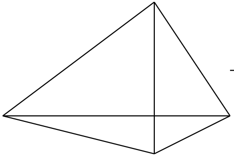
$C_*^{\overline{p}}(X) = \{$cadenas de $\overline{p}$-intersección filtradas$\}$

calcula

$H_*^{\overline{p}}(X)$ Homología de intersección

Cocadenas

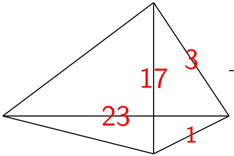
1-Cocadenas

$\Delta =$  $\xrightarrow{\sigma} X$

Cocadenas

1-Cocadenas

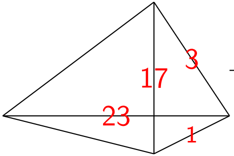
$\Delta =$



$\xrightarrow{\sigma} X$

Cocadenas

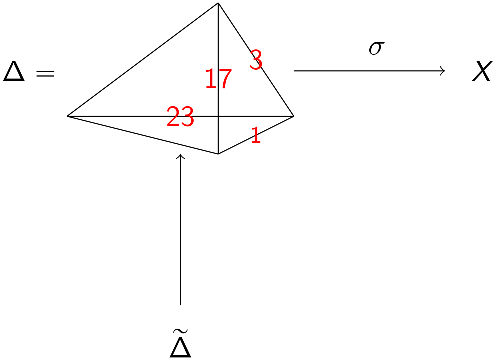
1-Cocadenas

$\Delta =$ 

$\xrightarrow{\sigma} X$

más compatibilidad

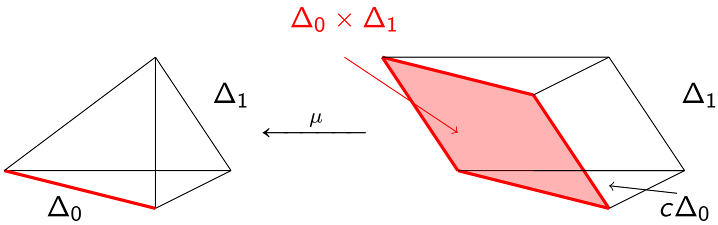
1-Cocadenas



más compatibilidad

COHOMOLOGIA EXPLOTADA

Explosión de un símplice filtrado



$\Delta = \Delta_0 * \Delta_1$ tiene por explosión $\widetilde{\Delta} = c\Delta_0 \times \Delta_1$

$$\partial\widetilde{\Delta} = \widetilde{\partial\Delta} + \text{ Caras ocultas}$$

Explosión de un símplice filtrado

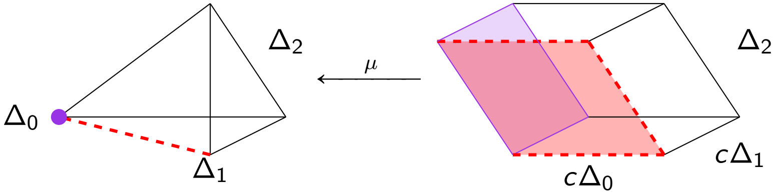
El prisma

$$\widetilde{\Delta} = c\Delta_0 \times \cdots \times c\Delta_{n-1} \times \Delta_n$$

es la explosión de

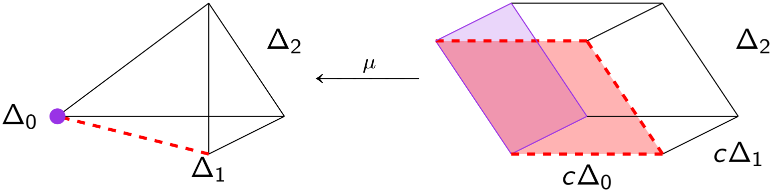
$$\Delta = \Delta_0 * \cdots * \Delta_n$$

