Filtered Intersection (co)-homology and Poincaré Duality

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Poincaré Duality. Goresky & McPherson

$$\pitchfork : H_*^{\overline{p}}(X;\mathbb{Q}) \times H_{n-*}^{D\overline{p}}(X;\mathbb{Q}) \to \mathbb{Q}$$

intersection pairing non degenerate

$\overline{p}$ GM-perversity, $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$

Poincaré Duality : homology/cohomology

$$\cap\,[X]\colon H^{*}_{D\overline{p}}(X;R) \xrightarrow{\cong} H^{\overline{p}}_{n-*}(X;R)$$

Poincaré Duality : homology/cohomology

$$\cap\,[X]\colon H^{*}_{D\overline{p}}(X;R) \xrightarrow{\cong} H^{\overline{p}}_{n-*}(X;R)$$

- $[X] \in H^{\overline{0}}_{n}(X;R)$ fundamental class.
- $\overline{p} \leqslant \overline{t}$ (top perversity).
- R is a field (Friedman-McClure)
- $(X;R)$ is a locally $(\overline{p};R)$-torsion free (Friedman)
- $(X;R) = (\Sigma\mathbb{RP}^3;\mathbb{Z})$ no Poincaré duality.

Goal

Introduce a new version of the intersection cohomology: *Thom-Witney cohomology.*

$$\cap\,[X]\colon H^*_{TW,\overline{p}}(X;R) \xrightarrow{\cong} H^{\overline{p}}_{n-*}(X;R),$$

- R any coefficient ring, $\overline{p} \leqslant \overline{t}$.

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Introduce a new version of the intersection cohomology: *Thom-Witney cohomology.*

$$\cap\,[X]\colon H^{*}_{TW,\overline{p}}(X;R) \xrightarrow{\cong} H^{\overline{p}}_{n-*}(X;R),$$

- R any coefficient ring, $\overline{p} \leqslant \overline{t}$.
- $H^{*}_{TW,\overline{p}}(X;R) = H^{*}_{D\overline{p}}(X;R)$ in the "locally torsion free" case.
- Cup/Cap product, Lefschetz Duality, topological invariance, Sullivan minimal models, ...

Idea of the construction of $H^*_{TW,\overline{p}}(X;R)$

Locally system over a simplicial set of Halperin

$$\sigma : \Delta \to X$$

$$\rightsquigarrow \textit{compatible way}$$

"Simplicial sheaf" *Cochain complex*

Idea of the construction of $H^*_{TW,\overline{p}}(X;R)$

Locally system over a simplicial set of Halperin

Linear filtration $\quad\quad \sigma : \Delta \to X \quad\quad$ *Filtered simplex*

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ *compatible way*

"*Simplicial sheaf*" $\quad\quad$ *Cochain complex*

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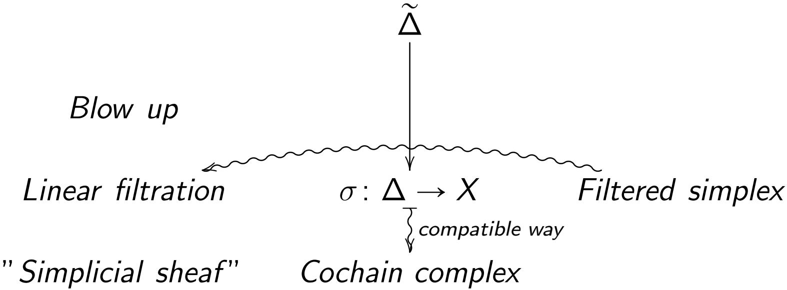
Locally system over a simplicial set of Halperin



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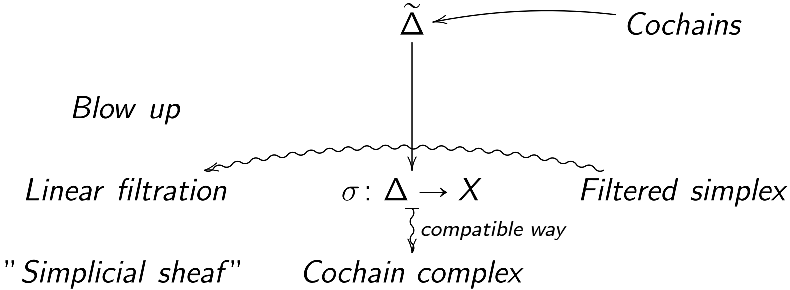

