On Bode's "Ideal cut-off characteristics" and non-rational feedback laws

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Abstract *On the base of a specific analysis and published examples, we show how the so-called problem of "insensitivity of damping ratio to load changes", devoted to robust control and with origin in the notion of "cut-off characteristics" introduced by Bode, can easily run in unacceptable controls with internal instabilities. This is in particular the case when "fractional" feedbacks, of transfer function* $k\,s^n$ *with n non integer, are used to control systems involving second order dynamics such as inertial ones. Such instabilities are in general not obvious at first sight and thus all the more pernicious.*

I. Introduction

As reminded by Åström in [1], a key objective of feedback control is to reduce the effects of disturbances under the constraint of stability of the controlled system. In this context, Bode introduced in [5] the definition of "ideal cut-off characteristics", a nice notion relating to the phase margin of the open-loop controlled transfer and possibly involving non rational feedbacks, for example with transfer function $k\,s^{-\alpha}$, $0 \leq \alpha < 1$ when controlling simple first order systems. Thanks to the particularly simple phase margin characteristics introduced by Bode, this notion may have induced a few decades later some kind of ideology : the so-called "insensitivity of the damping ratio to load changes", which have been recently investigated in control of mechanical systems by some authors following various approaches [2], [4], [6], [7], [8], [14], [16]. Such a characteristic is interpreted by Oustaloup *et al.* [17] as "insensitivity of the damping ratio" and attributed to the fractional differentiation[1] properties, these operators thus being central from their point of view. On their side, Devy-Vareta and Montseny [7,11] explain that it is in fact only a particular case of invariance of the closed loop dynamic response up to a suitable scale transformation, which in general results from more sophisticated non rational feedback operators of pseudodifferential type [10], [12], [13].

The main objective of this note is to highlight, with the help of published examples, that problems relating to this kind of insensitivity, when considering it as an *ideal* property devoted to "robust control" strategies, can easily run, in some situations, into major drawbacks somewhat hidden by the non-rational nature of the employed feedbacks. The practical consequence lies in the internal instability of the controlled system. Beyond their initial nice and simple formulations (sometimes whith straightforward solutions), such rather unusual problems must in fact be tackled with much care, namely by explicitly taking into account the *stability constraint* on *any of the dynamic transfers* involved in the controlled system, as recalled in [18] in the general framework of robust control. Of course, less simple solutions should be expected in such a case. Unfortunately and in opposite to more classical robust control approaches [19], this essential stability constraint is often insufficiently pointed out in some approaches based on non-rational feedbacks, or even sometimes simply forgotten with unacceptable resulting solutions.

As a consequence of the present work, insensitivity of the phase margin (that is "ideal cut-off characteristics" devoted to *load changes*) reveals itself rather *incompatible* with internal stability: in

[1] Such operators have the symbolic expression s^α, with α non integer.

the framework of robust control of dynamic systems, this too strong notion should therefore be appropriately modified to be really suitable for control purposes.

The paper is organized as follows : the second section recalls the concept of ideal cut-off characteristics of Bode. In section 3, the property of phase margin insensitivity is analyzed in terms of robust control. Some undesirable consequences of the approach, such as internal instability, are analyzed in the section 4, and illustrated in section 5 by means of concrete example. Some remarks are given in conclusion

II. The ideal cut-off characteristics.

In his well-known book "Network Analysis and Feedback Amplifier Design" [5], Hendrick W. Bode introduced new analysis tools for the specification and design of feedback systems. The most used of them is certainly the "Bode diagram", which has been very largely described in every lectures on frequency domain representation of dynamic linear systems. The main contribution of Bode in this context must be the asymptotic construction of the frequency gain and phase shift on a logarithmic scale of frequency.
In the first chapters, the main topics on feedback are introduced, through the definition of sensitivity transfer function : the motivation for feedback is to reduce the effect of disturbances and perturbations on plant. The sensitivity is as small as open loop gain can be large.
After the statement of the closed loop stability, the two last chapters of the book are dedicated to the design of stable feedback amplifiers. In these chapters, H. W. Bode introduced the concept of "ideal cut-off characteristics", deduced from the gain and phase margins introduced earlier by Nyquist [15].
For open loop Single-Input-Single-Output systems they describe the optimal trade-off between a desired phase margin (assumed to be independent of the open-loop gain) and the most rapid cut-off. As the phase shift and the rate of gain decrease are related trough Bode's integral relation, a non-integer variable y is introduced to specify this trade-off :