$\partial\widetilde{\Delta} = \widetilde{\partial\Delta} +$ caras ocultas



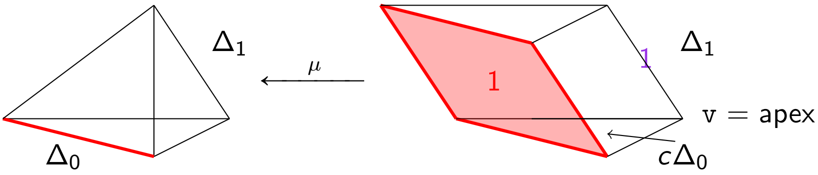
Cocadenas locales

$$\widetilde{N}^*(\Delta) = N^*(c\Delta_0) \otimes \cdots \otimes N^*(c\Delta_{n-1}) \otimes N^*(\Delta_n)$$



$$\Delta = \Delta_0 * \Delta_1 * \Delta_2 \qquad \widetilde{\Delta} = c\Delta_0 \times c\Delta_1 \times \Delta_2$$

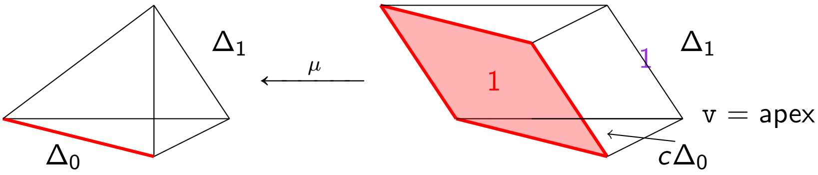
Grado perverso de la cocadena $\widetilde{N}^*(\Delta)$



$\Delta = \Delta_0 * \Delta_1$

$\widetilde{\Delta} = c\Delta_0 \times \Delta_1$

Grado perverso de la cocadena $\widetilde{N}^*(\Delta)$

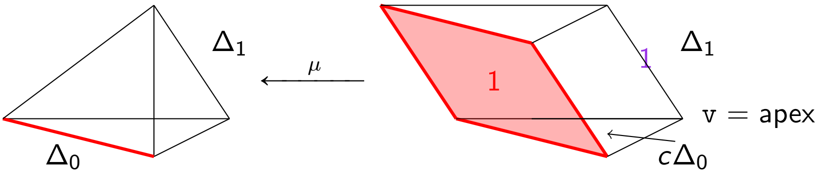


$\Delta = \Delta_0 * \Delta_1$ $\qquad\qquad$ $\widetilde{\Delta} = c\Delta_0 \times \Delta_1$

- $||\mathbf{1}_{\mathrm{v}\times\Delta_1}|| = -\infty$ $\qquad$ Ya que $\mathrm{v} \times \Delta_1$ no está en una cara oculta
- $||\mathbf{1}_{\Delta_0\times\Delta_1}|| = \dim \Delta_1$ $\qquad$ Ya que $\Delta_0 \times \Delta_1$ está en una cara oculta

Cohomología de intersección explotada

Grado perverso de la cocadena $\widetilde{N}^*(\Delta)$



$\Delta = \Delta_0 * \Delta_1$ $\qquad\qquad$ $\widetilde{\Delta} = c\Delta_0 \times \Delta_1$

1

$$\widetilde{N}^*(\Delta) = N^*(c\Delta_0) \otimes N^*(\Delta_1).$$

$$||\alpha \otimes \omega|| = \begin{cases} |\omega| & \text{Para cocadenas ocultas: } \alpha \otimes \omega \in N^*(\Delta_0) \otimes N^*(\Delta_1) \\ -\infty & \text{Para cocadenas no ocultas} \end{cases}$$

$||\mathbf{1}_{\Delta_0} \otimes \mathbf{1}_{\Delta_1}|| = |\mathbf{1}_{\Delta_1}| = \dim \Delta_1$ $\qquad$ $||\mathbf{1}_{\mathrm{v}} \otimes \mathbf{1}_{\Delta_1}|| = -\infty$