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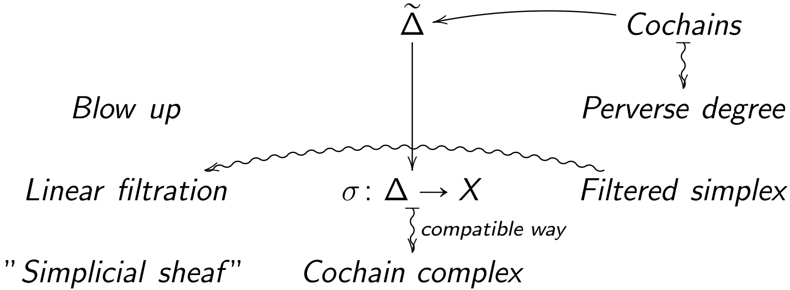

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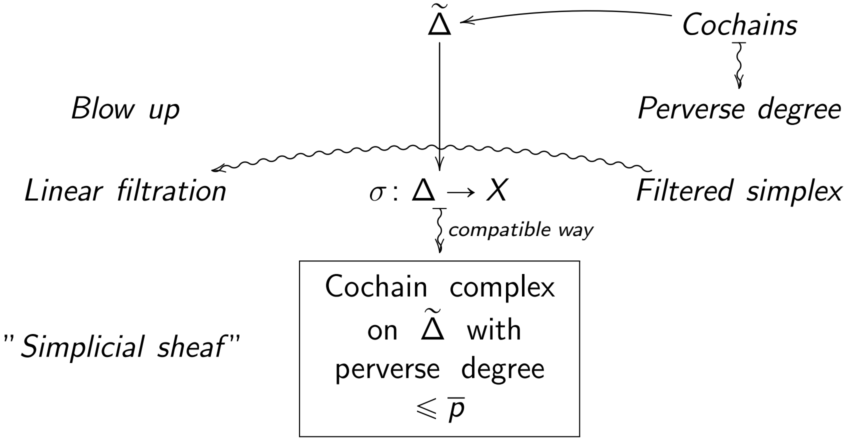


TW-cohomology construction : Filtered simplexes

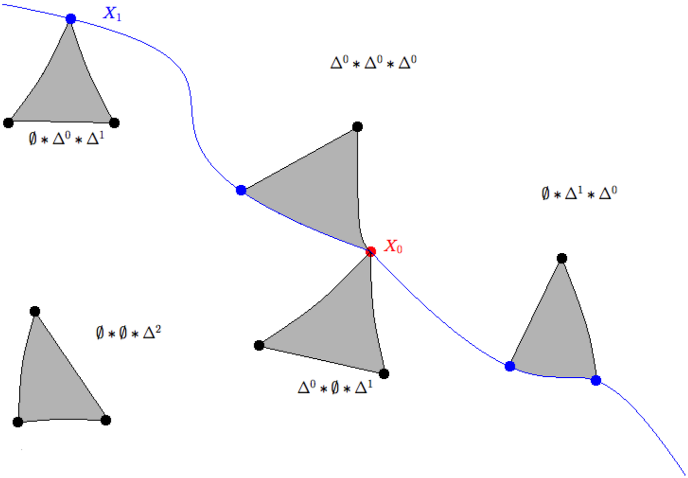
$$\sigma\colon \Delta \to X$$

- $\emptyset \subset X_0 \subset \cdots \subset X_{n-1} \subsetneq X_n = X$, filtered space.
- $\sigma^{-1}(X_k)$ is a face of Δ
- $\Delta = \underbrace{\Delta_0 * \cdots * \Delta_k}_{\sigma^{-1}(X_k)} * \cdots * \Delta_n$

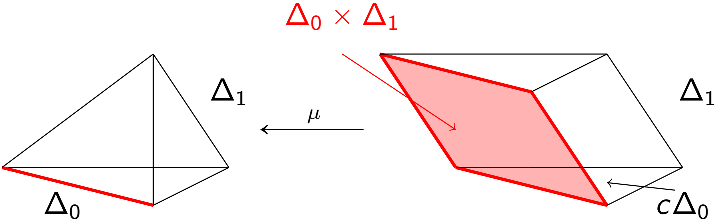
Intersection homology $H_*^{\overline{p}}(X;R)$ is computed with filtered simplexes.

$\overline{p}$ perversity. R coefficient ring.

TW-cohomology construction : Filtered simplexes



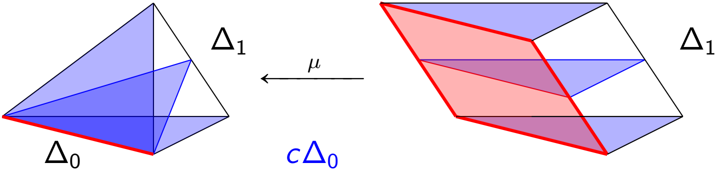
TW-cohomology construction : Blow up



$$\Delta = \Delta_0 * \Delta_1 \quad \text{has for blow-up} \quad \widetilde{\Delta} = c\Delta_0 \times \Delta_1$$

First example. First approach.