$$\varphi_M = y\pi \, rad \qquad - 1$$

This ideal transfer function is realized in a bounded frequency range by a passive low-pass filter whose utility is "to secure a loop cut-off as soon as possible to avoid the difficulties and uncertainties of design which parasitic elements in the circuit introduce at high frequencies"[2]. The corresponding slope of gain decrease in Bode plane is then $-(1-y)40$ *dB/dec.*
The corresponding transfer function can be written as in eq. 2:

$$\beta_{id}(s) = \frac{K}{\left(\sqrt{1+\frac{s^2}{\omega_0^2}}+\frac{s}{\omega_0}\right)^{2(1-y)}} \qquad - 2$$

The figure 1 below represents the Bode plot of this ideal transfer function for $K = 100$ and $\omega_0 = 1$ *rad/s* (+ plot) together with the Bode plot of $\beta(s) = \frac{K}{(1+2s)^{2(1-y)}}$.

Isaac M. Horowitz used this ideal cut-off in his Quantitative Feedback Theory [3], although his interpretation is slightly different. The concept of robustness to parameter uncertainty that he introduced, is related to worst case guarantee rather than insensitivity. From Horowitz point of view, it is seen as the boundary of a domain ("template") inside which the open loop locus must not enter.

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[2] This cut-off would be called « roll-off constraint » in a modern paper : it is an early intuition of the constraint on robust stability facing up to neglected dynamics. This has been clearly stated later by the application of the small gain theorem [19] on the Linear Fractional Transformation model of uncertain system.