Cocadenas perversas

- $\widetilde{N}^*(\Delta) = N^*(c\Delta_0) \otimes \cdots \otimes N^*(c\Delta_{n-1}) \otimes N^*(\Delta_n)$
- $\widetilde{N}^*_{\overline{p}}(\Delta) = \left\{ \omega \in \widetilde{N}^*(\Delta) / \max(||\omega||_k, ||d\omega||_k) \leqslant p_k \right\}$
- $\widetilde{N}^*_{\overline{p}}(X)$ es el haz simplicial

  $\left\{ (\omega_\sigma) \ / \ \sigma \colon \Delta \to X \text{ símplice filtrado}, \omega_\sigma \in \widetilde{N}^*_{\overline{p}}(\Delta), \text{ compatibles} \right\}.$
- $\mathscr{H}^*_{\overline{p}}(X)$ cohomología de intersección explotada. Coincide con la clásica en un cuerpo.
- $\mathscr{H}^*_{\overline{0}}(X) = H^*(X)$ cuando X es normal.

Producto cup

$$\widetilde{N}^{i}_{\overline{p}}(X) \otimes \widetilde{N}^{j}_{\overline{q}}(X) \xrightarrow{\smile} \widetilde{N}^{i+j}_{\overline{p}+\overline{q}}(X).$$

$$(\omega \smile \eta)_\sigma = \omega_\sigma \smile \eta_\sigma$$

$\smile$ está definido localmente:

$$\widetilde{N}^*(\Delta) = N^*(c\Delta) \otimes \cdots \otimes N^*(c\Delta_{n-1}) \otimes N^*(\Delta_n),$$

donde $\sigma \colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$ es un símplice filtrado.

$$\underbrace{(\alpha_1 \otimes \cdots \otimes \alpha_n)}_{\omega_\sigma} \smile \underbrace{(\beta_1 \otimes \cdots \otimes \beta_n)}_{\eta_\sigma} = \underbrace{\pm(\alpha_1 \smile \beta_1) \otimes \cdots \otimes (\alpha_n \smile \beta_n)}_{\omega_\sigma \smile \eta_\sigma}$$

Primeras propiedades

- Producto cup: $\mathscr{H}^{i}_{\overline{p}}(X) \otimes \mathscr{H}^{j}_{\overline{q}}(X) \xrightarrow{\smile} \mathscr{H}^{i+j}_{\overline{p}+\overline{q}}(X)$.

- Producto cap: $\mathscr{H}^{i}_{\overline{p}}(X) \otimes H^{\overline{q}}_{j}(X) \xrightarrow{\frown} H^{\overline{p}+\overline{q}}_{j-i}(X)$.

- Mayer-Vietoris. Sea $\{U, V\}$ un recubrimiento abierto de X:

$$\cdots \to \mathscr{H}^{i-1}_{\overline{p}}(U \cap V) \to \mathscr{H}^{i}_{\overline{p}}(X) \to \mathscr{H}^{i}_{\overline{p}}(U) \oplus \mathscr{H}^{i}_{\overline{p}}(V) \to \mathscr{H}^{i}_{\overline{p}}(U \cap V) \to \cdots$$

- $\mathscr{H}^{*}_{\overline{p}}\left(\mathbb{R}^k \times X\right) = \mathscr{H}^{*}_{\overline{p}}(X)$ y $\mathscr{H}^{i}_{\overline{p}}(\mathring{c}L) = \begin{cases} \mathscr{H}^{i}_{\overline{p}}(L) & \text{si } i \leqslant \overline{p}_{\dim L+1} \\ 0 & \text{si no} \end{cases}$

Independencia de la filtración

- Una pseudovariedad estratificada pude soportar diferentes filtraciones. La cohomología de intersección explotada no depende de la elección.
- $\mathbb{R}^k = \mathring{c}\mathbb{S}^{k-1}$.
- $\mathbb{R}^k \times \mathring{c}L = \mathring{c}(\mathbb{S}^{k-1} * L)$ (genérico)
- X^{Sull} estratificación de Sullivan: $x \sim y \iff \exists (U, x) \cong (V, y)$. No depende de la filtración de X.
- $\mathscr{H}^*_{\overline{p}}(X) \cong \mathscr{H}^*_{\overline{p}}(X^{Sull})$.