TW-cohomology construction : Blow up



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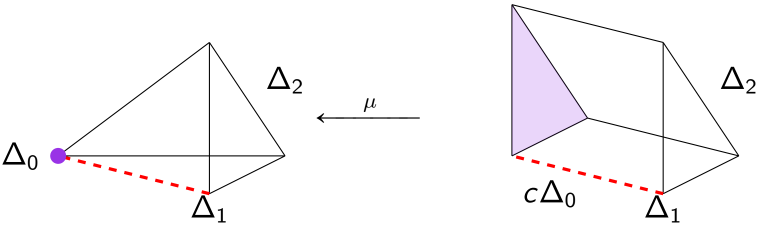
First example. Second approach.

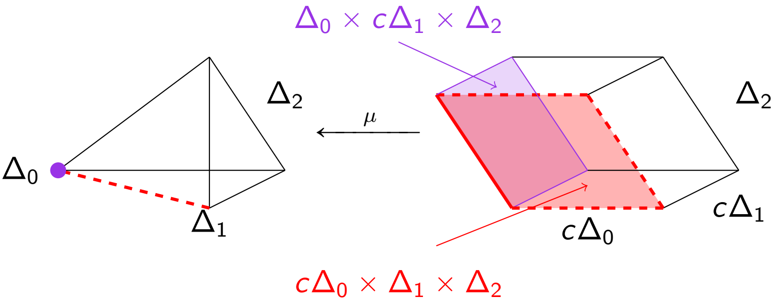
TW-cohomology construction : Blow up



$$\Delta = \Delta_0 * \Delta_1 * \Delta_2$$

Second example. First step

TW-cohomology construction : Blow up



$\Delta = \Delta_0 * \Delta_1 * \Delta_2$ has for blow-up $\widetilde{\Delta} = c\Delta_0 \times c\Delta_1 \times \Delta_2$

Second example. Second step

TW-cohomology construction : Blow up

$$\mu\colon c\Delta_0 \times \cdots \times c\Delta_{n-1} \times \Delta_n \to \Delta_0 * \cdots * \Delta_n$$

- $\mu(\text{Face}) = (\text{Face})$ same dimension except

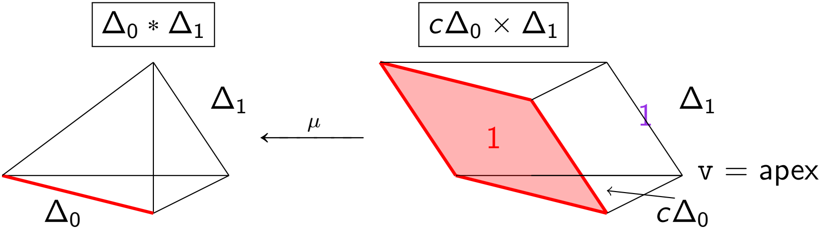
- $$\underbrace{\overbrace{c\Delta_0 \times \cdots \times c\Delta_{k-1} \times \Delta_k}^{blow\ \ up\ \ of\,\Delta_0*\cdots*\Delta_k} \times \overbrace{c\Delta_{k+1} \times \cdots \times c\Delta_{n-1} \times \Delta_n}^{fiber}}_{Hidden\ \ faces}$$

$$\stackrel{\mu}{\longrightarrow}$$

$$\Delta_0 * \cdots * \Delta_k$$

- $\partial\widetilde{\Delta} = \widetilde{\partial\Delta} +$ Hidden faces

TW-cohomology construction : Perverse degree



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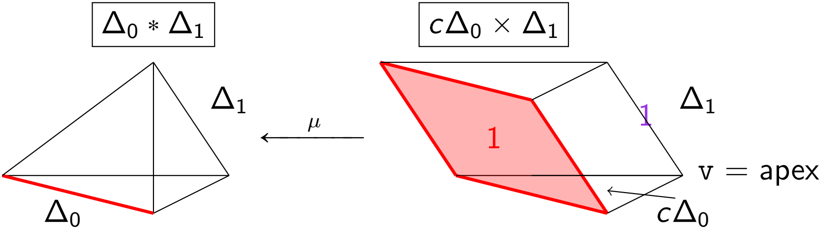

- $||\mathbf{1}_{\mathrm{v}\times\Delta_1}|| = -\infty$ Not hidden faces

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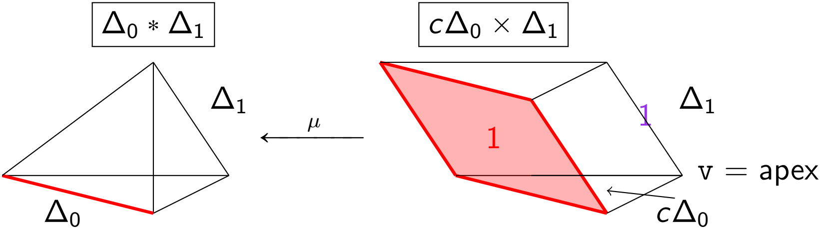

