# Intersection cohomologies

*New York, August 2018*

- Poincaré Duality : Two cohomologies
- Minimal models.
- Steenrod Squares

*David Chataur* (Amiens, France)
*Martintxo Saralegi Aranguren* (Orator) (Artois, France)
*Daniel Tanré* (Lille, France)

# INTRODUCTION

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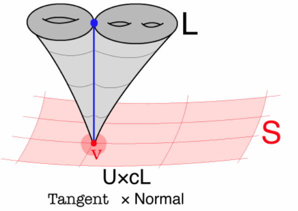

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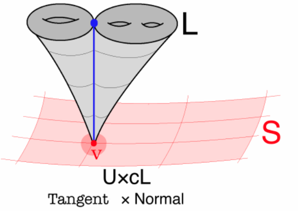
Links 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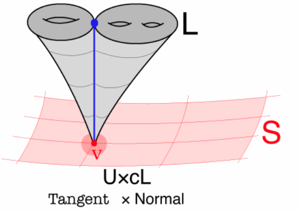
Links 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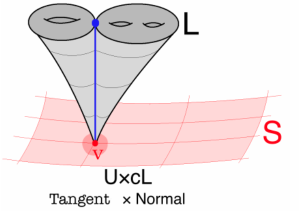


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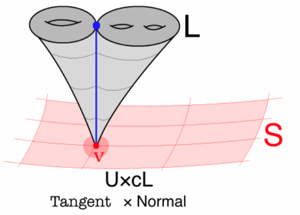
Links 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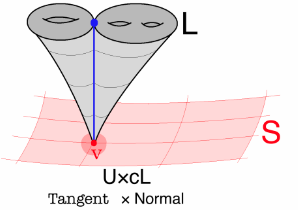


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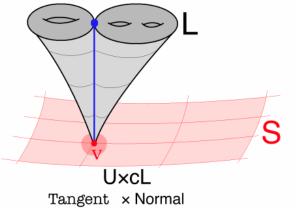


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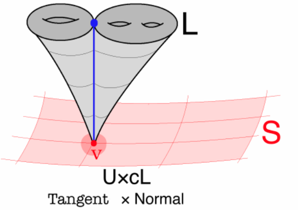


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Intersection (co)homology : complexes of sheaves or **(co)chain complexes**

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Program for a Manifold $M$

$$C_*(M;\mathbb{Z})$$

$$\rightsquigarrow$$

$$C^*(M;\mathbb{Z}) = \mathrm{Hom}(C_*(M;\mathbb{Z});\mathbb{Z})$$

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| $H^*(M;\mathbb{Z}) \cong H_{n-*}(M;\mathbb{Z})$ | $H^*_{\overline{p}}(X;\mathbb{Z}) \cong H^{D\overline{p}}_{n-*}(X;\mathbb{Z})$ |

If $X$ locally $\overline{p}$-torsion-free :

$$\mathrm{Tors}\, H^{\overline{p}}_{\overline{p}(\dim L+1)}(Link;\mathbb{Z}) = 0$$

$$((X = \Sigma\mathbb{R}\mathbb{P}^3))$$

**Goal of the talk**

- Construct a cohomology developing the previous program: The blown-up cohomology $\mathscr{H}_{\overline{p}}^{*}(X;\mathbb{Z})$.

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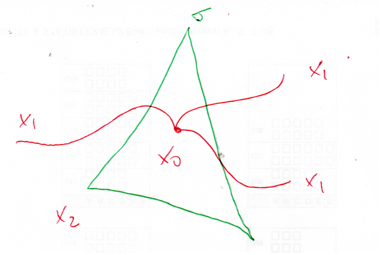
- Compare both cohomologies: $H^*_{\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$
- Extend the Sullivan minimal model theory to this context.
- Goresky-Pardon's conjecture.

# INTERSECTION HOMOLOGY

# Intersection homology

- $\sigma \colon \Delta \to X$.



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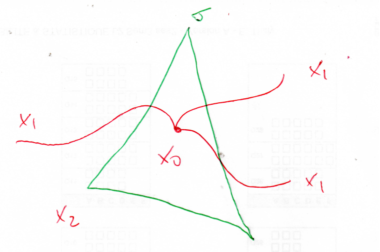

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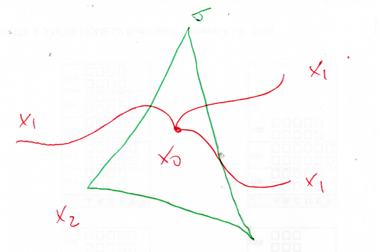

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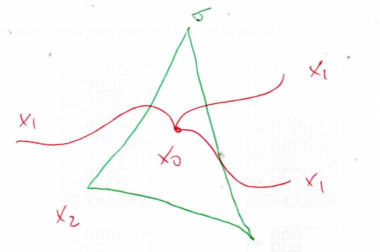

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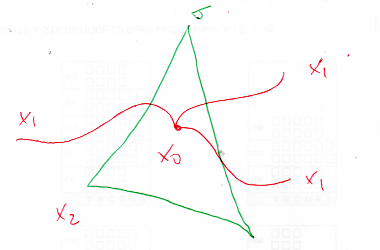

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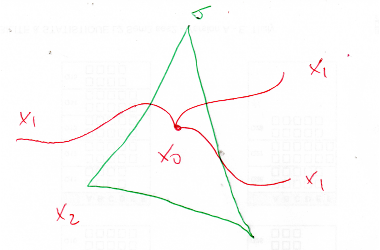


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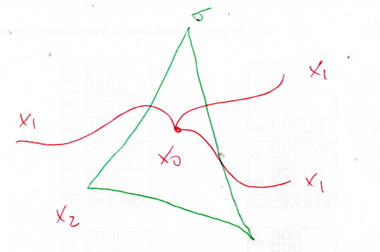


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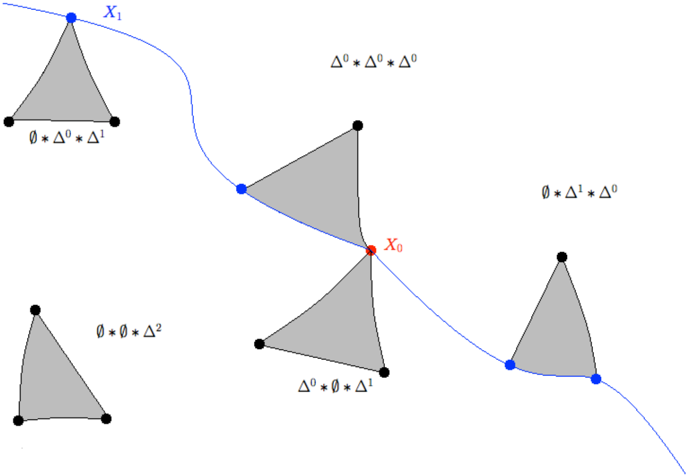
- Intersection homology $H_*^{\overline{p}}(X;\mathbb{Z})$.
- $H_*^{\overline{t}}(X;\mathbb{Z}) = H_*(X;\mathbb{Z})$ if $X$ is normal.
- $0 \to \mathrm{Ext}\left(H_{*-1}^{\overline{p}}(X;\mathbb{Z}),\mathbb{Z}\right) \to H^*_{\overline{p}}(X;\mathbb{Z}) \to \mathrm{Hom}\left(H_*^{\overline{p}}(X;\mathbb{Z}),\mathbb{Z}\right) \to 0$.

Filtered simplices : $\sigma\colon \Delta \to X$ with $\sigma^{-1}(X_k)$ a face of $\Delta$

$$\Delta = \underbrace{\Delta_0 * \cdots * \Delta_k}_{\sigma^{-1}(X_k)} * \cdots * \Delta_n$$



$X_1$
$\Delta^0 * \Delta^0 * \Delta^0$
$\emptyset * \Delta^0 * \Delta^1$
$\emptyset * \Delta^1 * \Delta^0$
$X_0$
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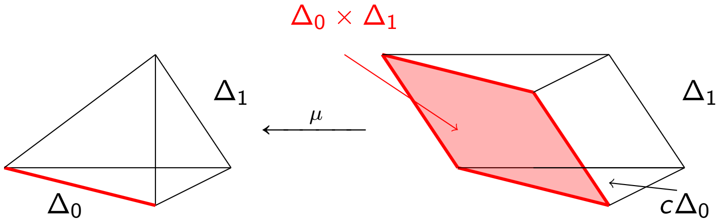
$C_*^{\overline{p}}(X;\mathbb{Z}) = \{\text{filtered } \overline{p}\text{-intersection chains}\}$ computes

$H_*^{\overline{p}}(X;\mathbb{Z})$ intersection homology and $H^*_{\overline{p}}(X;\mathbb{Z})$ intersection cohomology

# BLOWN-UP COHOMOLOGY

# Blown-up cohomology

## Blow up of a filtered simplex



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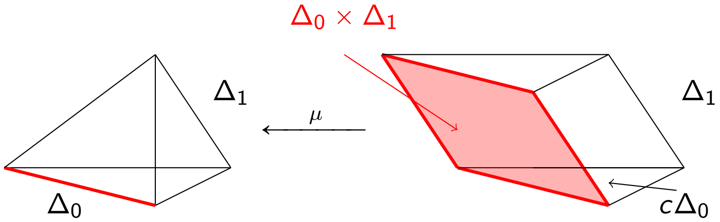



$$\Delta = \Delta_0 * \Delta_1 \text{ has for blow-up } \widetilde{\Delta} = c\Delta_0 \times \Delta_1$$

$$\partial\widetilde{\Delta} = \widetilde{\partial\Delta} + \text{ Hidden faces}$$