Dualidad de Poincaré

$$\mathscr{H}_{\overline{p}}^{*}(X) \xrightarrow{\frown [\gamma_X]} H^{\overline{p}}_{n-*}(X)$$

Dualidad de Poincaré

$$\mathscr{H}^{*}_{\overline{p},c}(X) \xrightarrow{\frown [\gamma_X]} H^{\overline{p}}_{n-*}(X)$$

Dualidad de Poincaré

$$\mathscr{H}_{\overline{p}}^{*}(X) \xrightarrow{\frown [\gamma_X]} H_{n-*}^{BM,\overline{p}}(X)$$

Dualidad de Poincaré

Dualidad de Lefschetz: $\mathscr{H}^{*}_{\overline{p}}(X, \partial_1 X) \xrightarrow{\frown [\gamma x]} H^{\overline{p}}_{n-*}(X, \partial_2 X)$

Dualidad de Poincaré ordinaria versus Homología de intersección

$$
\begin{tikzcd}
H^k(X) \arrow[r, "\frown[\gamma x]"] \arrow[d] & H_{n-k}(X) \\
\mathscr{H}^k_{\overline{p}}(X) \arrow[r, "\frown[\gamma x]", "\cong"'] & H^{\overline{p}}_{n-k}(X) \arrow[u]
\end{tikzcd}
$$

Cohomología de intersección explotada

Dualidad de Poincaré ordinaria versus Homología de intersección

$$
\begin{tikzcd}
H^{k}(X) \arrow[r, "\frown[\gamma X]"] \arrow[dd, "\cong"'] & H_{n-k}(X) & \\
& & H^{\bar{t}}_{n-k}(X) \arrow[ul, "\cong"'] \\
\mathscr{H}^{k}_{\bar{0}}(X) \arrow[r, "\frown[\gamma X]", "\cong"'] & H^{\bar{0}}_{n-k}(X) \arrow[ur] &
\end{tikzcd}
$$

Dualidad de Poincaré ordinaria versus Homología de intersección

$$\begin{tikzcd}
H^k(X) \arrow[r, "\frown[\gamma x]"] \arrow[d, "\cong"'] & H_{n-k}(X) & \\
& & H^{\overline{t}}_{n-k}(X) \arrow[ul, "\cong"'] \\
\mathscr{H}^k_{\overline{0}}(X) \arrow[r, "\frown[\gamma x]", "\cong"'] & H^{\overline{0}}_{n-k}(X) \arrow[ur] &
\end{tikzcd}$$

Dualidad de Poincaré versus Homología de intersección

Si X es una pseudovariedad compact y normal de dimensión n entonces

$$X \text{ verifica la dualidad de Poincaré} \iff H^*_{\overline{0}}(X) \xrightarrow[\cong]{} H^*_{\overline{t}}(X).$$

$H^*_{D\overline{p}}(X)$ y $\mathscr{H}^*_{\overline{p}}(X)$ son diferentes

$H^k_{D\overline{p}}(cM;\mathbb{Z})$	k	$\mathscr{H}^k_{\overline{p}}(cM;\mathbb{Z})$
$H^k(M;\mathbb{Z})$	$\leqslant \overline{p}(m+1)$	$H^k(M;\mathbb{Z})$
$\mathrm{Tors}\, H^k(M;\mathbb{Z})$	$\overline{p}(m+1)+1$	0
0	$\geqslant \overline{p}(m+1)+2$	0

con M variedad, $\mathtt{v}$ el apex del cono cM y $m = \dim M$.