- $||\mathbf{1}_{\mathrm{v}\times\Delta_1}|| = -\infty$ Not hidden faces
- $||\mathbf{1}_{\Delta_0\times\Delta_1}|| = \dim \Delta_1$ Hidden faces

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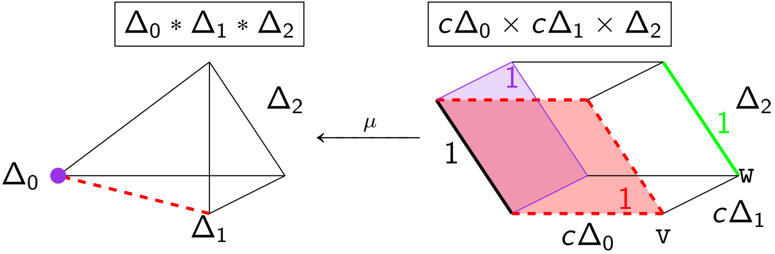

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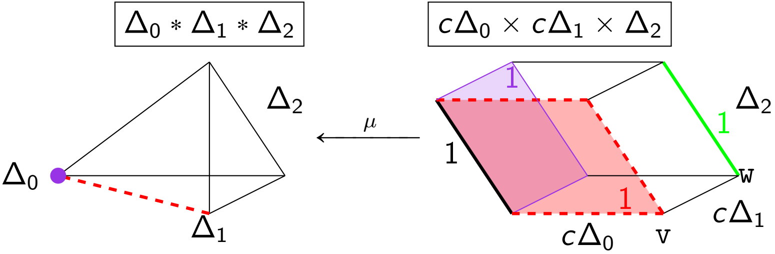

- $||\mathbf{1}_{\mathtt{v}\times\mathtt{w}\times\Delta_2}|| = (-\infty, -\infty)$

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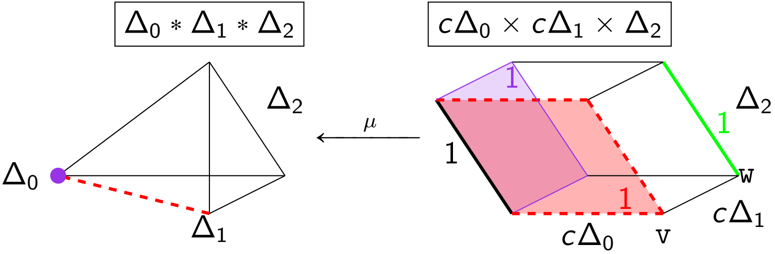

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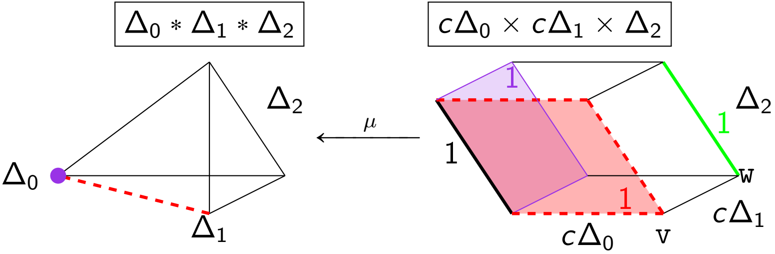

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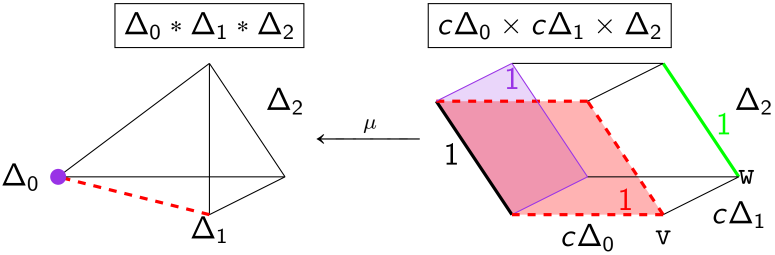

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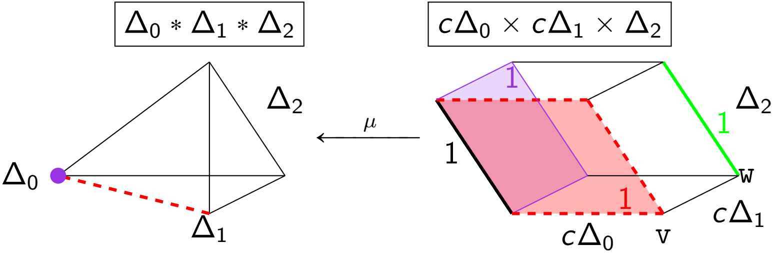

