Cohomology of $\mathbb{S}^3$-actions and duality

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Scenario

- M smooth (C^∞), closed (compact, without boundary) manifold;
- Any smooth action

$$\Phi \colon \mathbb{S}^3 \times M \longrightarrow M$$

- Singular cohomologies (real coefficients)

$$H^*(M), H^*(M/\mathbb{S}^3)$$

- Exact sequence (Gysin sequence) and spectral sequences relating those cohomologies.

Sketch

We construct a Gysin sequence for (singular) actions of $\mathbb{S}^3$.

1. Gysin sequences.
2. Our result.
3. Examples of singular actions of $\mathbb{S}^3$.
4. Duality.

Gysin sequence (classical)

For any

$$\pi \colon M \longrightarrow B$$

fiber bundle of fiber $\mathbb{S}^k$, we have the Gysin sequence:

$$\cdots \to H^i(B) \xrightarrow{\pi^*} H^i(M) \xrightarrow{\oint} H^{i-k}(B) \xrightarrow{\varepsilon} H^{i+1}(B) \to \cdots ,$$

where $\oint$ comes from the integration along the fibers,
and $\varepsilon \in H^{k+1}(B)$ multiplication by the Euler class.

Gysin sequences

- Particular cases: free actions of $\mathbb{S}^1$ and $\mathbb{S}^3$;
- More Gysin sequences (generalizations):
  - isometric or Riemannian flows ($\mathbb{R}$-actions);
  - singular actions of $\mathbb{S}^1$;
  - semi-free actions of $\mathbb{S}^3$;
  - singular Riemannian flows;
  - two orbit types torus actions;
  - ...

Particular case: free $\mathbb{S}^1$-action

If we have a smooth free action

$$\mathbb{S}^1 \times M \longrightarrow M,$$

the orbit space $M/\mathbb{S}^1$ is a smooth manifold, and we have a fibre bundle

$$\pi \colon M \longrightarrow M/\mathbb{S}^1$$

so, it is the previous situation for $k = 1$. The Gysin sequence is:

$$\cdots \to H^i(M/\mathbb{S}^1) \xrightarrow{\pi^*} H^i(M) \xrightarrow{\int} H^{i-1}(M/\mathbb{S}^1) \xrightarrow{\varepsilon} H^{i+1}(M/\mathbb{S}^1) \to \cdots$$

$\varepsilon \in H^2(M/\mathbb{S}^1)$

Particular case: free $\mathbb{S}^3$-action

If we have a smooth free action

$$\mathbb{S}^3 \times M \longrightarrow M,$$

the orbit space $M/\mathbb{S}^3$ is a smooth manifold, and we have a fibre bundle

$$\pi \colon M \longrightarrow M/\mathbb{S}^3$$

so, it is the previous situation for $k = 3$. The Gysin sequence is:

$$\cdots \to H^i(M/\mathbb{S}^3) \xrightarrow{\pi^*} H^i(M) \xrightarrow{\int} H^{i-3}(M/\mathbb{S}^3) \xrightarrow{\varepsilon} H^{i+1}(M/\mathbb{S}^3) \to \cdots$$

$\varepsilon \in H^4(M/\mathbb{S}^1)$

Generalizations

Isometric $\mathbb{R}$-actions

- Induce an oriented 1-dimensional foliation $\mathscr{F}$ (flow)
- $M/\mathbb{R}$ is not a manifold, but we can study the basic cohomology $H^*(M/\mathscr{F})$
- We can construct a *Gysin sequence:* [Kamber-Tondeur, 1983]

$$\cdots \to H^i(M/\mathscr{F}) \xrightarrow{\pi^*} H^i(M) \xrightarrow{\int} H^{i-1}(M/\mathscr{F}) \xrightarrow{\varepsilon} H^{i+1}(M/\mathscr{F}) \to \cdots$$

In the case of a *Riemannian* flow, we have [JIRP, 2000]:

$$\cdots \to H^i(M/\mathscr{F}) \xrightarrow{\pi^*} H^i(M) \xrightarrow{\int} H_\kappa^{i-1}(M/\mathscr{F}) \xrightarrow{\varepsilon} H^{i+1}(M/\mathscr{F}) \to \cdots$$

- $\varepsilon \in H^2_{-\kappa}(M/\mathbb{S}^1)$ (twisted cohomology, κ mean cutrvature)