## Blow up of a filtered simplex

The prism

$$c\Delta_0 \times \cdots \times c\Delta_{n-1} \times \Delta_n$$

is the blow up of

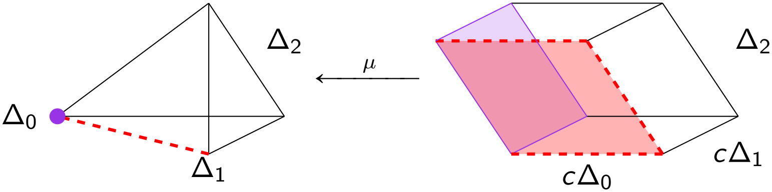
$$\Delta_0 * \cdots * \Delta_n$$

$\partial\widetilde{\Delta} = \widetilde{\partial\Delta} +$ Hidden faces

# Blown-up cohomology

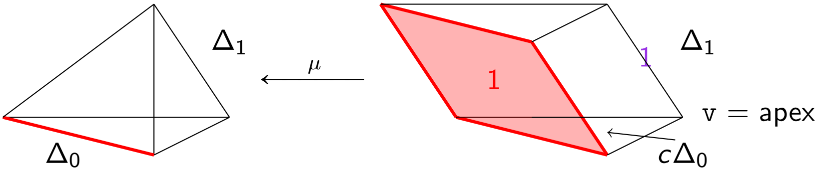
Local cochains

$$\widetilde{N}^*(\Delta) = N^*(c\Delta_0) \otimes \cdots \otimes N^*(c\Delta_{n-1}) \otimes N^*(\Delta_n)$$




$$\Delta = \Delta_0 * \Delta_1 * \Delta_2 \qquad \widetilde{\Delta} = c\Delta_0 \times c\Delta_1 \times \Delta_2$$

Perverse degree of a cochain of $\widetilde{N}^*(\Delta)$




$\Delta = \Delta_0 * \Delta_1$ $\qquad$ $\widetilde{\Delta} = c\Delta_0 \times \Delta_1$

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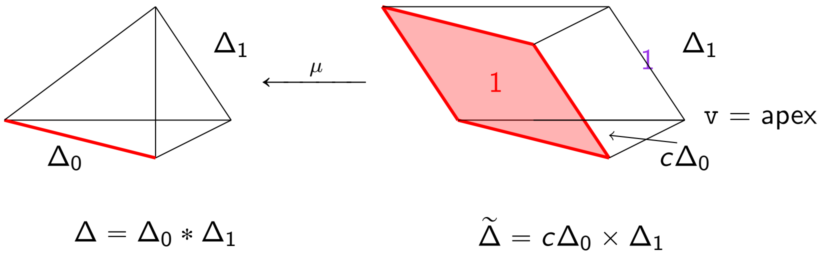


- $||\mathbf{1}_{\mathrm{v}\times\Delta_1}|| = -\infty$ — Since $\mathrm{v} \times \Delta_1$ not hidden face
- $||\mathbf{1}_{\Delta_0\times\Delta_1}|| = \dim \Delta_1$ — Since $\Delta_0 \times \Delta_1$ hidden face

Perverse cochains

- $\widetilde{N}^*(\Delta) = N^*(c\Delta_0) \otimes \cdots \otimes N^*(c\Delta_{n-1}) \otimes N^*(\Delta_n)$
- $\widetilde{N}^*_{\overline{p}}(\Delta) = \left\{\omega \in \widetilde{N}^*(\Delta) / \max(||\omega||_k, ||d\omega||_k) \leqslant p_k\right\}$

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- $\widetilde{N}^*_{\overline{p}}(X)$ is the simplicial sheaf

  $\left\{(\omega_\sigma) \ / \ \sigma \colon \Delta \to X \text{ filtered simplex}, \omega_\sigma \in \widetilde{N}^*_{\overline{p}}(\Delta), \ \text{compatible}\right\}.$

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- $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ blown-up cohomology
- $\mathscr{H}^*_{\overline{0}}(X;\mathbb{Z}) = H^*(X;\mathbb{Z})$, cohomology, when $X$ is normal.

## Cup product

$$\widetilde{N}^{i}_{\overline{p}}(X) \otimes \widetilde{N}^{j}_{\overline{q}}(X) \xrightarrow{\ \cup\ } \widetilde{N}^{i+j}_{\overline{p}+\overline{q}}(X).$$

$$(\omega \cup \eta)_\sigma = \omega_\sigma \cup \eta_\sigma$$

This cup product is defined locally in

$$\widetilde{N}^*(\Delta) = N^*(c\Delta) \otimes \cdots \otimes N^*(c\Delta_{n-1}) \otimes N^*(\Delta_n),$$

where $\sigma\colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$ is a filtered simplex.

$$\underbrace{(\alpha_1 \otimes \cdots \otimes \alpha_n)}_{\omega_\sigma} \cup \underbrace{(\beta_1 \otimes \cdots \otimes \beta_n)}_{\eta_\sigma} = \underbrace{\pm(\alpha_1 \cup \beta_1) \otimes \cdots \otimes (\alpha_n \cup \beta_n)}_{\omega_\sigma \cup\ \eta_\sigma}$$

## Properties

- Cup product: $\mathscr{H}^{i}_{\overline{p}}(X;\mathbb{Z}) \otimes \mathscr{H}^{j}_{\overline{q}}(X;\mathbb{Z}) \xrightarrow{\quad\cup\quad} \mathscr{H}^{i+j}_{\overline{p}+\overline{q}}(X;\mathbb{Z})$ .
- Cap product: $\mathscr{H}^{i}_{\overline{p}}(X;\mathbb{Z}) \otimes H^{\overline{q}}_{j}(X;\mathbb{Z}) \xrightarrow{\quad\cap\quad} H^{\overline{p}+\overline{q}}_{j-i}(X;\mathbb{Z})$.
- $\mathscr{H}^{*}_{\overline{p}}(X;\mathbb{Z})$ independent of the stratification.
- Poincaré Duality: $\mathscr{H}^{*}_{\overline{p}}(X;\mathbb{Z}) \xrightarrow{\cap\,[\gamma x]} H^{\overline{p}}_{n-*}(X;\mathbb{Z})$
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## Properties

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- $\mathscr{H}^{*}_{\overline{p}}(X;\mathbb{Z})$ independent of the stratification.
- Poincaré Duality: $\mathscr{H}^{*}_{\overline{p},c}(X;\mathbb{Z}) \xrightarrow{\cap\ [\gamma_X]} H^{\overline{p}}_{n-*}(X;\mathbb{Z})$
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- Cup product: $\mathscr{H}^{i}_{\overline{p}}(X;\mathbb{Z}) \otimes \mathscr{H}^{j}_{\overline{q}}(X;\mathbb{Z}) \xrightarrow{\quad\cup\quad} \mathscr{H}^{i+j}_{\overline{p}+\overline{q}}(X;\mathbb{Z})$ .
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- $\mathscr{H}^{*}_{\overline{p}}(X;\mathbb{Z})$ independent of the stratification.
- Poincaré Duality: $\mathscr{H}^{*}_{\overline{p}}(X;\mathbb{Z}) \xrightarrow{\cap\ [\gamma x]} H^{BM,\overline{p}}_{n-*}(X;\mathbb{Z})$
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# Blown-up cohomology

Properties

- Cup product: $\mathscr{H}^{i}_{\overline{p}}(X;\mathbb{Z}) \otimes \mathscr{H}^{j}_{\overline{q}}(X;\mathbb{Z}) \xrightarrow{\quad\cup\quad} \mathscr{H}^{i+j}_{\overline{p}+\overline{q}}(X;\mathbb{Z})$ .

- Cap product: $\mathscr{H}^{i}_{\overline{p}}(X;\mathbb{Z}) \otimes H^{\overline{q}}_{j}(X;\mathbb{Z}) \xrightarrow{\quad\cap\quad} H^{\overline{p}+\overline{q}}_{j-i}(X;\mathbb{Z})$.

- $\mathscr{H}^{*}_{\overline{p}}(X;\mathbb{Z})$ independent of the stratification.