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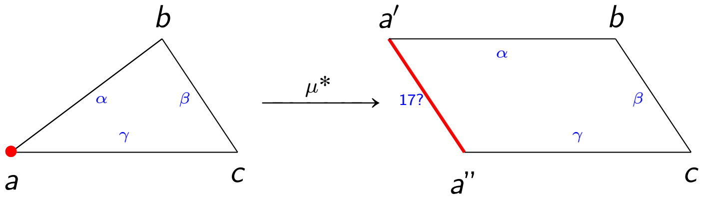
- $||\mathbf{1}_{\Delta_0\times\Delta_1\times\Delta_2}|| = (\dim \Delta_1 + \dim \Delta_2, \dim \Delta_2)$
- $||-|| = (||-||_0, ||-||_1)$.

TW-cohomology construction:TW cochains on Δ

- $\widetilde{N}^*(\Delta; R) = N^*(c\Delta_0) \otimes \cdots \otimes N^*(c\Delta_{n-1}) \otimes N^*(\Delta_n)$
- $\widetilde{N}^*_{\overline{p}}(\Delta; R) = \left\{\omega \in \widetilde{N}^*(\Delta; R) / \max(||\omega||_k, ||d\omega||_k) \leqslant p_k\right\}$
- $\widetilde{N}^*_{\overline{0}}(\Delta; R) \cong N^*(\Delta; R)$.

TW-cohomology construction:TW cochains on Δ

$$N^*(\Delta; R) \qquad \cong \qquad \widetilde{N}^*_{\bar{0}}(\Delta; R)$$



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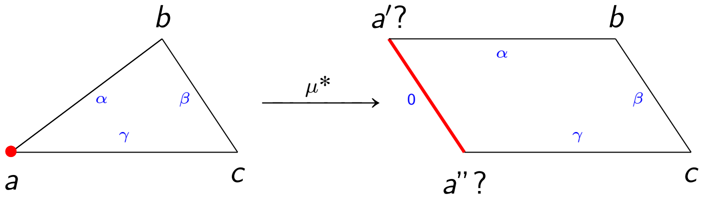

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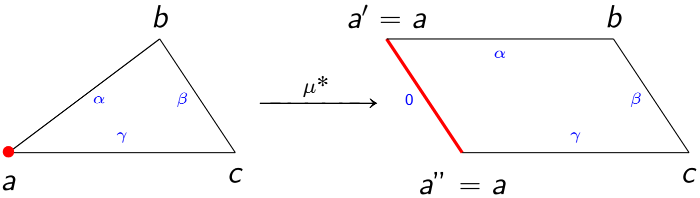

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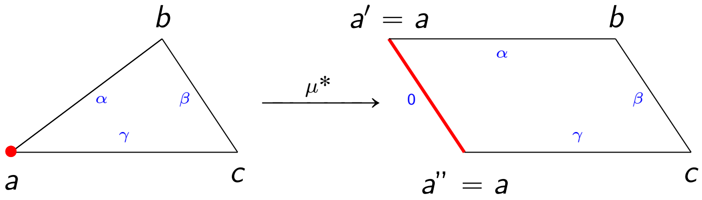

TW-cohomology construction:TW cochains on X

- $\widetilde{N}^*_{\overline{p}}(X;R)$ is the simplicial sheaf

$$\left\{ (\omega_\sigma) \ / \ \omega_\sigma \in \widetilde{N}^*_{\overline{p}}(\Delta;R), \sigma\colon \Delta \to X \text{ regular simplex} \right\}.$$

  regular = filtered + $(\Delta_n \neq \emptyset)$.

- $H^*_{TW,\overline{p}}(X;R)$ Thom-Witney cohomology

- $H^*_{TW,\overline{0}}(X;R) = H^*(X;R)$, cohomology, when X normal.

TW-cohomology versus intersection cohomology

$$H^*_{TW,\overline{p}}(X;R) = H^*_{D\overline{p}}(X;R)$$

if $(X;R)$ is a $(\overline{p};R)$-torsion free.