Singular actions of $\mathbb{S}^1$

[G. Hector-MS, 1993]

- Isotropy groups: finite ones and $\mathbb{S}^1$. F fixed point set.
- $M/\mathbb{S}^1$ is **not** a manifold, but a stratified pseudomanifold.
- Intersection cohomology: $H^*_{\overline{p}}(M/\mathbb{S}^1)$ (Poincaré Duality)

$$\cdots \to H^i_{\overline{p}}(M/\mathbb{S}^1) \xrightarrow{\pi^*} H^i(M) \xrightarrow{\oint} H^{i-1}_{\overline{p}-\overline{2}}(M/\mathbb{S}^1) \xrightarrow{\varepsilon} H^{i+1}_{\overline{p}}(M/\mathbb{S}^1) \to \cdots$$

- In particular ($\overline{p} = \overline{0}$), the classical result :

$$\cdots \to H^i(M/\mathbb{S}^1) \xrightarrow{\pi^*} H^i(M) \xrightarrow{\oint} H^{i-1}(M/\mathbb{S}^1, F) \xrightarrow{\varepsilon} H^{i+1}(M/\mathbb{S}^1) \to \cdot$$

- $\varepsilon \in H^2_{\overline{2}}(M/\mathbb{S}^1)$.

Semi-free actions of $\mathbb{S}^3$

[MS, 1993]

- Isotropy groups: $\{1\}$ and $\mathbb{S}^3$.
- F fixed point set.
- $M/\mathbb{S}^3$ is **not** a manifold, but a stratified pseudomanifold.
- Intersection cohomology: $H^*_{\overline{p}}(M/\mathbb{S}^3)$

$$\cdots \to H^i_{\overline{p}}(M/\mathbb{S}^3) \xrightarrow{\pi^*} H^i(M) \xrightarrow{\oint} H^{i-3}_{\overline{p}-\overline{4}}(M/\mathbb{S}^3) \xrightarrow{\varepsilon} H^{i+1}_{\overline{p}}(M/\mathbb{S}^3) \to \cdots$$

- In particular ($\overline{p} = \overline{0}$), the classical result :

$$\cdots \to H^i(M/\mathbb{S}^3) \xrightarrow{\pi^*} H^i(M) \xrightarrow{\oint} H^{i-3}(M/\mathbb{S}^3, F) \xrightarrow{\varepsilon} H^{i+1}(M/\mathbb{S}^3) \to \cdot$$

- $\varepsilon \in H^4_{\overline{4}}(M/\mathbb{S}^1)$.

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Gysin sequence for any smooth $\mathbb{S}^3$-action

[JIRP-MS 2014]
Given any smooth action

$$\mathbb{S}^3 \times M \longrightarrow M$$

then, we have the following *Gysin sequence*:

$$\longrightarrow H^i(M/\mathbb{S}^3) \longrightarrow H^i(M) \longrightarrow H^{i-3}(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3) \oplus \left(H^{i-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$$

$$\longrightarrow H^{i+1}(M/\mathbb{S}^3) \longrightarrow,$$

- $\Sigma \equiv$ points whose isotropy is not finite;
- the $\mathbb{Z}^2$-action is induced by $j \in \mathbb{S}^3$;
- $(-)^{-\mathbb{Z}_2}$ stands for the subspace of antisymmetric elements.

Stratification of M induced by the $\mathbb{S}^3$-action

- Only possible isotropy subgroups (up to conjugation) [G.E. Bredon, 1972]:
  - finite;
  - $\mathbb{S}^1$
  - $N(\mathbb{S}^1) \cong \mathbb{S}^1 \cup j \cdot \mathbb{S}^1$;
  - $\mathbb{S}^3$
- $F = M^{\mathbb{S}^3}$; $\Sigma = \{x \in M : \ dim\ \mathbb{S}^3_x \geq 1\}$
- Stratification:

$$F \subseteq \Sigma \subseteq M$$

- Σ is not a smooth manifold, but F, $M \backslash \Sigma$ and $\Sigma \backslash F$ are smooth submanifolds.
- Same situation on $M/\mathbb{S}^3$.

Tool for singularities: Verona cohomology.