- Lefschetz Duality: $\mathscr{H}^{*}_{\overline{p}}(X,\partial_1 X;\mathbb{Z}) \xrightarrow{\cap\ [\gamma x]} H^{\overline{p}}_{n-*}(X,\partial_2 X;\mathbb{Z})$

- There is no Universal Coefficient Theorem

Ordinary Poincaré Duality versus intersection homology

$$\begin{array}{ccc} H^k(X;\mathbb{Z}) & \xrightarrow{\cap[\gamma x]} & H_{n-k}(X;\mathbb{Z}) \\ \downarrow & & \uparrow \\ \mathscr{H}^k_{\overline{p}}(X;\mathbb{Z}) & \xrightarrow[\cong]{\cap[\gamma x]} & H^{\overline{p}}_{n-k}(X;\mathbb{Z}) \end{array}$$

# Blown-up cohomology

## Ordinary Poincaré Duality versus intersection homology

$$
\begin{array}{ccccc}
H^k(X;\mathbb{Z}) & \xrightarrow{\cap[\gamma x]} & H_{n-k}(X;\mathbb{Z}) & \xleftarrow{\cong} & \\
\downarrow \cong & & & & H^{\overline{t}}_{n-k}(X;\mathbb{Z}) \\
\color{red}{\mathscr{H}^k_{\overline{0}}(X;\mathbb{Z})} & \underset{\cong}{\xrightarrow{\cap[\gamma x]}} & H^{\overline{0}}_{n-k}(X;\mathbb{Z}) & \longrightarrow & 
\end{array}
$$

# Blown-up cohomology

## Ordinary Poincaré Duality versus intersection homology

$$
\begin{array}{ccccc}
H^k(X;\mathbb{Z}) & \xrightarrow{\cap[\gamma x]} & H_{n-k}(X;\mathbb{Z}) & \xleftarrow{\cong} & \\
\downarrow{\cong} & & & & H^{\overline{t}}_{n-k}(X;\mathbb{Z}) \\
\mathscr{H}^k_{\overline{0}}(X;\mathbb{Z}) & \xrightarrow[\cong]{\cap[\gamma x]} & H^{\overline{0}}_{n-k}(X;\mathbb{Z}) & \longrightarrow & 
\end{array}
$$

**Poincaré Duality versus intersection homology**

Let $X$ be a normal compact oriented $n$-dimensional pseudomanifold. Then

$$H^*(X;\mathbb{Z}) \xrightarrow[\cong]{\cap[\gamma x]} H_{n-*}(X;\mathbb{Z}) \Longleftrightarrow H^*_{\overline{0}}(X;R) \xrightarrow[\cong]{} H^*_{\overline{t}}(X;\mathbb{Z}).$$

# Blown-up cohomology

Ordinary Poincaré Duality versus intersection homology

$$
\begin{array}{ccccc}
H^k(X;\mathbb{Z}) & \xrightarrow{\cap[\gamma x]} & H_{n-k}(X;\mathbb{Z}) & \xleftarrow{\cong} & \\
\downarrow \cong & & & & H^{\overline{t}}_{n-k}(X;\mathbb{Z}) \\
\mathscr{H}^k_{\overline{0}}(X;\mathbb{Z}) & \xrightarrow[\cong]{\cap[\gamma x]} & H^{\overline{0}}_{n-k}(X;\mathbb{Z}) & \longrightarrow & 
\end{array}
$$

$X = \Sigma\mathbb{T}^2$ .................................... does not verify Ordinary Poincaré Duality

$X = \Sigma\mathbb{T}^2 \times [0,1]/A$ with $A \in SL_2(\mathbb{Z})$ ............... verifies Ordinary Poincaré Duality
since $A\colon H_2(\Sigma\mathbb{T}^2;\mathbb{Z}) = H_1(\mathbb{T}^2;\mathbb{Z}) \circlearrowleft$ has no fixed points.

Ordinary Poincaré Duality versus intersection homology

$$
\begin{array}{ccccc}
H^{k}(X;\mathbb{Z}) & \xrightarrow{\ \cap[\gamma x]\ } & H_{n-k}(X;\mathbb{Z}) & & \\
\Big\downarrow \cong & & & \nwarrow \cong & \\
& & & & H^{\bar t}_{n-k}(X;\mathbb{Z}) \\
& & & \nearrow & \\
{\color{red}\mathscr{H}^{k}_{\bar 0}(X;\mathbb{Z})} & \xrightarrow[\cong]{\ \cap[\gamma x]\ } & H^{\bar 0}_{n-k}(X;\mathbb{Z}) & &
\end{array}
$$

$$\cdots \longrightarrow H^{\bar 0}_{k}(X;\mathbb{Z}) \longrightarrow H^{\bar t}_{k}(X;\mathbb{Z}) \longrightarrow H^{\bar t/\bar 0}_{k}(X;\mathbb{Z}) \longrightarrow H^{\bar 0}_{k}(X;\mathbb{Z}) \longrightarrow \cdots$$

$FH^{\bar t/\bar 0}_{*}(X;\mathbb{Z}) \otimes FH^{\bar t/\bar 0}_{n+1-*}(X;\mathbb{Z}) \to \mathbb{Z}$ non-singular.

$\mathrm{Tors}\, H^{\bar t/\bar 0}_{*}(X;\mathbb{Z}) \otimes \mathrm{Tors}\, H^{\bar t/\bar 0}_{n-*}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ non-singular.

# $H^*_{D\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ are different

| $H^k_{D\overline{p}}(cM;\mathbb{Z})$ | $k$ | $\mathscr{H}^k_{\overline{p}}(cM;\mathbb{Z})$ |
|---|---|---|
| $H^k(M;\mathbb{Z})$ | $\leqslant \overline{p}(m+1)$ | $H^k(M;\mathbb{Z})$ |
| $\mathrm{Ext}\,(H_{k-1}(M;\mathbb{Z});\mathbb{Z})$ | $\overline{p}(m+1)+1$ | $0$ |
| $0$ | $\geqslant \overline{p}(m+1)+2$ | $0$ |

with $M$ manifold, v the apex of $cM$ and $m = \dim M$.

$H^*_{D\overline{p}}(cM;\mathbb{Z}) \cong \mathscr{H}^*_{\overline{p}}(cM;\mathbb{Z})$ if $cM$ locally $\overline{p}$-free torsion: $H_{\overline{p}(\dim M+1)}(M;\mathbb{Z}) = 0$

## $H^*_{D\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ are different

- The intersection cohomology verifies the Universal Coefficient Theorem:

$$0 \to \mathrm{Ext}\,(H^{\overline{p}}_{k-1}(X;\mathbb{Z}),\mathbb{Z}) \to H^k_{\overline{p}}(X;\mathbb{Z}) \to \mathrm{Hom}\,(H^{\overline{p}}_k(X;\mathbb{Z}),\mathbb{Z}) \to 0$$

- The blown-up cohomology verifies Poincaré Duality without LTF-condition.

$$\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z}) \xrightarrow{\cap\ [\gamma x]} H^{\overline{p}}_{n-*}(X;\mathbb{Z})$$

The natural map

$$\chi\colon \widetilde{N}^*_{\overline{p}}(X;\mathbb{Z}) \to C^*_{D\overline{p}}(X;\mathbb{Z}) = \mathrm{Hom}\left(C^{D\overline{p}}_*(X;\mathbb{Z}),\mathbb{Z}\right) := \omega \mapsto (\sigma \mapsto \varepsilon(\omega \cap \sigma))$$

verifies the following properties.

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$$\chi\colon \widetilde{N}^*_{\overline{p}}(X;\mathbb{Z}) \to C^*_{D\overline{p}}(X;\mathbb{Z}) = \mathrm{Hom}\,(C^{D\overline{p}}_*(X;\mathbb{Z}),\mathbb{Z}) := \omega \mapsto (\sigma \mapsto \varepsilon(\omega \cap \sigma))$$

verifies the following properties.

- $\chi$ is a quasi-isomorphism over $\mathbb{Q}$.

# $H^*_{D\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ are not so different

The natural map

$$\chi\colon \widetilde{N}^*_{\overline{p}}(X;\mathbb{Z}) \to C^*_{D\overline{p}}(X;\mathbb{Z}) = \mathrm{Hom}\,(C^{D\overline{p}}_*(X;\mathbb{Z}),\mathbb{Z}) := \omega \mapsto (\sigma \mapsto \varepsilon(\omega \cap \sigma))$$

verifies the following properties.

- $\chi$ is a quasi-isomorphism over $\mathbb{Q}$.
- $\chi^*$ fits into the following exact sequence

$$\dots \to \mathscr{H}^k_{\overline{p}}(X;\mathbb{Z}) \xrightarrow{\chi^*} H^k_{D\overline{p}}(X;\mathbb{Z}) \to \mathscr{R}^k_{\overline{p}}(X;\mathbb{Z}) \to \mathscr{H}^{k+1}_{\overline{p}}(X;\mathbb{Z}) \to \dots$$

# $H^*_{D\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ are not so different

The natural map

$$\chi \colon \widetilde{N}^*_{\overline{p}}(X;\mathbb{Z}) \to C^*_{D\overline{p}}(X;\mathbb{Z}) = \operatorname{Hom}\left(C^{D\overline{p}}_*(X;\mathbb{Z}),\mathbb{Z}\right) := \omega \mapsto (\sigma \mapsto \varepsilon(\omega \cap \sigma))$$

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where $\mathscr{R}^*_{\overline{p}}(X;\mathbb{Z})$ is the cohomology of the Goresky-Siegel's Peripheral term

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$$\chi\colon \widetilde{N}^*_{\overline{p}}(X;\mathbb{Z}) \to C^*_{D\overline{p}}(X;\mathbb{Z}) = \mathrm{Hom}\left(C^{D\overline{p}}_*(X;\mathbb{Z}),\mathbb{Z}\right) := \omega \mapsto (\sigma \mapsto \varepsilon(\omega \cap \sigma))$$

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where $\mathscr{R}^*_{\overline{p}}(X;\mathbb{Z})$ is the cohomology of the Goresky-Siegel's Peripheral term

- $\mathscr{R}^*_{\overline{p}}(X;\mathbb{Z})$ is a torsion term.

The natural map

$$\chi\colon \widetilde{N}^*_{\overline{p}}(X;\mathbb{Z}) \to C^*_{D\overline{p}}(X;\mathbb{Z}) = \mathrm{Hom}\left(C^{D\overline{p}}_*(X;\mathbb{Z}),\mathbb{Z}\right) := \omega \mapsto (\sigma \mapsto \varepsilon(\omega \cap \sigma))$$

verifies the following properties.