TW-cohomology versus intersection cohomology

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Long exact sequence

$$\cdots \to H^*_{TW,\overline{p}}(X;R) \to H^*_{D\overline{p}}(X;R) \to \begin{array}{c}\text{Goresky-Siegel's}\\ \text{Peripheral term}\end{array} \to \dots$$

TW-cohomology versus intersection cohomology

X with isolated singularities $\{x_i\}_I$, $\dim X = n$, $\overline{p} = p$,

$$0 \longrightarrow H^{p+1}_{TW,\overline{p}}(X;R) \longrightarrow H^{p+1}_{D\overline{p}}(X;R) \longrightarrow \bigoplus_{i\in I} Ext\left(H_p(L_i;R);R\right)$$

$$\longrightarrow H^{p+2}_{TW,\overline{p}}(X;R) \longrightarrow H^{p+2}_{D\overline{p}}(X;R) \longrightarrow 0$$

X locally torsion free: $\bigoplus_{i\in I} Ext\left(H_p(L_i;R);R\right) = 0$

TW-cohomology properties : Cup product

$$\widetilde{N}^{i}_{\overline{p}}(X;R) \otimes \widetilde{N}^{j}_{\overline{q}}(X;R) \xrightarrow{\quad \cup \quad} \widetilde{N}^{i+j}_{\overline{p}+\overline{q}}(X;R).$$

$$(\omega \cup \eta)_\sigma = \omega_\sigma \cup \eta_\sigma$$

This cup product is computed in

$$\widetilde{N}^*(\Delta;R) = N^*(c\Delta_0;R) \otimes \cdots \otimes N^*(c\Delta_{n-1};R) \otimes N^*(\Delta_n;R),$$

where $\sigma\colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$ regular simplex.

$$\underbrace{(\alpha_1 \otimes \cdots \otimes \alpha_n)}_{\omega_\sigma} \cup \underbrace{(\beta_1 \otimes \cdots \otimes \beta_n)}_{\eta_\sigma} = \underbrace{\pm(\alpha_1 \cup \beta_1) \otimes \cdots \otimes (\alpha_n \cup \beta_n)}_{\omega_\sigma \cup \ \eta_\sigma}$$

TW-cohomology properties : Cap product

In a similar way, we define the cap product

$$H^{i}_{TW,\overline{p}}(X;R) \otimes H^{\overline{q}}_{j}(X;R) \xrightarrow{\quad\cap\quad} H^{\overline{p}+\overline{q}}_{j-i}(X;R).$$

TW-cohomology properties : Dualities

Poincaré

$$\cap[X]\colon H^{*}_{TW,\overline{p}}(X;R) \xrightarrow{\cong} H^{\overline{p}}_{n-*}(X;R)$$

X is a n-dimensional oriented compact pseudomanifold.
$[X] \in H^{\overline{0}}_{n}(X;R)$ fundamental class and R is a ring and $\overline{p} \leqslant \overline{t}$.

TW-cohomology properties : Dualities

Lefschetz

$$\cap[X]\colon H^{*}_{TW,\overline{p}}(X,\partial X;R) \xrightarrow{\cong} H^{\overline{p}}_{n-*}(X;R)$$

X is a n-dimensional oriented compact ∂-pseudomanifold.
$[X] \in H^{\overline{0}}_{n}(X,\partial X;R)$ fundamental class and R is a ring and $\overline{p} \leqslant \overline{t}$.

TW-cohomology properties:Topological Invariance

(1) $H^*_{TW,\overline{p}}(X;R)$, independent of the stratification.

with

(1) $\overline{p}$ GM-perversity: $\overline{p}(k) \leqslant \overline{p}(k+1) \leqslant \overline{p}(k)+1$.

TW-cohomology properties:Topological Invariance

(1) $H^*_{TW,\overline{p}}(X;R)$, independent of the stratification.

(2) $H^*_{TW,\overline{p}}(X;R) \cong H^*_{TW,\overline{p}*}(X^*;R)$, X^* intrinsic stratification.

$H^*_{TW,\overline{p}}(X;R) \cong H^*_{TW,\overline{p}'}(X';R)$, X' refinement of X.

with

(1) $\overline{p}$ GM-perversity: $\overline{p}(k) \leqslant \overline{p}(k+1) \leqslant \overline{p}(k)+1$.

(2) $\overline{p}$ general perversity:

$$\overline{p}(S') \leqslant \overline{p}(S) \leqslant \overline{p}(S') + \mathrm{codimS} - \mathrm{codimS'}$$

if $S \preceq S'$.