- $M/\mathbb{S}^3$ stratified pseudomanifold.
- We can use *Verona cohomology*: $\omega \in \Omega^*(M \backslash \Sigma)$ is a Verona form if there exist
  - $\omega_1 \in \Omega^*(\Sigma \backslash F)$
  - $\omega_0 \in \Omega^*(F)$

  which are compatible in some sense (blow up, tubular neighborhoods ...)
- Verona forms: $\Omega_v^*(M)$, $\Omega_v^*(M/\mathbb{S}^3)$
- $H_v^*(M) \cong H^*(M)$, $H_v^*(M/\mathbb{S}^3) \cong H^*(M/\mathbb{S}^3)$ [A. Verona, 1971]
- Analogously, $H_v^*(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3) \cong H^*(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3)$

How do $H^*(M)$ and $H^*(M/\mathbb{S}^3)$ appear?

- We use $\mathbb{S}^3$-invariant forms: $\Omega^*(M)^{\mathbb{S}^3}$
- $H^*(\Omega^*_v(M)^{\mathbb{S}^3}) \cong H^*(M)$
- **Gysin problem**: The short exact sequence

$$0 \longrightarrow \Omega^*_v(M/\mathbb{S}^3) \xhookrightarrow{I} \Omega^*_v(M)^{\mathbb{S}^3} \longrightarrow \mathrm{Coker}\, I \longrightarrow 0$$

- gives the long exact sequence

$$\longrightarrow H^i(M) \longrightarrow H^i(\mathrm{Coker}\, I) \longrightarrow H^{i+1}(M/\mathbb{S}^3) \longrightarrow H^{i+1}(M) \longrightarrow \cdots$$

How do $H^*(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3)$ and $\left(H^{i-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$ appear?

Integration along the fibres operator:

$$\int : \mathrm{Coker}\, I \longrightarrow \Omega_v^{*-3}(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3)$$

we obtain the short exact sequence:

$$0 \longrightarrow \mathrm{Ker} \int \hookrightarrow \mathrm{Coker}\, I \xrightarrow{\int} \Omega_v^{*-3}(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3) \longrightarrow 0$$

By a suitable decomposition of invariant forms, we get that this short exact sequence splits, and so we just have to prove that

$$H^*\left(\mathrm{Ker} \int\right) = \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$$

Excision

Using excision techniques, we get (here, $\operatorname{Ker} f = A(M)$):

$$H^*(A^*(M)) = H^*(A^*(\Sigma)) = H^*(A^*(\Sigma, F)) = \frac{H^*(\mathscr{E}, U)}{H^*(\mathscr{E}/\mathbb{S}^3, U/\mathbb{S}^3)}$$

where $\mathscr{E}$ is the subset of points whose isotropy group is conjugated to $\mathbb{S}^1$ and U a suitable neighborhood of F.

Künneth

We have the homeomorphisms:

$$\mathscr{E} = \mathbb{S}^3 \times_{N(\mathbb{S}^1)} \mathscr{E}^{\mathbb{S}^1} = (\mathbb{S}^3/\mathbb{S}^1) \times_{N(\mathbb{S}^1)/\mathbb{S}^1} \mathscr{E}^{\mathbb{S}^1} = \mathbb{S}^2 \times_{\mathbb{Z}_2} \mathscr{E}^{\mathbb{S}^1}$$

Hopf action $\varphi \colon \mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{S}^3$ given by

$$z \cdot (z_1, z_2) = (z \cdot z_1, z \cdot z_2),$$

The quotient $\mathbb{S}^3/\mathbb{S}^1 \cong \mathbb{S}^2$.
The group $N(\mathbb{S}^1)/\mathbb{S}^1 = \mathbb{Z}_2$ acts on $\mathbb{S}^2$ by:

$$j \cdot (x_0, x_1, x_2) = (x_0, x_1, -x_2).$$

Non orientable!

Künneth

We have the homeomorphisms:

$$\mathscr{E} = \mathbb{S}^3 \times_{N(\mathbb{S}^1)} \mathscr{E}^{\mathbb{S}^1} = (\mathbb{S}^3/\mathbb{S}^1) \times_{N(\mathbb{S}^1)/\mathbb{S}^1} \mathscr{E}^{\mathbb{S}^1} = \mathbb{S}^2 \times_{\mathbb{Z}_2} \mathscr{E}^{\mathbb{S}^1}$$