- $\chi$ is a quasi-isomorphism over $\mathbb{Q}$.
- $\chi^*$ fits into the following exact sequence

$$\dots \to \mathscr{H}^k_{\overline{p}}(X;\mathbb{Z}) \xrightarrow{\chi^*} H^k_{D\overline{p}}(X;\mathbb{Z}) \to \mathscr{R}^k_{\overline{p}}(X;\mathbb{Z}) \to \mathscr{H}^{k+1}_{\overline{p}}(X;\mathbb{Z}) \to \dots$$

where $\mathscr{R}^*_{\overline{p}}(X;\mathbb{Z})$ is the cohomology of the Goresky-Siegel's Peripheral term

- $\mathscr{R}^*_{\overline{p}}(X;\mathbb{Z})$ is a torsion term.
- $\mathscr{R}^*_{\overline{p}}(X;\mathbb{Z}) \otimes \mathscr{R}^{n-*}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ non-singular pairing

# $H^*_{D\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ are not so different

When are they equal?

## $H^*_{D\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ are not so different

When are they equal?

- When $X$ is locally $\overline{p}$-torsion free (LTF) ............. $\mathrm{Tors}\, H^{\overline{p}}_{\overline{p}(\dim Link+1)}(Link;\mathbb{Z}) = 0$

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- When the Peripheral term is acyclic (PTA) ........................ $\mathscr{R}^*_{\overline{p}}(X,\mathbb{Z}) = 0$

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We have LTF $\Longrightarrow$ PTA but PTA $\not\Longrightarrow$ LTF.

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- When the Peripheral term is acyclic (PTA) ........................ $\mathscr{R}^*_{\overline{p}}(X,\mathbb{Z}) = 0$

**We have LTF $\Longrightarrow$ PTA** but PTA $\not\Longrightarrow$ LTF.

---

$$\text{LTF} \iff \mathscr{R}^*_{\overline{p}}(cL) = 0, \forall L \iff \mathscr{R}^*_{\overline{p}}(U) = 0, \forall \text{ open chart } U \Longrightarrow \mathscr{R}^*_{\overline{p}}(X) = 0 \iff \text{PTA}$$

# $H^*_{D\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ are not so different

When are they equal?

- When $X$ is locally $\overline{p}$-torsion free (LTF) . . . . . . . . . . . . . $\operatorname{Tors} H^{\overline{p}}_{\overline{p}(\dim Link+1)}(Link;\mathbb{Z}) = 0$
- When the Peripheral term is acyclic (PTA) . . . . . . . . . . . . . . . . . . . . . . . . . $\mathscr{R}^*_{\overline{p}}(X,\mathbb{Z}) = 0$

We have LTF $\Longrightarrow$ PTA but **PTA** $\not\Longrightarrow$ **LTF.**

---

$$X = \Sigma(\underbrace{\mathbb{RP}^3 \times \mathbb{T}^2}_{L}) \times [0,1]/A \text{ with } A \in SL_2(\mathbb{Z})$$

~~LTF~~ $\mathscr{R}^*_{\overline{2}}(cL) = \mathscr{R}^3_{\overline{2}}(cL) = H^3(L) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \neq \varnothing \Rightarrow X$ is not LTF.

PTA $\mathscr{R}^*_{\overline{2}}(X) = 0$ since $A$: $\mathscr{R}^3_{\overline{2}}(\Sigma L) = \mathscr{R}^3_{\overline{2}}(cL) \oplus \mathscr{R}^3_{\overline{2}}(cL) \circlearrowleft$ has no fixed points.

# $H^*_{D\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ are intertwined

We have the pairing

$$\mathcal{D}\colon \widetilde{N}^*_{\overline{p}}(X;\mathbb{Z}) \otimes C^{n-*}_{\overline{p}}(X;\mathbb{Z}) \to I^*_{\mathbb{Z}} := \omega \otimes c \mapsto c(\omega \cap \gamma_X)$$

- $[\gamma_X]$ : Fundamental class of $X$.
- $I^*_{\mathbb{Z}}$ : Injective resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

## $H^*_{D\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ are intertwined

We have the pairing

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inducing the pairings

$$F\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z}) \otimes FH^{n-*}_{\overline{p}}(X;\mathbb{Z}) \to \mathbb{Z}$$

$$\mathrm{Tors}\,\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{Tors}\,H^{n+1-*}_{\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

Non-singular pairings

# $H^*_{D\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ are intertwined

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inducing the pairings

$$F\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z}) \otimes F H^{n-*}_{\overline{p}}(X;\mathbb{Z}) \to \mathbb{Z}$$

$$\mathrm{Tors}\, \mathscr{H}^*_{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{Tors}\, H^{n+1-*}_{\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

Non-singular pairings

Tools:

- Poincaré Duality
- Biduality

# $H^*_{D\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ are intertwined

We have the pairing

$$\mathcal{D}\colon \widetilde{N}^*_{\overline{p}}(X;\mathbb{Z}) \otimes C^{n-*}_{\overline{p}}(X;\mathbb{Z}) \to I^*_{\mathbb{Z}} := \omega \otimes c \mapsto c(\omega \cap \gamma_X)$$

**Duality in cohomology**

If the Peripheral term is acyclic the above pairing induces the two following non singular pairings

$$F\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z}) \otimes F\mathscr{H}^{n-*}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Z}$$

$$\mathrm{Tors}\,\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{Tors}\,\mathscr{H}^{n+1-*}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

## $H^*_{D\overline{p}}(X;\mathbb{Z})$ and $\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z})$ are intertwined

We have the pairing

$$\mathcal{D}\colon \widetilde{N}^*_{\overline{p}}(X;\mathbb{Z}) \otimes C^{n-*}_{\overline{p}}(X;\mathbb{Z}) \to I^*_{\mathbb{Z}} := \omega \otimes c \mapsto c(\omega \cap \gamma_X)$$

**Duality in cohomology**

If the Peripheral term is acyclic the above pairing induces the two following non singular pairings

$$F\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z}) \otimes F\mathscr{H}^{n-*}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Z}$$

$$\mathrm{Tors}\,\mathscr{H}^*_{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{Tors}\,\mathscr{H}^{n+1-*}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

- $X =$ Thom Space of to the unitary tangent bundle of $\mathbb{S}^2$.
- $X = \Sigma\mathbb{RP}^3$.

| $k$ | $\mathscr{H}^k_{\overline{0}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | 0 |
| 2 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

# $H^*_{D\bar{p}}(X)$ and $\mathscr{H}^*_{\bar{p}}(X)$ are intertwined: $X = \Sigma(\mathbb{S}^1 \times \mathbb{RP}^3)$

| $k$ | $\mathscr{H}^k_{\bar{0}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | 0 |
| 2 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

$$\downarrow \cong$$

| $k$ | $H^k_{D\bar{0}=\bar{3}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | 0 |
| 2 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

# $H^*_{D\bar{p}}(X)$ and $\mathscr{H}^*_{\bar{p}}(X)$ are intertwined: $X = \Sigma(\mathbb{S}^1 \times \mathbb{RP}^3)$

| $k$ | $\mathscr{H}^k_{\bar{0}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | 0 |
| 2 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

| $k$ | $\mathscr{H}^k_{\bar{1}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | 0 |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

$\downarrow \cong$

| $k$ | $H^k_{D\bar{0}=\bar{3}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | 0 |
| 2 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

# $H^*_{D\overline{p}}(X)$ and $\mathscr{H}^*_{\overline{p}}(X)$ are intertwined: $X = \Sigma(\mathbb{S}^1 \times \mathbb{RP}^3)$

| $k$ | $\mathscr{H}^k_{\overline{0}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | 0 |
| 2 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

$\downarrow \cong$

| $k$ | $H^k_{D\overline{0}=\overline{3}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | 0 |
| 2 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

| $k$ | $\mathscr{H}^k_{\overline{1}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | 0 |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

$\downarrow \ncong$

| $k$ | $H^k_{D\overline{1}=\overline{2}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}_2$ |
| 3 | 0 |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

# $H^*_{D\overline{p}}(X)$ and $\mathscr{H}^*_{\overline{p}}(X)$ are intertwined: $X = \Sigma(\mathbb{S}^1 \times \mathbb{RP}^3)$

| $k$ | $\mathscr{H}^k_{\overline{0}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | 0 |
| 2 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

$\downarrow \cong$

| $k$ | $H^k_{D\overline{0}=\overline{3}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | 0 |
| 2 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
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| $k$ | $\mathscr{H}^k_{\overline{1}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
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| 5 | $\mathbb{Z}$ |

$\downarrow \ncong$

| $k$ | $H^k_{D\overline{1}=\overline{2}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}_2$ |
| 3 | 0 |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

| $k$ | $\mathscr{H}^k_{\overline{2}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}_2$ |
| 3 | 0 |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

$\downarrow \ncong$

| $k$ | $H^k_{D\overline{2}=\overline{1}}(X)$ |
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| 0 | $\mathbb{Z}$ |
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| 3 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 4 | 0 |
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$\downarrow \cong$

| $k$ | $H^k_{D\overline{3}=\overline{0}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
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| 2 | $\mathbb{Z}_2$ |
| 3 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
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| 5 | $\mathbb{Z}$ |

# $H^*_{D\bar{p}}(X)$ and $\mathscr{H}^*_{\bar{p}}(X)$ are intertwined: $X = \Sigma(\mathbb{S}^1 \times \mathbb{RP}^3)$

| $k$ | $\mathscr{H}^k_{\bar{0}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | 0 |
| 2 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

$\downarrow \cong$

| $k$ | $H^k_{D\bar{0}=\bar{3}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | 0 |
| 2 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

