Minimality and Singular riemannian foliations

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Primus Hispanico-Gallicus Conventus de Scientiis Mathematicis.
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Geometrical minimality

The foliation $\mathcal{F}$ is taut if there is a metric μ on M such that every leaf of $\mathcal{F}$ is a minimal submanifold with respect to μ.

Cohomological minimality and regular riemannian foliations on compact manifolds

Let $\mathcal{F}$ be a (regular) riemannian foliation defined on a compact manifold M and let μ be a bundle-like metric.

+ The mean curvature form κ_μ can be supposed to be a basic (D.Domínguez) cycle (F.Kamber, P.Tondeur).
+ The tautness class $\boldsymbol{\kappa} = [\kappa_\mu] \in H^1(M/\mathcal{F})$ does not depend on the choice of μ (J.Álvarez López).

The following statements are equivalent:

- The foliation $\mathcal{F}$ is taut.
- The tautness class $\boldsymbol{\kappa}$ vanishes.
- $H^{codim \mathcal{F}}(M/\mathcal{F}) \neq 0$ ($\mathcal{F}$ transversally oriented) (X.Masa)

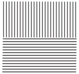
$$\Downarrow \quad \text{Poincaré Duality} \quad \left(\begin{array}{l} \text{F.Kamber, P.Tondeur} \\ \text{A. El Kacimi, G. Hector, V. Sergiescu} \end{array} \right)$$

- $H^0_{\kappa_\mu}(M/\mathcal{F}) \neq 0$.

Singular Riemannian Foliations (definition) Molino

A Singular Riemannian Foliation (SRF) on a manifold X is a partition $\mathcal{K}$ of X by submanifolds (leaves) verifying:

- The module of smooth vector fields tangent to the leaves is transitive on each leaf.
- There exists a bundle-like metric (i.e., a geodesic perpendicular to a leaf at a point remains perpendicular to every leaf it meets).



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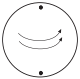
No



Yes

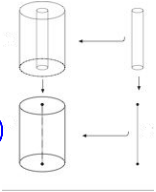
Singular riemannian foliations (properties)

Classifying the points of X by the dimension of the leaves, we get a stratification of X. The strata are ordered by

$$S_1 \preceq S_2 \Leftrightarrow S_1 \subseteq \overline{S_2}.$$

- The restriction of $\mathcal{K}$ to each stratum S is a regular riemannian foliation $\mathcal{K}_s$.
- The maximal stratum R, regular part, is an open dense subset of X.
- The other strata are called singular strata.
- The minimal strata are compact manifolds.
- Each stratum possesses a foliated tubular neighborhood
- Molino's desingularization

  Regular riemannian foliation $(\widetilde{X}, \widetilde{\mathcal{K}})$

  Singular riemannian foliation $(X, \mathcal{K})$

  S stratum

Cohomological minimality and singular riemannian foliations on non compact manifolds

A singular foliation on a compact manifold admitting an adapted Riemannian metric for which all leaves are minimal must be regular (V. Miquel,R. Wolak).

A regular foliation on a non compact manifold may not have a tautness class. (G.Cairns-R.Escobales).

Regular stratum. Theorem.

Let $\mathcal{K}$ be a singular riemann foliation defined on a compact manifold X and let R be the regular stratum. Then

- There exists a bundle-like metric μ on R whose mean curvature form κ_μ is a basic cycle.
- The tautness class $\boldsymbol{\kappa}_R = [\kappa_\mu] \in H^1(R/\mathcal{K})$ does not depend on the choice of μ

$\boxed{\text{The following statements are equivalent:}}$

- The foliation $(R, \mathcal{K})$ is taut.
- The tautness class $\boldsymbol{\kappa}_R$ vanishes.
- $H_c^{codim \mathcal{K}}(R/\mathcal{K}) \neq 0$ (when $(R, \mathcal{K})$ transversally oriented).
- $H^0_{\kappa_\mu}(R/\mathcal{K}) \neq 0$.
- The foliation $\left(\widetilde{X}, \widetilde{\mathcal{K}}\right)$ is taut.
- $H^{codim \mathcal{K}}(X/\mathcal{K}, \partial(X/\mathcal{K})) \neq 0$ (if $(R, \mathcal{K})$ trans. oriented).

Regular stratum. Theorem. Principle of the Proof

We lift the question to the blow up, which is a compact manifold endowed with a regular riemannian foliation.

Singular strata. Theorem.