So, by Künneth, the cohomlogy $H^*(\mathscr{E}, U)$ is:

$$\begin{aligned}
& H^*(\mathbb{S}^2 \times_{\mathbb{Z}_2} \mathscr{E}^{\mathbb{S}^1}, \mathbb{S}^2 \times_{\mathbb{Z}_2} U^{\mathbb{S}^1}) = \Big(H^*(\mathbb{S}^2) \otimes H^*(\mathscr{E}^{\mathbb{S}^1}, U^{\mathbb{S}^1})\Big)^{\mathbb{Z}_2} \\
= \; & \Big(H^0(\mathbb{S}^2) \otimes H^*(\mathscr{E}^{\mathbb{S}^1}, U^{\mathbb{S}^1})\Big)^{\mathbb{Z}_2} \oplus \Big(H^2(\mathbb{S}^2) \otimes H^{*-2}(\mathscr{E}^{\mathbb{S}^1}, U^{\mathbb{S}^1})\Big)^{\mathbb{Z}_2} \\
= \; & \Big(H^*(\mathscr{E}^{\mathbb{S}^1}, U^{\mathbb{S}^1})\Big)^{\mathbb{Z}_2} \oplus \Big(H^{*-2}(\mathscr{E}^{\mathbb{S}^1}, U^{\mathbb{S}^1})\Big)^{-\mathbb{Z}_2} = \cdots \\
= \; & H^*(\mathscr{E}/\mathbb{S}^3, U/\mathbb{S}^3) \oplus \Big(H^{*-2}\Big(M^{\mathbb{S}^1}\Big)\Big)^{-\mathbb{Z}_2}
\end{aligned}$$

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Multiplication. Action on $\mathbb{S}^2$

- The Hopf action $\varphi\colon \mathbb{S}^1 \times \mathbb{S}^3 \longrightarrow \mathbb{S}^3$ given by

$$z \cdot (z_1, z_2) = (z_1 \cdot z, z_2 \cdot z),$$

- The quotient $\mathbb{S}^2 \cong \mathbb{S}^3/\mathbb{S}^1$.
- The action $\varphi_\mu\colon \mathbb{S}^3 \times \mathbb{S}^2 \to \mathbb{S}^2$ given by

$$g \cdot < h >=< g \cdot h >$$

is a transitive action. All isotropy groups are conjugated to $\mathbb{S}^1$.

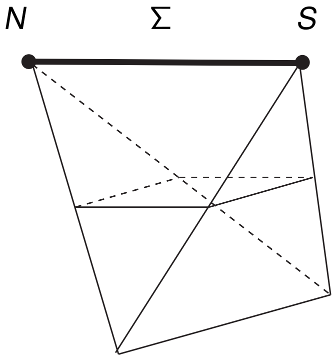
Singular action of $\mathbb{S}^3$ on itself

- $\mathbb{S}^3 = \Sigma\mathbb{S}^2 = \mathbb{S}^0 * \mathbb{S}^2$ (suspension).
- The action $\varphi\colon \mathbb{S}^3 \times \mathbb{S}^3 \longrightarrow \mathbb{S}^3$ is given by

$$g \cdot < t, x > = < t, g \cdot x > .$$

- $F = \{N, S\}$
- $M^{\mathbb{S}^1} \cong \mathbb{S}^1$
- $j(z) = \overline{z} \Rightarrow \boxed{\left(H^*(M^{\mathbb{S}^1}\right)^{-\mathbb{Z}_2} \cong H^1(\mathbb{S}^1)}$.

Action on $\mathbb{S}^7$

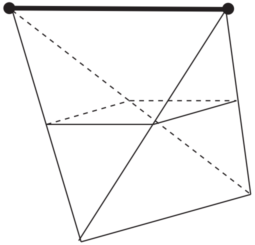


- $M = \mathbb{S}^7 = \mathbb{S}^3 * \mathbb{S}^3$ (join).
- The action $\psi : \mathbb{S}^3 \times \mathbb{S}^7 \longrightarrow \mathbb{S}^7$ is given by

$$g \cdot < z, \omega > = < g \cdot z, \varphi(g, \omega) > .$$

- $\Sigma = \mathbb{S}^3$
- $F = \{N, S\}$
- $M^{\mathbb{S}^1} \cong \mathbb{S}^1$.