$H_0\left(\mathbb{S}^1 \times \mathbb{RP}^3\right)$ free

| $k$ | $\mathscr{H}^k_{\bar{1}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | 0 |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

$\downarrow \ncong$

| $k$ | $H^k_{D\bar{1}=\bar{2}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}_2$ |
| 3 | 0 |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

$H_1\left(\mathbb{S}^1 \times \mathbb{RP}^3\right)$ torsion

| $k$ | $\mathscr{H}^k_{\bar{2}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}_2$ |
| 3 | 0 |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

$\downarrow \ncong$

| $k$ | $H^k_{D\bar{2}=\bar{1}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}_2$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z}$ |
| 5 | $\mathbb{Z}$ |

$H_2\left(\mathbb{S}^1 \times \mathbb{RP}^3\right)$ torsion

| $k$ | $\mathscr{H}^k_{\bar{3}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}_2$ |
| 3 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 4 | 0 |
| 5 | $\mathbb{Z}$ |

$\downarrow \cong$

| $k$ | $H^k_{D\bar{3}=\bar{0}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}_2$ |
| 3 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 4 | 0 |
| 5 | $\mathbb{Z}$ |

cm $H_3\left(\mathbb{S}^1 \times \mathbb{RP}^3\right)$ free

# $H^*_{D\bar{p}}(X)$ and $\mathscr{H}^*_{\bar{p}}(X)$ are intertwined: $X = \Sigma(\mathbb{S}^1 \times \mathbb{RP}^3)$

| $k$ | $\mathscr{H}^k_{\bar{0}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | 0 |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

| $k$ | $H^k_{D\bar{3}=\bar{0}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}_2$ |
| 3 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 4 | 0 |
| 5 | $\mathbb{Z}$ |

Duality

# $H^*_{D\bar{p}}(X)$ and $\mathscr{H}^*_{\bar{p}}(X)$ are intertwined: $X = \Sigma(\mathbb{S}^1 \times \mathbb{RP}^3)$

| $k$ | $\mathscr{H}^k_{\bar{0}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | 0 |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

| $k$ | $\mathscr{H}^k_{\bar{3}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}_2$ |
| 3 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 4 | 0 |
| 5 | $\mathbb{Z}$ |

| $k$ | $H^k_{D\bar{0}=\bar{3}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | 0 |
| 2 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 5 | $\mathbb{Z}$ |

| $k$ | $H^k_{D\bar{3}=\bar{0}}(X)$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}_2$ |
| 3 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 4 | 0 |
| 5 | $\mathbb{Z}$ |

Duality

# $H^*_{D\bar{p}}(X)$ and $\mathscr{H}^*_{\bar{p}}(X)$ are intertwined: $X = \Sigma(\mathbb{S}^1 \times \mathbb{RP}^3)$

| $k$ | $\mathscr{H}^k_{\bar{0}}(X)$ | $\mathscr{H}^k_{\bar{1}}(X)$ | $\mathscr{H}^k_{\bar{2}}(X)$ | $\mathscr{H}^k_{\bar{3}}(X)$ |
|---|---|---|---|---|
| 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 2 | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| 3 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ | 0 |
| 5 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |

$\downarrow \cong$ $\quad$ $\downarrow \ncong$ $\quad$ $\downarrow \ncong$ $\quad$ $\downarrow \cong$

| $k$ | $H^k_{D\bar{0}=\bar{3}}(X)$ | $H^k_{D\bar{1}=\bar{2}}(X)$ | $H^k_{D\bar{2}=\bar{1}}(X)$ | $H^k_{D\bar{3}=\bar{0}}(X)$ |
|---|---|---|---|---|
| 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 1 | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 2 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| 3 | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| 4 | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| 5 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |

Duality

# INTERSECTION HOMOLOGY AND DUALITY

# Coming back to intersection homology

Duality: Manifolds (PL)

$$\pitchfork : H_*(M;\mathbb{Q}) \otimes H_{n-*}(M;\mathbb{Q}) \to \mathbb{Q}$$ Non-singular

# Coming back to intersection homology

Duality: Manifolds (PL)

$\pitchfork : H_*(M;\mathbb{Q}) \otimes H_{n-*}(M;\mathbb{Q}) \to \mathbb{Q}$ Non-singular

$\pitchfork : FH_*(M;\mathbb{Z}) \otimes FH_{n-*}(M;\mathbb{Z}) \to \mathbb{Z}$ Non-singular

$\pitchfork : \mathrm{Tors}\, H_*(M;\mathbb{Z}) \otimes \mathrm{Tors}\, H_{n-1-*}(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ Non-singular

# Coming back to intersection homology

Duality: Pseudomanifolds (PL)

$$\pitchfork \colon H_*^{\overline{p}}(X;\mathbb{Q}) \otimes H_{n-*}^{D\overline{p}}(X;\mathbb{Q}) \to \mathbb{Q} \qquad \text{Non-singular}$$

# Coming back to intersection homology

Duality: Pseudomanifolds (PL)

$\pitchfork : H_*^{\overline{p}}(X;\mathbb{Q}) \otimes H_{n-*}^{D\overline{p}}(X;\mathbb{Q}) \to \mathbb{Q}$ Non-singular

$\pitchfork : FH_*^{\overline{p}}(X;\mathbb{Z}) \otimes FH_{n-*}^{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Z}$ Non-degenerate

$\pitchfork : \mathrm{Tors}\, H_*^{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{Tors}\, H_{n+1-*}^{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ ———

# Coming back to intersection homology

Duality : Pseudomanifolds

$$\mathcal{D}\colon \widetilde{N}^{*}_{\overline{p}}(X;\mathbb{Z}) \otimes \widetilde{N}^{n-*}_{D\overline{p}}(X;\mathbb{Z}) \to I^{*}_{\mathbb{Z}} := \omega \otimes \eta \mapsto \varepsilon((\omega \cup \eta) \cap \gamma_X))$$

$$\Downarrow$$

# Coming back to intersection homology

Duality : Pseudomanifolds

$$\mathcal{D}\colon \widetilde{N}^{*}_{\overline{p}}(X;\mathbb{Z}) \otimes \widetilde{N}^{n-*}_{D\overline{p}}(X;\mathbb{Z}) \to I^{*}_{\mathbb{Z}} := \omega \otimes \eta \mapsto \varepsilon((\omega \cup \eta) \cap \gamma_X))$$

$$\Downarrow$$

$$F\mathscr{H}^{*}_{\overline{p}}(X;\mathbb{Z}) \otimes F\mathscr{H}^{n-*}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Z}$$

$$\mathrm{Tors}\,\mathscr{H}^{*}_{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{Tors}\,\mathscr{H}^{n+1-*}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

$\Downarrow$ Poincaré Duality

## Coming back to intersection homology

Duality : Pseudomanifolds

$$F\mathscr{H}^{*}_{\overline{p}}(X;\mathbb{Z}) \otimes F\mathscr{H}^{n-*}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Z}$$

$$\mathrm{Tors}\,\mathscr{H}^{*}_{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{Tors}\,\mathscr{H}^{n+1-*}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

$\Downarrow$ Poincaré Duality

$$FH^{\overline{p}}_{*}(X;\mathbb{Z}) \otimes FH^{D\overline{p}}_{n-*}(X;\mathbb{Z}) \to \mathbb{Z}$$

$$\mathrm{Tors}\,H^{\overline{p}}_{*}(X;\mathbb{Z}) \otimes \mathrm{Tors}\,H^{D\overline{p}}_{n-1-*}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

# Coming back to intersection homology

Duality : Pseudomanifolds

$$FH_*^{\overline{p}}(X;\mathbb{Z}) \otimes FH_{n-*}^{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Z}$$

$$\operatorname{Tors} H_*^{\overline{p}}(X;\mathbb{Z}) \otimes \operatorname{Tors} H_{n-1-*}^{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

**Duality in homology**

If the Peripheral term is acyclic then the above pairings are non singular.

# Coming back to intersection homology

Duality defect: Components of the peripheral complex

# Coming back to intersection homology

Duality defect: Components of the peripheral complex

$$0 \longrightarrow FH_*^{\overline{p}}(X) \xrightarrow{(Free)} \mathrm{Hom}\left(FH_{n-*}^{D\overline{p}}(X);\mathbb{Z}\right) \longrightarrow \mathcal{A}_*^{\overline{p}}(X)$$

(Free) always non degenerate pairing.

# Coming back to intersection homology

Duality defect: Components of the peripheral complex

$$\mathcal{B}_*^{\overline{p}}(X) \longrightarrow \mathrm{Tors}\, H_*^{\overline{p}}(X) \xrightarrow{(Tor)} \mathrm{Hom}\,(\mathrm{Tors}\, H_{n-1-*}^{D\overline{p}}(X); \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathcal{C}_*^{\overline{p}}(X)$$

(Tor) degenerate pairing or non singular pairing since $\mathcal{B}_*^{\overline{p}}(X) \cong \mathcal{C}_{n-1-*}^{\overline{p}}(X)$.

# Coming back to intersection homology

Duality defect: Components of the peripheral complex

$$0 \longrightarrow FH_*^{\overline{p}}(X) \xrightarrow{(Free)} \mathrm{Hom}\,(FH_{n-*}^{D\overline{p}}(X);\mathbb{Z}) \longrightarrow \mathcal{A}_*^{\overline{p}}(X)$$

$$\mathcal{B}_*^{\overline{p}}(X) \longrightarrow \mathrm{Tors}\, H_*^{\overline{p}}(X) \xrightarrow{(Tor)} \mathrm{Hom}\,(\mathrm{Tors}\, H_{n-1-*}^{D\overline{p}}(X);\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathcal{C}_*^{\overline{p}}(X)$$

(Free) always non degenerate pairing.