$$H^*_{D\overline{p}}(cM;\mathbb{Z}) \cong \mathscr{H}^*_{\overline{p}}(cM;\mathbb{Z}) \text{ if } cM \text{ localmente libre de } \overline{p}\text{-torsion:}$$

$$\mathrm{Tors}\, H^{\overline{p}(\dim M+1)}(M;\mathbb{Z}) = 0$$

Relación entre $H^*_{D\overline{p}}(X;\mathbb{Z})$ y $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$

- $H^*_{D\overline{p}}(X;\mathbb{Z}) \cong \mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ si X localmente libre de $\overline{p}$-torsion (o coeficientes en un cuerpo)
- Forma bilineales no degenaradas

$$F\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z}) \otimes FH^{n-*}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Z}$$

$$\mathrm{Tors}\,\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{Tors}\,H^{n+1-*}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

CUADRADOS DE STEENROD

Conjetura de Goresky & Pardon

$$
\begin{tikzcd}
 & H^{r+i}_{\mathcal{L}(\overline{p},i)}(X;\mathbb{Z}_2) \arrow[d] \\
H^{r}_{\overline{p}}(X;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^i"] \arrow[ur, dashed] & H^{r+i}_{2\overline{p}}(X;\mathbb{Z}_2)
\end{tikzcd}
$$

$$\mathcal{L}(\overline{p}, i)(\ell) = \min(2\overline{p}(\ell), \overline{p}(\ell) + i)$$

Conjetura de Goresky & Pardon

$$\begin{tikzcd} & \mathscr{H}^{r+i}_{\mathcal{L}(\overline{p},i)}(X;\mathbb{Z}_2) \arrow[d] \\ \mathscr{H}^{r}_{\overline{p}}(X;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^i"] \arrow[ur, dashed] & \mathscr{H}^{r+i}_{2\overline{p}}(X;\mathbb{Z}_2) \end{tikzcd}$$

$$\mathcal{L}(\overline{p},i)(\ell) = \min(2\overline{p}(\ell), \overline{p}(\ell)+i)$$

Avance efectivo :

$0 \neq \mathrm{Sq}^2 \in \mathscr{H}^*_{\overline{p}(\ell)+2}(X)$ y $0 = \mathrm{Sq}^2 \in \mathscr{H}^*_{2\overline{p}(\ell)}(X)$.

M variedad y $p = \overline{p}(\dim M + 1)$

$$\begin{tikzcd}
\mathscr{H}^{*}_{\overline{p}}(cM;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^{\,i}_{cM}"] \arrow[d, "\cong"] & \mathscr{H}^{*}_{2\overline{p}}(cM;\mathbb{Z}_2) \arrow[d, "\cong"] \\
H^{\leqslant p}(M;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^{\,i}_{M}"] & H^{\leqslant 2p}(M;\mathbb{Z}_2)
\end{tikzcd}$$

M variedad y $p = \overline{p}(\dim M + 1)$

$$\begin{tikzcd}
\mathscr{H}^{*}_{\overline{p}}(cM;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^{i}_{cM}"] \arrow[d, "\cong"] & \mathscr{H}^{*}_{2\overline{p}}(cM;\mathbb{Z}_2) \arrow[d, "\cong"] \\
H^{\leqslant p}(M;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^{i}_{M}"] \arrow[dr, "\mathrm{Sq}^{i}_{M}"'] & H^{\leqslant 2p}(M;\mathbb{Z}_2) \\
& H^{\leqslant p+i}(M;\mathbb{Z}_2)
\end{tikzcd}$$