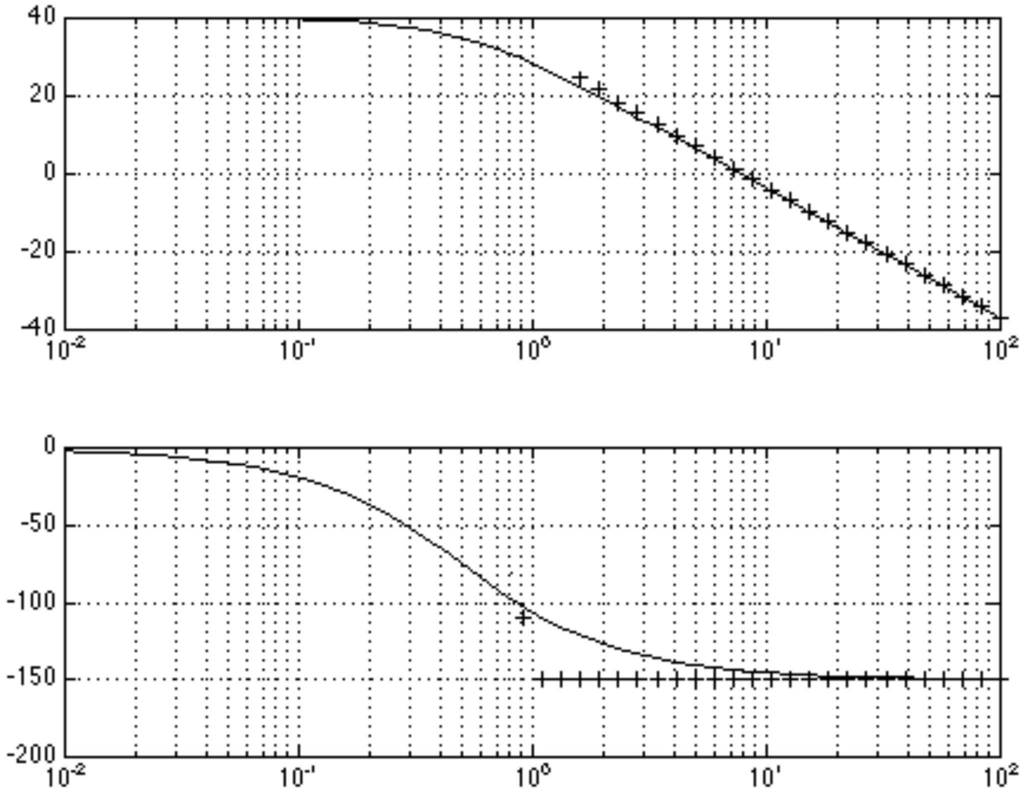

Figure 1

In "Model Uncertainty and Robust Control" [4], Karl J. Åström relates all these features, but he unfortunately represents the ideal cut-off characteristics by the ideal transfer function of eq. - 3

$$L_{ideal}(s) = \left(\frac{s}{\omega_0}\right)^n, \; -2 < n < -1 \qquad - 3$$

Such representation of ideal loop transfer function can lead to conceptual errors in the design of feedback insensitive control design, as it will be shown in the sequel. Such errors are avoided if all the objectives of feedback are taken into account in a good representation. Checking the stability of all the four sensitivity functions described in Fig. 2 (that is *S, T, CS, PS* here-after stated in (4)) is indeed at least necessary, according to internal stability theorem [1], [18].

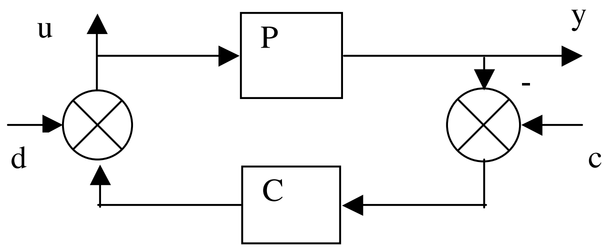

Figure 2

The 4 sensitivity functions are:

$$S = \frac{u}{d} = \frac{1}{1+PC}$$
$$T = \frac{y}{c} = \frac{PC}{1+PC}$$
$$CS = \frac{u}{c} = \frac{C}{1+PC} \qquad - 4$$
$$PS = \frac{y}{d} = \frac{P}{1+PC}$$

III. Insensitivity and robustness

In [1], Åström gives an example of 2^{nd} order plant under the form of the stable transfer function of eq. 5:

$$P(s) = \frac{k}{s(\omega_1 + s)} \qquad - 5$$

As the desired phase margin is 45°, applying eq. 3 results in :

$$C(s) = \frac{s+\omega_1}{\sqrt{s}} \qquad - 6$$

which is improper[3], so that a high frequency low-pass filter will be needed to make the controller feasible. The necessary robustness to un-modeled dynamics imposes in fact that the roll-off in the frequency response of the open loop must be sufficient to verify the small gain theorem.

However in this case, due to the fact that, in the low frequency range, the plant behaves like a first order integrator, the controller has a low frequency range behavior of (1/2) order integrator.

The main drawback of the controller is that its zero cancels a pole of the plant, making it uncontrollable, which is only possible if this pole is in the strict left half plane. This is the case for example in [4].

Let $-\omega_2$ be the actual pole of the plant, the 4 sensitivity functions are [4]:

$$S = \frac{\frac{s^{3/2}}{k}\left(\frac{s+\omega_2}{s+\omega_1}\right)}{\frac{s^{3/2}}{k}\left(\frac{s+\omega_2}{s+\omega_1}\right)+1} \underset{\omega_2 \to \omega_1}{\rightarrow} \frac{\frac{s^{3/2}}{k}}{\frac{s^{3/2}}{k}+1} \qquad - 7$$

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[3] That is $\lim_{\omega \to \infty} \left|C(j\omega)\right| = +\infty$.