(Tor) degenerate pairing or non singular pairing since $\mathcal{B}_*^{\overline{p}}(X) \cong \mathcal{C}_{n-1-*}^{\overline{p}}(X)$.

# Coming back to intersection homology

Duality defect: Components of the peripheral complex

$$0 \longrightarrow FH^{\overline{p}}_*(X) \xrightarrow{(Free)} \mathrm{Hom}\,(FH^{D\overline{p}}_{n-*}(X);\mathbb{Z}) \longrightarrow \mathcal{A}^{\overline{p}}_*(X)$$

$$\mathcal{B}^{\overline{p}}_*(X) \longrightarrow \mathrm{Tors}\, H^{\overline{p}}_*(X) \xrightarrow{(Tor)} \mathrm{Hom}\,(\mathrm{Tors}\, H^{D\overline{p}}_{n-1-*}(X);\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathcal{C}^{\overline{p}}_*(X)$$

(Free) always non degenerate pairing.

(Tor) degenerate pairing or non singular pairing since $\mathcal{B}^{\overline{p}}_*(X) \cong \mathcal{C}^{\overline{p}}_{n-1-*}(X)$.

$$0 \longrightarrow \mathcal{A}^{\overline{p}}_*(X) \longrightarrow \mathscr{R}^*_{\overline{p}}(X)/\mathcal{C}^{\overline{p}}_*(X) \longrightarrow \mathcal{B}^{\overline{p}}_*(X) \longrightarrow 0$$

We also have ...

$$0 \longrightarrow F\mathscr{H}^{*}_{\overline{p}}(X) \xrightarrow{(Free)} \operatorname{Hom}\left(F\mathscr{H}^{n-*}_{D\overline{p}}(X);\mathbb{Z}\right) \longrightarrow \mathcal{A}^{\overline{p}}_{*}(X)$$

$$\mathcal{B}^{\overline{p}}_{*}(X) \longrightarrow \operatorname{Tors}\mathscr{H}^{*}_{\overline{p}}(X) \xrightarrow{(Tor)} \operatorname{Hom}\left(\operatorname{Tors}\mathscr{H}^{n-1-*}_{D\overline{p}}(X);\mathbb{Z}\right) \longrightarrow \mathcal{C}^{\overline{p}}_{*}(X)$$

and ...

We also have ...

$$0 \longrightarrow F\mathscr{H}^{*}_{\overline{p}}(X) \xrightarrow{(Free)} \mathrm{Hom}\left(F\mathscr{H}^{n-*}_{D\overline{p}}(X);\mathbb{Z}\right) \longrightarrow \mathcal{A}^{\overline{p}}_{*}(X)$$

$$\mathcal{B}^{\overline{p}}_{*}(X) \longrightarrow \mathrm{Tors}\,\mathscr{H}^{*}_{\overline{p}}(X) \xrightarrow{(Tor)} \mathrm{Hom}\left(\mathrm{Tors}\,\mathscr{H}^{n-1-*}_{D\overline{p}}(X);\mathbb{Z}\right) \longrightarrow \mathcal{C}^{\overline{p}}_{*}(X)$$

and ...

$$0 \longrightarrow F\mathscr{H}^{*}_{\overline{p}}(X) \xrightarrow{(Free)} FH^{*}_{D\overline{p}}(X) \longrightarrow \mathcal{A}^{\overline{p}}_{*}(X)$$

$$\mathcal{B}^{\overline{p}}_{*}(X) \longrightarrow \mathrm{Tors}\,\mathscr{H}^{*}_{\overline{p}}(X) \xrightarrow{(Tor)} \mathrm{Tors}\,H^{*}_{D\overline{p}}(X) \longrightarrow \mathcal{C}^{\overline{p}}_{*}(X)$$

# MINIMAL MODELS

# Sullivan minimal models

$$A^*_{PL}(\Delta, \mathbb{Q}) = \Lambda(x_1, \ldots, x_m, dx_1, \ldots, dx_m),$$

where $(x_0, \ldots, x_m)$ are the barycentric coordinates of $\Delta$.

$A^*_{PL}(X)$ is the simplicial sheaf

$$\left\{(\omega_\sigma) \ / \ \sigma \colon \Delta \to X \text{ singular simplex}, \omega_\sigma \in A^*_{PL}(\Delta, \mathbb{Q}), \text{compatible}\right\}$$

# Sullivan minimal models

$X$ simply connected and of finite type.

- $A^*_{PL}(X)$ is a DGCA computing $H^*(X;\mathbb{Q})$
- There exists a minimal model $(\Lambda V, d) \xrightarrow{\cong} A^*_{PL}(X)$.
- It contains the rational cohomology of $X$ since $H^*(\Lambda V, d) = H^*(X;\mathbb{Q})$.
- It contains the rational homotopy of $X$ since $\hom_{\mathbb{Q}}(V^k, \mathbb{Q}) = \pi_k(X) \otimes \mathbb{Q}$.

Example

$$H^k(\Sigma\mathbb{CP}^2) = \begin{cases} \mathbb{Q} & \text{if } k = 0 \\ \mathbb{Q}[e \wedge dt] & \text{if } k = 3 \\ \mathbb{Q}[e^2 \wedge dt] & \text{if } k = 5 \\ 0 & \text{if not} \end{cases}$$

Minimal model:

$$\varphi\colon \Lambda(a_3, b_5, c_7, \ldots) \to A^*_{PL}(X)$$

with

# Sullivan minimal models

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Minimal model:

$$\varphi \colon \Lambda(a_3, b_5, c_7, \ldots) \to A^*_{PL}(X)$$

with

$$\begin{array}{cccc} da_3 = 0 & db_5 = 0 & dc_7 = a_3 b_5 & \\ \varphi(a_3) = e \wedge dt & \varphi(b_5) = e^2 \wedge dt & \varphi(c_7) = 0 & \cdots \end{array}$$

Example

$$H^k(\Sigma\mathbb{CP}^2) = \begin{cases} \mathbb{Q} & \text{if } k = 0 \\ \mathbb{Q}[e \wedge dt] & \text{if } k = 3 \\ \mathbb{Q}[e^2 \wedge dt] & \text{if } k = 5 \\ 0 & \text{if not} \end{cases}$$

Minimal model:

$$\varphi\colon \Lambda(a_3, b_5, c_7, \ldots) \to A^*_{PL}(X)$$

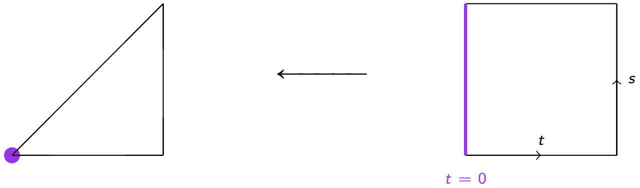
with

$$da_3 = 0 \qquad db_5 = 0 \qquad dc_7 = a_3 b_5$$

Minimality: $dV \subset \Lambda^+ V \cdot \Lambda^+ V$

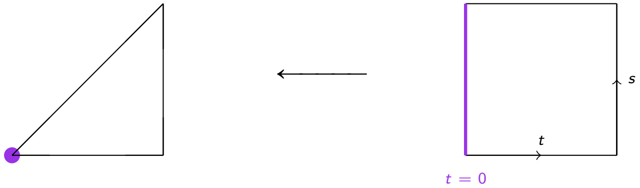
Local differential forms

$\Delta = \Delta_0 * \Delta_1$ $\qquad$ $\widetilde{\Delta} = c\Delta_0 \times \Delta_1$



## Local differential forms

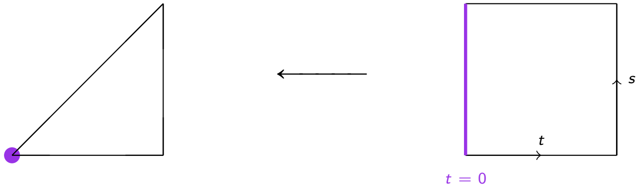
$$\Delta = \Delta_0 * \Delta_1 \qquad \qquad \widetilde{\Delta} = c\Delta_0 \times \Delta_1$$




$$\widetilde{A}^*_{PL}(\Delta) = \{P_0(s,t) + P_1(s,t)\ ds + P_2(s,t)\ dt + P_3(s,t)\ ds \wedge dt\}$$

## Local differential forms

$$\Delta = \Delta_0 * \Delta_1 \qquad \widetilde{\Delta} = c\Delta_0 \times \Delta_1$$




$$\widetilde{A}^*_{PL}(\Delta) = \{P_0(s,t) + P_1(s,t)\ ds + P_2(s,t)\ dt + P_3(s,t)\ ds \wedge dt\}$$

$$||tP(s,t)|| = -\infty \qquad ||P(s)|| = 0 \qquad ||P(s,t)\ dt|| = -\infty \qquad ||P(s)\ ds|| = 1$$

$$||P(s,t)\ ds \wedge dt|| = -\infty$$

Global differential forms

- $A^*_{PL}(\Delta) = \Lambda(x_1,\ldots,x_m, dx_1,\ldots,dx_m)$, with $(x_0,\ldots,x_m)$ barycentric coordinates of $\Delta$
- $\widetilde{A}^*_{PL}(\Delta) = A^*_{PL}(c\Delta_0) \otimes \cdots \otimes A^*_{PL}(c\Delta_{n-1}) \otimes A^*_{PL}(\Delta_n)$, with $\Delta = \Delta_0 * \cdots * \Delta_n$.
- $\widetilde{A}^*_{PL,\overline{p}}(\Delta) = \{\omega \in \widetilde{A}^*_{PL}(\Delta) / \max(||\omega||_k, ||d\omega||_k) \leqslant p_k\}$.