Goresky & Pardon conjecture

$$\begin{array}{ccc} & & H^{r+i}_{\mathcal{L}(\overline{p},i)}(X;\mathbb{Z}_2) \\ & \nearrow & \downarrow \\ H^r_{\overline{p}}(X;\mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^i} & H^{r+i}_{2\overline{p}}(X;\mathbb{Z}_2) \end{array}$$

$$\mathcal{L}(\overline{p},i)(\ell) = \min(2\overline{p}(\ell), \overline{p}(\ell) + i)$$

Effective improvement

Goresky & Pardon conjecture

$$\begin{array}{ccc} & & H^{r+i}_{TW,\mathcal{L}(\overline{p},i)}(X;\mathbb{Z}_2) \\ & \nearrow & \downarrow \\ H^{r}_{TW,\overline{p}}(X;\mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^i} & H^{r+i}_{TW,2\overline{p}}(X;\mathbb{Z}_2) \end{array}$$

$$\mathcal{L}(\overline{p},i)(\ell) = \min(2\overline{p}(\ell), \overline{p}(\ell) + i)$$

Effective improvement

Sullivan minimal models : Regular case

$$A^*_{PL}(\Delta, \mathbb{Q}) = \Lambda(x_1, \ldots, x_m, dx_1, \ldots, dx_m),$$

where $(x_0, \ldots, x_m)$ are the barycentric coordinates of Δ.

$A^*_{PL}(X; \mathbb{Q})$ is the simplicial sheaf

$$\left\{(\omega_\sigma) \;/\; \omega_\sigma \in A^*_{PL}(\Delta, \mathbb{Q}), \sigma \colon \Delta \to X \text{ singular simplex}\right\}$$

Sullivan minimal models : Regular case

- $A^*_{PL}(X;\mathbb{Q})$ is a DGCA computing $H^*(X;\mathbb{Q})$
- There exists a minimal model $(\Lambda V, d) \xrightarrow{\cong} A^*_{PL}(X;\mathbb{Q})$.
- It contains the rational cohomology of X since $H^*(\Lambda V, d) = H^*(X;\mathbb{Q})$.
- It contains the rational homotopy of X since $\hom_{\mathbb{Q}}(V^k, \mathbb{Q}) = \pi_k(X) \otimes \mathbb{Q}$.

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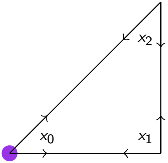
What about intersection homotopy?

Sullivan minimal models : Singular case

$$\Delta = \Delta_0 * \Delta_1$$



Differential forms on Δ			
$-x_2 dx_1 + x_1 dx_2$			

Sullivan minimal models : Singular case

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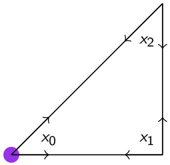
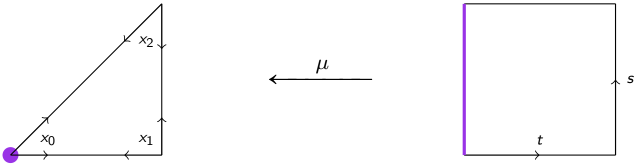


Differential forms on Δ			
$-x_2 dx_1 + x_1 dx_2$			
$\dfrac{-x_2 dx_1 + x_1 dx_2}{(x_1 + x_2)^2}$			

Sullivan minimal models : Singular case

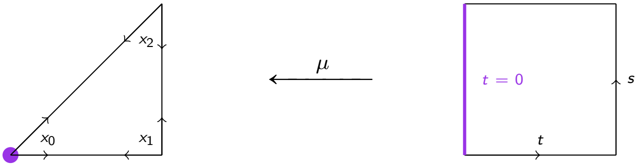
$$\Delta = \Delta_0 * \Delta_1 \qquad \widetilde{\Delta} = c\Delta_0 \times \Delta_1$$



Differential forms on Δ	μ^*	Differential forms on $\widetilde{\Delta}$	
$-x_2 dx_1 + x_1 dx_2$	$\to$	ds	
$\dfrac{-x_2 dx_1 + x_1 dx_2}{(x_1 + x_2)^2}$	$\to$	$t^2\ ds$	

Sullivan minimal models : Singular case

$$\Delta = \Delta_0 * \Delta_1 \qquad\qquad \widetilde{\Delta} = c\Delta_0 \times \Delta_1$$



Differential forms on Δ	μ^*	Differential forms on $\widetilde{\Delta}$	Perverse degree $(t = 0)$
$-x_2 dx_1 + x_1 dx_2$	$\rightarrow$	ds	1
$\dfrac{-x_2 dx_1 + x_1 dx_2}{(x_1 + x_2)^2}$	$\rightarrow$	$t^2\ ds$	$-\infty$

Minimal models : Singular case

- $\widetilde{A}^*_{PL}(\Delta;\mathbb{Q}) = A^*_{PL}(c\Delta_0;\mathbb{Q}) \otimes \cdots \otimes A^*_{PL}(c\Delta_{n-1};\mathbb{Q}) \otimes A^*_{PL}(\Delta_n;\mathbb{Q})$.

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Perverse minimal models

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Any nodal hypersurface of $\mathbb{CP}^4$ is intersection-formal. •

Intersection cohomology with differential forms

- $\mathcal{L}^2$- differential forms (Cheeger).