[4] The CS sensitivity appears to be improper, which would be enough to reject such control. Actually, the necessary low-pass filter added to C(s) to make it feasible will ensure properness of this sensitivity.

$$T = \frac{1}{\frac{s^{3/2}}{k}\left(\frac{s+\omega_2}{s+\omega_1}\right)+1} \underset{\omega_2 \to \omega_1}{\rightarrow} \frac{1}{\frac{s^{3/2}}{k}+1} \qquad - 8$$

$$CS = \frac{\frac{s}{k}\left(s+\omega_2\right)}{\frac{s^{3/2}}{k}\left(\frac{s+\omega_2}{s+\omega_1}\right)+1} \underset{\omega_2 \to \omega_1}{\rightarrow} \frac{\frac{s}{k}\left(s+\omega_2\right)}{\frac{s^{3/2}}{k}+1} \qquad - 9$$

$$PS = \frac{\left(\frac{s^{1/2}}{s+\omega_1}\right)}{\frac{s^{3/2}}{k}\left(\frac{s+\omega_2}{s+\omega_1}\right)+1} \underset{\omega_2 \to \omega_1}{\rightarrow} \frac{\left(\frac{s^{1/2}}{s+\omega_1}\right)}{\frac{s^{3/2}}{k}+1} \qquad - 10$$

The actual open-loop transfer function, *$L_{actual}(s)$* is given by eq. 11.

$$L_{actual}(s) = \frac{k}{s^{3/2}} \frac{s+\omega_1}{s+\omega_2} \qquad - 11$$

Its only quality is in fact (quasi)-invariance of phase margin with respect to the gain uncertainty (i.e. parameter k). It is possible to evaluate the phase margin error $\varepsilon_{M\phi}$ due to the uncertainty on the parameter ω_2:

$$\varepsilon_{M\varphi} = \angle L_{ideal}(j\omega_u) - \angle L_{actual}(j\omega_u) = Arc\tan\left[\frac{\omega_u(\omega_1-\omega_2)}{\omega_1\omega_2+\omega_u^2}\right] \qquad - 12$$

where $\omega_u \propto k^{-\frac{2}{3}}$ is the crossover frequency. This error, which is close to 0 if $|\omega_1 - \omega_2|$ is small enough, is of course negative if $\omega_2 < \omega_1$ (worst phase margin).
For given values of ω_1 and ω_2, the phase margin error is as small as possible if and only if ω_u is as great as possible. This is an illustration of insensitivity of closed loop through large gain of the open loop.
In [4] and [6] similar results are presented using the so-called diffusive representation, an infinite-dimensional technique allowing concrete approximate realizations of the controller.

IV. Internal stability versus insensitivity

Let us now consider that ω_2 in (5). This is namely the case for inertial systems with mass (or momentum) $M = \frac{1}{k}$. The above analysis then leads to the proper feedback:

$$C(s) = \frac{s}{\sqrt{s}\left(1+\frac{s}{\omega_h}\right)} = \frac{\sqrt{s}}{\left(1+\frac{s}{\omega_h}\right)}. \qquad - 13$$

Other phase margins can be chosen, so that more general open-loop transfers of the form:

$$L_{ideal}(s) = \left(\frac{\omega_u}{s}\right)^n \frac{1}{\left(1+\frac{s}{\omega_h}\right)}, 1 < n < 2 \qquad - 14$$

in which $\omega_u := \left(\frac{1}{M\omega_0^{2-n}}\right)^{\frac{1}{n}}$, if ω_h is chosen much greater than ω_u, are straightforwardly obtained by means of the following "fractional" feedback:

$$C(s) = \left(\frac{s}{\omega_0}\right)^{2-n}, 0 < 2 - n < 2, \qquad - 15$$

The suspension system can be represented by the closed loop of the figure 3 below, in which *d* is a force disturbance appearing on the car, resulting either from a variation of the suspended weight, or from a force generated by an acceleration or a sudden brake.