M variedad y $p = \overline{p}(\dim M + 1)$

$$\begin{tikzcd}
\mathscr{H}^{*}_{\overline{p}}(cM;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^{i}_{cM}"] \arrow[d, "\cong"] & \mathscr{H}^{*}_{2\overline{p}}(cM;\mathbb{Z}_2) \arrow[d, "\cong"] & \\
H^{\leqslant p}(M;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^{i}_{M}"] \arrow[ddr, "\mathrm{Sq}^{i}_{M}"'] & H^{\leqslant 2p}(M;\mathbb{Z}_2) \arrow[dr, squiggly] & \\
 & & H^{\leqslant \min(2p,p+i)}(M;\mathbb{Z}_2) = \mathscr{H}^{*}_{\mathcal{L}(\overline{p},i)}(cM;\mathbb{Z}_2) \\
 & H^{\leqslant p+i}(M;\mathbb{Z}_2) \arrow[ur, squiggly] &
\end{tikzcd}$$

$$\boxed{||c||_k \leqslant p_k \Rightarrow ||\mathrm{Sq}^i(c)||_k \leqslant \min(2p_k, p_k + i)}$$

$c \in \widetilde{N}^*_{\overline{p}}(X; \mathbb{Z}_2)$

$$\boxed{||c||_k \leqslant p_k \Rightarrow ||\mathrm{Sq}^i(c)||_k \leqslant \min(2p_k, p_k + i)}$$

$c \in \widetilde{N}^*_{\bar{p}}(X; \mathbb{Z}_2)$

$$\boxed{||\mathrm{Sq}^i(c)||_k \leqslant \min(2||c||_k, ||c||_k + i)}$$

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$c \in \widetilde{N}^{*}_{\overline{p}}(X;\mathbb{Z}_2)$

$$\boxed{||\mathrm{Sq}^i(c)||_k \leqslant \min(2||c||_k, ||c||_k + i)}$$

$$c \in \widetilde{N}^*_{\overline{p}}(X; \mathbb{Z}_2)$$

$$\boxed{||\mathrm{Sq}^i(\alpha \otimes \omega)||_k \leqslant \min(2|\omega|, |\omega| + i)}$$

$$c = \alpha \otimes \omega \in \widetilde{N}^*(\Delta)$$

$$||c||_k = ||\alpha \otimes \omega||_k = |\omega|$$

$$\boxed{||\mathrm{Sq}^i(\alpha \otimes \omega)||_k \leqslant \min(2|\omega|, |\omega| + i)}$$

$$\alpha \otimes \omega \in \widetilde{N}^*(\Delta)$$

$$\boxed{||\mathrm{Sq}^i(\alpha \otimes \omega)||_k \leqslant \min(2|\omega|, |\omega| + i)}$$

$$\mathrm{Sq}^i(\alpha \otimes \omega) = \sum_{j=0}^{|\alpha\otimes\omega|-i} (\alpha \cup_{|\alpha\otimes\omega|-i-j} \alpha) \otimes (\omega \cup_j \omega)$$

$$||\mathrm{Sq}^i(\alpha \otimes \omega)||_k \leqslant |\omega \cup_j \omega| \text{ (máx)}$$

Propiedad del producto $\smile_\ell$: $|A \smile_\ell B| = |A| + |B| - \ell$.

$$||\mathrm{Sq}^i(\alpha \otimes \omega)||_k \leqslant 2|\omega| - j.$$

$$\boxed{2|\omega| - j \leqslant \min(2|\omega|, |\omega| + i)}$$

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Propiedad del producto $\smile_\ell$: $A \smile_\ell B \neq 0 \Longrightarrow \ell \leqslant \min(|A|, |B|)$.

$$\mathrm{Sq}^i(\alpha \otimes \omega) = \sum_{j=0}^{|\alpha\otimes\omega|-i} \underbrace{(\alpha \cup_{|\alpha\otimes\omega|-i-j} \alpha)}_{\neq 0} \otimes (\omega \cup_j \omega)$$

$$\alpha \cup_{|\alpha\otimes\omega|-i-j} \alpha \neq 0 \Longrightarrow \underbrace{|\alpha \otimes \omega|}_{|\alpha|+|\omega|} - i - j \leqslant |\alpha| \Longrightarrow |\omega| - j \leqslant i \Longrightarrow 2|\omega| - j \leqslant |\omega| + i$$

ESKERRIK ASKO