$\mathcal{L}^2$-intersection cohomology

$$H^*_{(2)}(X\backslash\Sigma, \mu_{\overline{p}}) \quad \cong \quad H^*_{\overline{p}}(X, \mathbb{R})$$

Metric $\mu_{\overline{p}}$ locally defined on

Regular part $\left(]-1,1[^{n-k} \times \mathring{c}L\right) =]-1,1[^{n-k} \times (L\backslash\Sigma_{_L}) \times]-1,1[$

by

$$\mu_{\overline{p}} \overset{q.i.}{\sim} dx_1^2 + \cdots + dx_{n-k}^2 + r^{2a_k}\mu^{^L}_{\overline{p}} + dr^2,$$

a_k depending on $\overline{p}_k$.

Intersection cohomology with differential forms

- $\mathcal{L}^2$- differential forms (Cheeger).
- Controlled differential forms using a Thom-Mather neighborhoods system (Brylinsky-Goresky-McPherson).
- Desingularized differential forms (MS). •

deRham intersection cohomology

Unfoldable stratified pseudomanifolds

Local structure | Blow up

$$\begin{array}{ccc}
\mathbb{R}^m \times \widetilde{L} \times]-1,1[& \xrightarrow{\tilde{\varphi}} & \widetilde{X} \\
\Big\downarrow Q & & \Big\downarrow \mathcal{L} \\
\mathbb{R}^m \times \mathring{c}L & \xrightarrow{\varphi} & X
\end{array}$$

smooth manifold

continuous map

stratified pseudomanifold

deRham intersection cohomology

Unfoldable stratified pseudomanifolds

$$
\begin{array}{ccc}
\widetilde{X} & \hookleftarrow & \mathcal{L}^{-1}(X\backslash\Sigma) \\
\Big\downarrow \mathcal{L} & & \Big\downarrow \mathcal{L} \\
X & \hookleftarrow & X\backslash\Sigma
\end{array}
$$

smooth manifold

smooth trivial covering

smooth manifold

deRham intersection cohomology

Liftable differential forms

$\widetilde{\omega}$

$$
\begin{array}{ccccc}
\widetilde{X} & \hookleftarrow & \mathcal{L}^{-1}(X\backslash\Sigma) & & \mathcal{L}^*\omega \\
\downarrow \mathcal{L} & & \downarrow \mathcal{L} & & \\
X & \hookleftarrow & X\backslash\Sigma & & \omega
\end{array}
$$

deRham intersection cohomology

Perverse degree

Bundle with fiber $\widetilde{L}$

Blow up

$$\begin{array}{ccc}
\mathcal{L}^{-1}(S) & \xrightarrow{\tilde{\varphi}} & \widetilde{X} \\
\downarrow \mathcal{L} & & \downarrow \mathcal{L} \\
S & \xrightarrow{\varphi} & X
\end{array}$$

$\widetilde{\omega}$ lifting

ω liftable differential form

stratum

deRham intersection cohomology

Perverse degree

Bundle with fiber $\widetilde{L}$ Blow up

$$\begin{array}{ccc} \mathcal{L}^{-1}(S) & \xrightarrow{\ \tilde{\varphi}\ } & \widetilde{X} \\ \Big\downarrow{\scriptstyle \mathcal{L}} & & \Big\downarrow{\scriptstyle \mathcal{L}} \\ S & \xrightarrow{\ \varphi\ } & X \end{array}$$

stratum

$\widetilde{\omega}$ lifting

$$||\omega||_S = \text{ degree of } \widetilde{\omega} \text{ on } \widetilde{L}$$

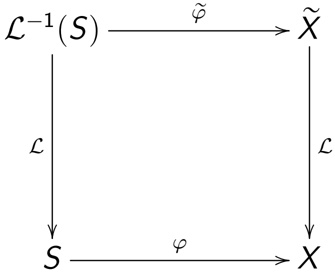
ω liftable differential form

deRham intersection cohomology

Intersection differential forms

Bundle with fiber $\widetilde{L}$

Blow up



stratum

$\widetilde{\omega}$ lifting

$||\omega||_S =$ degree of $\widetilde{\omega}$ on $\widetilde{L}$

ω liftable differential form

$$\Omega^*_{\overline{p}}(X) = \{\omega / ||\omega||_S, ||d\omega||_S \leqslant \overline{p}(S)\}$$

deRham intersection cohomology

$$H^*\left(\Omega^{\cdot}_{\overline{p}}(X)\right) \quad \cong \quad H^*_{TW,\overline{p}}(X,\mathbb{R})$$

$$\wedge \qquad \leftrightsquigarrow \qquad \cup$$

THANKS FOR YOUR ATTENTION !