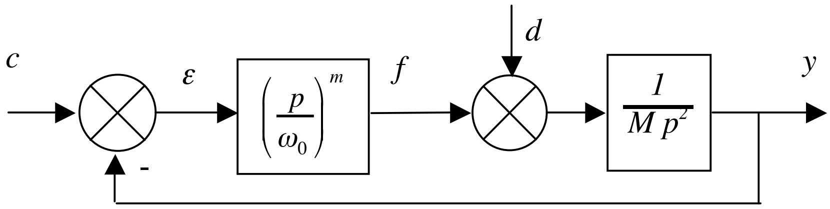

Figure 3

On such a diagram, the sensitivity function from the disturbance *d* to the position error *ε* is :

$$\frac{\varepsilon}{d} = -PS = -\frac{1}{M\omega_0^m}\frac{1}{p^{2-m}} \qquad - 16$$

which is unstable as $0 < 2 - m < 1$. A constant disturbance *d* generates an error *ε* which tends to - ∞ as the power – *n* of the time.

In the second version of this suspension, as presented in [5], the ideal loop transfer function of eq. 13 is replaced by the actual one:

$$L_{actual}(p) = C_0\left(1 + \frac{p}{\omega_b}\right)^m \left(1 + \frac{p}{\omega_h}\right)^{1-m} \left(\frac{1}{M p^2}\right) \qquad - 17$$

The objective is to realize the constant phase shift of the ideal open loop, at *(1-m)π rad*, in a limited frequency range between 2 specified values ω_A and ω_B.
Due to the equation 12, a small value of the phase margin error $\varepsilon_{M\phi}$ implies a small value of ω_b versus the crossover frequency ω_0, even in the worst case of the greatest value of the suspended mass *M*.
The closed loop sensitivity function between *d* and *ε* becomes:

$$\frac{\varepsilon}{d} = -PS = -\frac{1}{M p^2 + C_0\left(1 + \frac{p}{\omega_b}\right)^m \left(1 + \frac{p}{\omega_h}\right)^{1-m}} \qquad - 18$$

This transfer function is stable, but, due to the low value of ω_b, the steady state value of the controller, the stiffness coefficient $C_{0,}$ is assumed to have a very low value of 215 *N/m* so that the steady state

value of sensitivity *PS* is unacceptable. Any additional system, resulting in an external position integral control as it is possible with hydraulic suspension system, cannot avoid this drawback.
In [6], Devy-Vareta and Montseny present a similar controller for the position tracking of a flexible beam. As the model of the flexible beam in the useful frequency range is a double integrator, the controller must be of the form of eq. 15, i. e. a m^{th} order differentiator, which results in the same problem of internal instability.

Conclusion

In the book of Bode [1], the fractional order transfer function is introduced to guarantee the degree of stability of feedback amplifier, despite uncertainties on its gain. Hence the controller is a low pass filter whose frequency response has a constant argument around the crossover frequency. The value of this argument is related to the fractional order of the ideal transfer function.
The description by Åström [4] of the fractional control presents the objective of time invariance, as in Oustaloup [5], while Devy-Vareta and Montseny [6] realize pseudo-invariance. In fact, this objective introduces a modification of the ideal cut-off characteristics: it results in a frequency domain open loop reference model under the form of a fractional integrator. To obtain this transfer function, it is necessary to cancel one pole of the plant by a zero of the controller : this is possible, within the approximation of eq. 12, only if this pole is stable!
In the case of vehicle suspension [5] or flexible beam [6], where the model is a double integrator, the pole to cancel is unstable, so that the closed loop is not internally stable. The approximate controller given in [5] give good insensitivity of phase margin to variations of the mass of the vehicle only in the case where the steady state gain of the suspension, which is in fact the stiffness of the equivalent spring, is very small.
Introduced by Bode in the very specific context of feedback amplifiers, the concept of ideal loop transfer function remains interesting as long as the objective is to represent a certain insensitivity of phase margin to variations of gain. With this point of view, no particular property of robustness can be given to fractional order systems : the realization of the ideal loop transfer function needs always that the open loop has a great gain (or, which is the same, a high crossover frequency). Moreover, the representation by fractional differentiation should be avoided, as it produces often internal instability. The use of ideal loop transfer function must be reserved to the case where it can be obtained by a fractional integrator controller.

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