Gysin sequence of ψ



$$H^0(M/\mathbb{S}^3) \cong H^0(M)$$

$$H^4(M/\mathbb{S}^3) \cong H^1\left(M^{\mathbb{S}^1}\right)^{-\mathbb{Z}_2} \cong H^1(\mathbb{S}^1)$$

$$H^{else}(M/\mathbb{S}^3) \cong 0$$

$$H^4(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3) \cong H^7(\mathbb{S}^7)$$

$$H^{else}(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3) \cong 0$$

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Spectral sequence of $\mathscr{F}$. Duality

$(M, \mathscr{F})$ Riemannian foliation (fiber bundle, free compact Lie group action....)

Filtering $\Omega^*(M)$ by the basic (transverse) degree, we get a first quadrant spectral sequence:

$$E_r^{s,t}(\mathscr{F}) \Rightarrow H^*(M)$$

If the Riemannian foliation (of dimension k and codimension l) is regular, then the duality property is satisfied [J.A.Álvarez, 1989]

$$E_2^{s,t}(\mathscr{F}) \cong E_2^{k-s,l-t}(\mathscr{F})$$

Singular Riemannian foliation

$(M, \mathscr{F})$ singular Riemannian foliation (compact Lie group action, ...).

This duality exists?

Singular Riemannian foliation

$(M, \mathscr{F})$ singular Riemannian foliation (compact Lie group action, ...).

This duality exists?

Yes for $\mathbb{S}^1$-actions.

Singular Riemannian foliation

$$
\begin{tikzcd}
H^*_{\overline{-1}}(M/\mathbb{S}^1) \arrow[r] \arrow[rrr, bend left] \arrow[d, equal] & H^*_{\overline{0}}(M/\mathbb{S}^1) \arrow[r, "\cdots\cdots"] \arrow[r, bend left, "\textit{Intersection Poincaré Duality}"] \arrow[d, equal] & H^*_{\overline{t}}(M/\mathbb{S}^1) \arrow[r] & H^*_{\overline{\infty}}(M/\mathbb{S}^1) \arrow[d, equal] \\
H^*(M/\mathbb{S}^1, F) \arrow[rrr, bend right, "\textit{Lefschetz Duality}"'] & H^*(M/\mathbb{S}^1) & & H^*((M\backslash F)/\mathbb{S}^1)
\end{tikzcd}
$$

Singular Riemannian foliation

The second term $E_2^{s,t}(\mathscr{F})$ of the spectral sequence associated to an action $\varphi \colon \mathbb{S}^1 \times M \to M$ is

t

$t = 1$ $\qquad H^{*-2}(M/\mathbb{S}^1, F)$ $\qquad\qquad$ $H_{\bar{t}}^{*-2}(M/\mathbb{S}^1)$

$t = 0$ $\qquad H^{*}(M/\mathbb{S}^1)$ $\qquad\qquad$ $H^{*}((M \backslash F)/\mathbb{S}^1)$

s $\qquad\qquad$ s

$\overline{p} = \overline{0}$ $\qquad\qquad$ $\overline{p} = \overline{t}$

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$(M, \mathscr{F})$ singular Riemannian foliation (compact Lie group action, ...).

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Singular Riemannian foliation

$(M, \mathscr{F})$ singular Riemannian foliation (compact Lie group action, ...).

This duality exists?

Not in general...

The second term $E_2^{s,t}(\mathscr{F})$ of the spectral sequence associated to an action $\varphi\colon \mathbb{S}^3 \times M \to M$ is

$$\begin{tikzcd}
t & \\
t = 3 & H^{*-3}(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3) \arrow[ddd, bend left] \\
t = 2 & \left(H^{*-2}(M^{\mathbb{S}^1})\right)^{-\mathbb{Z}_2} \arrow[d, bend right] \\
t = 1 & 0 \\
t = 0 & H^*(M/\mathbb{S}^3) \\
 & & s
\end{tikzcd}$$

The second term $E_2^{s,t}(\mathscr{F})$ of the spectral sequence associated to an action $\varphi\colon \mathbb{S}^3 \times M \to M$ is

$$
\begin{array}{c|c}
t & \\
t=3 & H^{*-3}(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3) \\
t=2 & \left(H^{*-2}(M^{\mathbb{S}^1})\right)^{-\mathbb{Z}_2} \\
t=1 & 0 \\
t=0 & H^*(M/\mathbb{S}^3) \\
\hline
 & s
\end{array}
$$

Duals by using intersection cohomology

The second term $E_2^{s,t}(\mathscr{F})$ of the spectral sequence associated to an action $\varphi\colon \mathbb{S}^3 \times M \to M$ is