Let $\mathcal{K}$ be a singular riemann foliation defined on a compact manifold X and let S be a stratum. Then

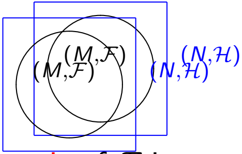
- There exists a bundle-like metric μ on S whose mean curvature form κ_μ is a basic cycle.
- The tautness class $\boldsymbol{\kappa}_S = [\kappa_\mu] \in H^1(S/\mathcal{K})$ does not depend on the choice of μ

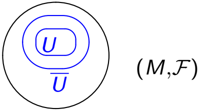
$\boxed{\text{The following statements are equivalent:}}$

- The foliation $(S, \mathcal{K})$ is taut.
- The tautness class $\boldsymbol{\kappa}_S$ vanishes.
- $H_c^{codim \mathcal{K}}(S/\mathcal{K}) \neq 0$ (when $(S, \mathcal{K})$ transversally oriented).
- $H^0_{\kappa_\mu}(S/\mathcal{K}) \neq 0$.

Let $\mathcal{F}$ be a regular riemannian foliation on a manifold M.

- A zipper of $\mathcal{F}$ is a compact manifold N endowed with a riemannian foliation $\mathcal{H}$ verifying the following property:
  - The manifold M is a saturated open subset of N and $\mathcal{H}_M = \mathcal{F}$.





- A reppiz of $\mathcal{F}$ is a saturated open subset U of M verifying the following properties:
  - The closure $\overline{U}$ (in M) is compact.
  - The inclusion $U \hookrightarrow M$ gives $H^*(U/\mathcal{F}) \cong H^*(M/\mathcal{F})$.

Why does it work? Theorem 1

Let $(M, \mathcal{F})$ be a CERF.

- There exists a bundle-like metric μ on M whose mean curvature form κ_μ is a basic cycle.
- The tautness class $\boldsymbol{\kappa}_M = [\kappa_\mu] \in H^1(M/\mathcal{F})$ does not depend on the choice of μ.

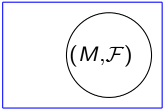
$\boxed{\text{The following statements are equivalent:}}$

- The foliation $(M, \mathcal{F})$ is taut.
- The tautness class $\boldsymbol{\kappa}_M$ vanishes.
- $H_c^{codim \mathcal{F}}(M/\mathcal{F}) \neq 0$ (when $(M, \mathcal{F})$ transversally oriented).
- $H^0_{\kappa_\mu}(M/\mathcal{F}) \neq 0$.

Why does it work? Theorem 1. Idea of the proof.

- There exists a bundle-like metric μ on M whose mean curvature form κ_{μ} is a basic cycle.

Because a such metric exists on the zipper $(N, \mathcal{H})$.



$(N,\mathcal{H})$

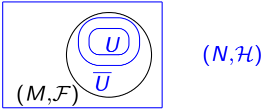
Why does it work? Theorem 1. Idea of the proof.

- The tautness class $\boldsymbol{\kappa}_M = [\kappa_\mu] \in H^1(M/\mathcal{F})$ does not depend on the choice of μ.

Consider two candidates μ_1 and μ_2. The first condition of the reppiz

The closure $\overline{U}$ (in M) is compact

implies the existence of two "good" extensions $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ on N with $\widetilde{\mu}_1 = \mu_1$ and $\widetilde{\mu}_2 = \mu_2$ on U.



The uniqueness of the tautness class on N gives the uniqueness of the tautness class on U and therefore $[\kappa_{\mu_1}] = [\kappa_{\mu_1}]$ on U. The second condition of the reppiz

$$H^*(U/\mathcal{F}) \cong H^*(M/\mathcal{F})$$

implies that $[\kappa_{\mu_2}] = [\kappa_{\mu_2}]$ on M.

Why does it work? Theorem 2

Each stratum of a SRF
defined on a compact manifold
is a CERF.

Back to Singular Riemannian Foliations

Let $\mathcal{K}$ be a singular riemannian foliation defined on a compact manifold X. The tautness class of $\mathcal{K}$ is the class $\boldsymbol{\kappa}_X \in H^1(X/\mathcal{K})$ verifying

$$\iota_S^* \boldsymbol{\kappa}_X = \boldsymbol{\kappa}_S, \text{for each stratum } S \text{ of } \mathcal{K},$$

where $\iota_S \colon S \hookrightarrow X$ is the natural inclusion.

Back to Singular Riemannian Foliations

Some important facts

- The tautness class exists and it is unique.
- The following facts are equivalent
  - The tautness class $\boldsymbol{\kappa}_X$ vanishes.
  - For each stratum S of $\mathcal{K}$ the foliation $\mathcal{K}_S$ is taut.
  - The foliation $\mathcal{K}_R$ is taut.
- The tautness class vanishes when the manifold X is simply connected (E.Ghys for the regular case).
- The tautness class vanishes when the codimension of $\mathcal{K}$ is one. (J.Álvarez for the regular case).
- If $S_1 \preceq S_2$ and $\mathcal{K}_{S_2}$ is taut then $\mathcal{K}_{S_1}$ is taut.
- It is possible to find a stratum S_1 with $\mathcal{K}_{S_1}$ taut and a stratum S_2 with $\mathcal{K}_{S_2}$ no taut.(J.I. Royo Prieto).

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