# Sullivan minimal perverse models

Global differential forms

- $A^*_{PL}(\Delta)=\Lambda(x_1,\dots,x_m,dx_1,\dots,dx_m)$, with $(x_0,\dots,x_m)$ barycentric coordinates of $\Delta$
- $\widetilde{A}^*_{PL}(\Delta) = A^*_{PL}(c\Delta_0)\otimes\cdots\otimes A^*_{PL}(c\Delta_{n-1})\otimes A^*_{PL}(\Delta_n)$, with $\Delta=\Delta_0*\cdots*\Delta_n$.
- $\widetilde{A}^*_{PL,\overline{p}}(\Delta) = \{\omega\in\widetilde{A}^*_{PL}(\Delta) / \max(||\omega||_k,||d\omega||_k)\leqslant p_k\}$.
- $\widetilde{A}^*_{PL,\overline{p}}(X)$ is the simplicial sheaf

$$\left\{(\omega_\sigma)\ /\ \omega_\sigma\in\widetilde{A}^*_{PL,\overline{p}}(\Delta),\sigma\colon\Delta\to X \text{ filtered simplex compatible}\right\}.$$

Global differential forms

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$$\left\{(\omega_\sigma) \;/\; \omega_\sigma \in \widetilde{A}^*_{PL,\overline{p}}(\Delta), \sigma \colon \Delta \to X \text{ filtered simplex compatible}\right\}.$$

- $H^*\left(\widetilde{A}^{\cdot}_{PL,\overline{p}}(X)\right) = \mathscr{H}^*_{\overline{p}}(X;\mathbb{Q}) = H^*_{\overline{p}}(X;\mathbb{Q})$

Algebra?

| | $H^k(\Sigma\mathbb{CP}^2)$ | | $H^k(\mathbb{CP}^2)$ |
|---|---|---|---|
| $H^k_{\overline{p}}(\Sigma\mathbb{CP}^2)$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
| $k=0$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |
| $k=1$ | 0 | 0 | 0 |
| $k=2$ | 0 | $\mathbb{Q}[e]$ | $\mathbb{Q}[e]$ |
| $k=3$ | $\mathbb{Q}[e\wedge dt]$ | 0 | 0 |
| $k=4$ | 0 | 0 | $\mathbb{Q}[e^2]$ |
| $k=5$ | $\mathbb{Q}[e^2\wedge dt]$ | $\mathbb{Q}[e^2\wedge dt]$ | 0 |

# Sullivan minimal perverse models

| $H^k_{\overline{p}}(\Sigma\mathbb{CP}^2)$ | $\overline{p} = \overline{0} = \overline{1}$ | $\overline{p} = \overline{2} = \overline{3}$ | $\overline{p} = \overline{\infty}$ |
|---|---|---|---|
| $k = 0$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |
| $k = 1$ | $0$ | $0$ | $0$ |
| $k = 2$ | $0$ | $\mathbb{Q}[e]$ | $\mathbb{Q}[e]$ |
| $k = 3$ | $\mathbb{Q}[e \wedge dt]$ | $0$ | $0$ |
| $k = 4$ | $0$ | $0$ | $\mathbb{Q}[e^2]$ |
| $k = 5$ | $\mathbb{Q}[e^2 \wedge dt]$ | $\mathbb{Q}[e^2 \wedge dt]$ | $0$ |

Minimal perverse model $\Lambda(a_3, b_5, , \ldots)$

| $\Lambda V$ | $\overline{p} = \overline{0} = \overline{1}$ | $\overline{p} = \overline{2} = \overline{3}$ | $\overline{p} = \overline{\infty}$ |
|---|---|---|---|
| $k = 2$ | $0$ | | |
| $k = 3$ | $a_3$ | | |
| $k = 4$ | $0$ | | |
| $k = 5$ | $b_5$ | | |

# Sullivan minimal perverse models

| $H^k_{\overline{p}}(\Sigma\mathbb{CP}^2)$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
|---|---|---|---|
| $k=0$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |
| $k=1$ | $0$ | $0$ | $0$ |
| $k=2$ | $0$ | $\mathbb{Q}[e]$ | $\mathbb{Q}[e]$ |
| $k=3$ | $\mathbb{Q}[e\wedge dt]$ | $0$ | $0$ |
| $k=4$ | $0$ | $0$ | $\mathbb{Q}[e^2]$ |
| $k=5$ | $\mathbb{Q}[e^2\wedge dt]$ | $\mathbb{Q}[e^2\wedge dt]$ | $0$ |

Minimal perverse model $\Lambda(a_3, b_5, \alpha_2, e, \ldots)$

| $\Lambda V$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
|---|---|---|---|
| $k=2$ | $0$ | $\alpha_2, e$ | |
| $k=3$ | $a_3$ | | |
| $k=4$ | $0$ | | |
| $k=5$ | $b_5$ | | |

$d\alpha_2 = a_3,$

# Sullivan minimal perverse models

| $H^k_{\overline{p}}(\Sigma\mathbb{CP}^2)$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
|---|---|---|---|
| $k=0$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |
| $k=1$ | $0$ | $0$ | $0$ |
| $k=2$ | $0$ | $\mathbb{Q}[e]$ | $\mathbb{Q}[e]$ |
| $k=3$ | $\mathbb{Q}[e\wedge dt]$ | $0$ | $0$ |
| $k=4$ | $0$ | $0$ | $\mathbb{Q}[e^2]$ |
| $k=5$ | $\mathbb{Q}[e^2\wedge dt]$ | $\mathbb{Q}[e^2\wedge dt]$ | $0$ |

Minimal perverse model $\Lambda(a_3, b_5, \alpha_2, e, \ldots)$

| $\Lambda V$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
|---|---|---|---|
| $k=2$ | $0$ | $\alpha_2, e$ | |
| $k=3$ | $a_3$ | | |
| $k=4$ | $0$ | | |
| $k=5$ | $b_5$ | $a_3\alpha_2, a_3e$ | |

$d\alpha_2 = a_3,$

# Sullivan minimal perverse models

| $H^k_{\overline{p}}(\Sigma\mathbb{CP}^2)$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
|---|---|---|---|
| $k=0$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |
| $k=1$ | $0$ | $0$ | $0$ |
| $k=2$ | $0$ | $\mathbb{Q}[e]$ | $\mathbb{Q}[e]$ |
| $k=3$ | $\mathbb{Q}[e\wedge dt]$ | $0$ | $0$ |
| $k=4$ | $0$ | $0$ | $\mathbb{Q}[e^2]$ |
| $k=5$ | $\mathbb{Q}[e^2\wedge dt]$ | $\mathbb{Q}[e^2\wedge dt]$ | $0$ |

Minimal perverse model $\Lambda(a_3, b_5, \alpha_2, e, \ldots)$

| $\Lambda V$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
|---|---|---|---|
| $k=2$ | $0$ | $\alpha_2, e$ | |
| $k=3$ | $a_3$ | | |
| $k=4$ | $0$ | $\alpha_4, \beta_4$ | |
| $k=5$ | $b_5$ | $a_3\alpha_2, a_3e$ | |

$d\alpha_2 = a_3,$

# Sullivan minimal perverse models

| $H^k_{\overline{p}}(\Sigma\mathbb{CP}^2)$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
|---|---|---|---|
| $k=0$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |
| $k=1$ | $0$ | $0$ | $0$ |
| $k=2$ | $0$ | $\mathbb{Q}[e]$ | $\mathbb{Q}[e]$ |
| $k=3$ | $\mathbb{Q}[e\wedge dt]$ | $0$ | $0$ |
| $k=4$ | $0$ | $0$ | $\mathbb{Q}[e^2]$ |
| $k=5$ | $\mathbb{Q}[e^2\wedge dt]$ | $\mathbb{Q}[e^2\wedge dt]$ | $0$ |

Minimal perverse model $\Lambda(a_3, b_5, \alpha_2, e, \ldots)$

| $\Lambda V$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
|---|---|---|---|
| $k=2$ | $0$ | $\alpha_2, e$ | |
| $k=3$ | $a_3$ | | |
| $k=4$ | $0$ | $\alpha_4, \beta_4$ | $\alpha_2 e, \alpha_2^2, e^2,$ |
| $k=5$ | $b_5$ | $a_3\alpha_2, a_3 e$ | |

$d\alpha_2 = a_3,$ $d\alpha_4 = a_3\alpha_2,\ d\beta_4 = a_3 e,$

# Sullivan minimal perverse models

| $H^k_{\overline{p}}(\Sigma\mathbb{CP}^2)$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
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| $k=1$ | $0$ | $0$ | $0$ |
| $k=2$ | $0$ | $\mathbb{Q}[e]$ | $\mathbb{Q}[e]$ |
| $k=3$ | $\mathbb{Q}[e\wedge dt]$ | $0$ | $0$ |
| $k=4$ | $0$ | $0$ | $\mathbb{Q}[e^2]$ |
| $k=5$ | $\mathbb{Q}[e^2\wedge dt]$ | $\mathbb{Q}[e^2\wedge dt]$ | $0$ |

Minimal perverse model $\Lambda(a_3, b_5, \alpha_2, e, \alpha_4, \beta_4, \dots)$

| $\Lambda V$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
|---|---|---|---|
| $k=2$ | $0$ | $\alpha_2, e$ | |
| $k=3$ | $a_3$ | | |
| $k=4$ | $0$ | $\alpha_4, \beta_4$ | $\alpha_2 e, \alpha_2^2, e^2, \beta_4-\alpha_2 e, \alpha_4-2\alpha_2 e$ |
| $k=5$ | $b_5$ | $a_3\alpha_2, a_3 e$ | |

$d\alpha_2 = a_3,$ $d\alpha_4 = a_3\alpha_2,\ d\beta_4 = a_3 e,$

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|---|---|---|---|
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| $k=1$ | 0 | 0 | 0 |
| $k=2$ | 0 | $\mathbb{Q}[e]$ | $\mathbb{Q}[e]$ |
| $k=3$ | $\mathbb{Q}[e\wedge dt]$ | 0 | 0 |
| $k=4$ | 0 | 0 | $\mathbb{Q}[e^2]$ |
| $k=5$ | $\mathbb{Q}[e^2\wedge dt]$ | $\mathbb{Q}[e^2\wedge dt]$ | 0 |

Minimal perverse model $\Lambda(a_3, b_5, \alpha_2, e, \alpha_4, \beta_4, \ldots)$

| $\Lambda V$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
|---|---|---|---|
| $k=2$ | 0 | $\alpha_2, e$ | |
| $k=3$ | $a_3$ | | $x, y$ |
| $k=4$ | 0 | $\alpha_4, \beta_4$ | $\alpha_2 e, \alpha_2^2, e^2, \beta_4-\alpha_2 e, \alpha_4-2\alpha_2 e$ |
| $k=5$ | $b_5$ | $a_3\alpha_2, a_3 e$ | |

$d\alpha_2 = a_3,$ $d\alpha_4 = a_3\alpha_2,\ d\beta_4 = a_3 e,$ $dx = \beta_4 - \alpha_2 e\ ,\ dy = \alpha_4 - 2\alpha_2 e$

# Sullivan minimal perverse models

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|---|---|---|---|
| $k=0$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |
| $k=1$ | 0 | 0 | 0 |
| $k=2$ | 0 | $\mathbb{Q}[e]$ | $\mathbb{Q}[e]$ |
| $k=3$ | $\mathbb{Q}[e\wedge dt]$ | 0 | 0 |
| $k=4$ | 0 | 0 | $\mathbb{Q}[e^2]$ |
| $k=5$ | $\mathbb{Q}[e^2\wedge dt]$ | $\mathbb{Q}[e^2\wedge dt]$ | 0 |

Minimal perverse model $\Lambda(a_3, b_5, \alpha_2, e, \alpha_4, \beta_4, x, y \ldots)$

| $\Lambda V$ | $\overline{p}=\overline{0}=\overline{1}$ | $\overline{p}=\overline{2}=\overline{3}$ | $\overline{p}=\overline{\infty}$ |
|---|---|---|---|
| $k=2$ | 0 | $\alpha_2, e$ | |
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| $k=5$ | $b_5$ | $a_3\alpha_2, a_3 e$ | |

$d\alpha_2 = a_3,$ $d\alpha_4 = a_3\alpha_2,$ $d\beta_4 = a_3 e,$ $dx = \beta_4 - \alpha_2 e$ , $dy = \alpha_4 - 2\alpha_2 e$

Minimality: $||a_3|| < ||\alpha_2||$ $||\beta_4|| < ||x||$ $||\alpha_4|| < ||y||$

# Sullivan minimal perverse model

| Regular | Perverse |
|---|---|
| $A^*_{PL}(X)$ DGCA | $\{\tilde{A}^*_{PL,\overline{p}}(X)\}_{\overline{p}}$ DGCA |

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| $dV \subset \Lambda^+ V \cdot \Lambda^+ V$ | $dV_{\overline{p}} \subset \Lambda^+ V \cdot \Lambda^+ V + \oplus_{\overline{q}<\overline{p}} (\Lambda V)_{\overline{q}}$ |

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| $dV \subset \Lambda^+ V \cdot \Lambda^+ V$ | $dV_{\overline{p}} \subset \Lambda^+ V \cdot \Lambda^+ V + \oplus_{\overline{q}<\overline{p}} (\Lambda V)_{\overline{q}}$ |
| $H^*(\Lambda V, d) \xrightarrow{\cong} H^*(X;\mathbb{Q})$ | $H^*((\Lambda V)_{\leqslant \overline{p}}, d) \xrightarrow{\cong} \mathscr{H}^*_{\overline{p}}(X;\mathbb{Q}) = H^*_{\overline{p}}(X;\mathbb{Q})$ |

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| $\hom_{\mathbb{Q}}(V^k, \mathbb{Q}) \leftrightsquigarrow \pi_k(X) \otimes \mathbb{Q}$ | Intersection homotopy ? <br> Work in progress ... |

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| $\hom_{\mathbb{Q}}(V^k, \mathbb{Q}) \leftrightsquigarrow \pi_k(X) \otimes \mathbb{Q}$ | Intersection homotopy ? <br> Work in progress ... |

Any nodal hypersurface of $\mathbb{CP}^4$ is intersection-formal.

# STEENROD SQUARES

# Goresky & Pardon conjecture

$$\begin{array}{ccc} & & H^{r+i}_{\mathcal{L}(\overline{p},i)}(X;\mathbb{Z}_2) \\ & \nearrow & \downarrow \\ H^{r}_{\overline{p}}(X;\mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^i} & H^{r+i}_{2\overline{p}}(X;\mathbb{Z}_2) \end{array}$$

$$\mathcal{L}(\overline{p}, i)(\ell) = \min(2\overline{p}(\ell), \overline{p}(\ell) + i)$$

$$\begin{array}{ccc} & & \mathscr{H}^{r+i}_{\mathcal{L}(\overline{p},i)}(X;\mathbb{Z}_2) \\ & \nearrow & \downarrow \\ \mathscr{H}^{r}_{\overline{p}}(X;\mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^i} & \mathscr{H}^{r+i}_{2\overline{p}}(X;\mathbb{Z}_2) \end{array}$$

$$\mathcal{L}(\overline{p}, i)(\ell) = \min(2\overline{p}(\ell), \overline{p}(\ell) + i)$$

Effective improvement :

$0 \neq \mathrm{Sq}^2 \in \mathscr{H}^{*}_{\overline{p}(\ell)+2}(X)$ and $0 = \mathrm{Sq}^2 \in \mathscr{H}^{*}_{2\overline{p}(\ell)}(X)$.

# Goresky & Pardon conjecture: A cone

$M$ manifold and $p = \overline{p}(\dim M + 1)$

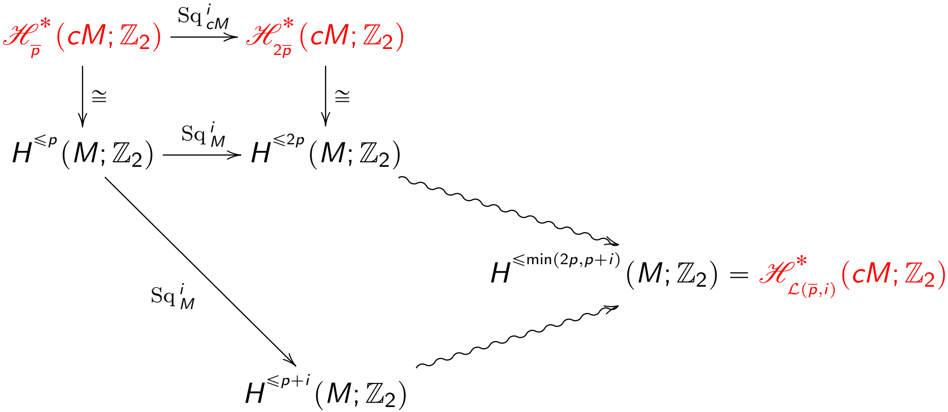
$$\begin{array}{ccc} \mathscr{H}^{*}_{\overline{p}}(cM;\mathbb{Z}_2) & \xrightarrow{\operatorname{Sq}^{i}_{cM}} & \mathscr{H}^{*}_{2\overline{p}}(cM;\mathbb{Z}_2) \\ \downarrow \cong & & \downarrow \cong \\ H^{\leqslant p}(M;\mathbb{Z}_2) & \xrightarrow{\operatorname{Sq}^{i}_{M}} & H^{\leqslant 2p}(M;\mathbb{Z}_2) \end{array}$$

# Goresky & Pardon conjecture: A cone

$M$ manifold and $p = \overline{p}(\dim M + 1)$

$$
\begin{array}{ccc}
\mathscr{H}^{*}_{\overline{p}}(cM;\mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^{i}_{cM}} & \mathscr{H}^{*}_{2\overline{p}}(cM;\mathbb{Z}_2) \\
\downarrow \cong & & \downarrow \cong \\
H^{\leqslant p}(M;\mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^{i}_{M}} & H^{\leqslant 2p}(M;\mathbb{Z}_2) \\
& \searrow^{\mathrm{Sq}^{i}_{M}} & \\
& & H^{\leqslant p+i}(M;\mathbb{Z}_2)
\end{array}
$$

$M$ manifold and $p = \overline{p}(\dim M + 1)$



THANKS FOR YOUR ATTENTION !